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Nordsieck representation of high order predictor-corrector Obreshkov methods and their implementation

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Abstract Predictor-corrector (PC) methods for the numerical solution of stiff ODEs can be extended to include the second derivative of the solution. In this paper, we consider second derivative PC methods with the three-step second derivative Adams-Bashforth as predictor and two-step second derivative Adams-Moulton as corrector which both methods have order six. Implementation of the proposed PC method is discussed by providing Nordsieck representation of the method and preparing an starting procedure, an estimate for local truncation error and a formula for changing stepsize. Efficiency and capability of the method are shown by some numerical experiments.

Keywords. PEC methods, Adams methods, Nordsieck representation, Local error estimation, Variable stepsize.

2010 Mathematics Subject Classification. 65L05.

1. Introduction

Many codes have been introduced for solving

$$y'(x) = f(x, y), \quad y(x_0) = y_0, \quad x \in I := [x_0, \overline{x}], \tag{1.1}$$

where $f : I \times \mathbb{R}^m \to \mathbb{R}^m$ and m is the dimensionality of the system, in the class of linear multistep methods (LMMs) which use first derivatives of the solution (for instance [6, 9, 15]). Adams methods [7, 14] for the numerical integration of (1.1) are an special case of LMMs which are usually implemented in predictor-corrector (PC) form.

For problems in which

$$g(x, y) := f_x(x, y) + f_y(x, y) \cdot f(x, y) = y'',$$

can be calculated along with $f(x, y)$, at a moderate additional cost, second derivative methods become feasible which can be of high order of accuracy. In this class of the

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methods, some successful methods have been introduced that have good properties, especially for stiff problems [1, 2, 3, 4, 5, 8, 10, 11, 12, 13, 16, 17]. Here, we will use second derivative Adams methods as PC pairs which k–step second derivative Adams-Bashforth method (ABM)

$$\begin{aligned} y_n = y_{n-1} &+ h(\hat{a}_1 f_{n-1} + \hat{a}_2 f_{n-2} + \cdots + \hat{a}_k f_{n-k}) \\ &+ h^2(\hat{b}_1 g_{n-1} + \hat{b}_2 g_{n-2} + \cdots + \hat{b}_k g_{n-k}), \end{aligned} \tag{1.2}$$

is used for predicted part and $(k-1)$–step second derivative Adams-Moulton method (AMM)

$$\begin{aligned} y_n = y_{n-1} &+ h(a_0 f_n + a_1 f_{n-1} + \cdots + a_{k-1} f_{n-k+1}) \\ &+ h^2(b_0 g_n + b_1 g_{n-1} + \cdots + b_{k-1} g_{n-k+1}), \end{aligned} \tag{1.3}$$

is used for corrected part. Here, $f_{n-i} = f(x_{n-i}, y_{n-i})$ and $g_{n-i} = g(x_{n-i}, y_{n-i})$, $i = 0, 1, \ldots, k$.

In this paper, we describe a nice approach to the implementation of the PECE (predict-evaluate-correct-evaluate) mode of PC formulas (1.2)–(1.3) with $k = 3$ in a variable stepsize environment using Nordsieck representation. For $k = 3$, the coefficients of the methods (1.2) and (1.3) are chosen so that these methods have order six. In this way, the resulting ABM and AMM take the forms

$$\begin{aligned} y_n = y_{n-1} &- \frac{949}{240} h f_{n-1} + \frac{38}{15} h f_{n-2} + \frac{581}{240} h f_{n-3} + \frac{637}{240} h^2 g_{n-1} \\ &+ \frac{9}{2} h^2 g_{n-2} + \frac{173}{240} h^2 g_{n-3}, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} y_n = y_{n-1} &+ \frac{101}{240} h f_n + \frac{8}{15} h f_{n-1} + \frac{11}{240} h f_{n-2} - \frac{13}{240} h^2 g_n \\ &+ \frac{1}{6} h^2 g_{n-1} + \frac{1}{80} h^2 g_{n-2}, \end{aligned} \tag{1.5}$$

respectively.

The paper is organized along the following lines. In Section 2, Nordsieck representation of the PC pairs (1.4) and (1.5) is obtained. This leads to a discussion of the variable stepsize mode of the proposed method and implementation issues including starting procedures, local error estimation and stepsize control. These are explained in Section 3. Finally, some numerical results are given in Section 4 to show efficiency of the method.

2. Nordsieck representation of the PC pairs

Nordsieck representation of the methods makes them very proper for implementation in a variable stepsize environment. Indeed, in this way, changing stepsize can be done very simple and inexpensive. Here we concentrate on a sixth order PC method consisting of the three step ABM (1.4) as a predictor and two step ABM (1.5) as a corrector.

For a Nordsieck method of order p, the input and output data at the step number n are in the form

$$N_{p,n-1} = \begin{bmatrix} y(x_{n-1}) \\ hy'(x_{n-1}) \\ \vdots \\ h^p y^{(p)}(x_{n-1}) \end{bmatrix} + O(h^{p+1}) \text{ and } N_{p,n} = \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ \vdots \\ h^p y^{(p)}(x_n) \end{bmatrix} + O(h^{p+1}),$$

respectively. So, for our method which is of order six, the output vector must have seven components including values at the current point x_n and previous points x_{n-1} and x_{n-2}. We choose the output vector as

$$Y_n = \begin{bmatrix} y_n \\ hf_n \\ h^2 g_n \\ hf_{n-1} \\ h^2 g_{n-1} \\ hf_{n-2} \\ h^2 g_{n-2} \end{bmatrix} = \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ h^2 y''(x_n) \\ hy'(x_{n-1}) \\ h^2 y''(x_{n-1}) \\ hy'(x_{n-2}) \\ h^2 y''(x_{n-2}) \end{bmatrix} + O(h^7). \tag{2.1}$$

Where $y(x)$ is the exact solution of the equation $y' = f(x, y)$ at the point x. By using Taylor expansion for each component of the vector in the right hand side of (2.1), we have

$$\begin{bmatrix} y(x_n) \\ hy'(x_n) \\ h^2 y''(x_n) \\ hy'(x_{n-1}) \\ h^2 y''(x_{n-1}) \\ hy'(x_{n-2}) \\ h^2 y''(x_{n-2}) \end{bmatrix} = T \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ \frac{h^2}{2!} y''(x_n) \\ \frac{h^3}{3!} y'''(x_n) \\ \frac{h^4}{4!} y^{(4)}(x_n) \\ \frac{h^5}{5!} y^{(5)}(x_n) \\ \frac{h^6}{6!} y^{(6)}(x_n) \end{bmatrix} + O(h^7), \tag{2.2}$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & -4 & 5 & -6 \\ 0 & 0 & 2 & -6 & 12 & -20 & 30 \\ 0 & 1 & -4 & 12 & -32 & 80 & -192 \\ 0 & 0 & 2 & -12 & 48 & -160 & 480 \end{bmatrix}.$$

So, we obtain the relation between the vectors Y_n and N_n given by

$$Y_n = T\, N_{6,n}. \tag{2.3}$$

Now, we derive Nordsieck form of the PC pairs (1.4) and (1.5) in the PECE mode.

- First stage
  At the first stage, we obtain an approximation using predictor equation

$$y_n^* = y_{n-1} - \frac{949}{240}hf_{n-1} + \frac{38}{15}hf_{n-2} + \frac{581}{240}hf_{n-3} + \frac{637}{240}h^2g_{n-1} + \frac{9}{2}h^2g_{n-2} + \frac{173}{240}h^2g_{n-3}. \tag{2.4}$$

  Since the values y_{n-1}, f_{n-1}, g_{n-1}, f_{n-2}, g_{n-2}, f_{n-3}, and g_{n-3} are available, the vector representing input and output data at the step number n, are in the form

$$Z_{n-1} \approx \begin{bmatrix} y(x_{n-1}) \\ hf(x_{n-1}) \\ h^2g(x_{n-1}) \\ hf(x_{n-2}) \\ h^2g(x_{n-2}) \\ hf(x_{n-3}) \\ h^2g(x_{n-3}) \end{bmatrix} \quad \text{and} \quad Z_n \approx \begin{bmatrix} y(x_n) \\ hf(x_n) \\ h^2g(x_n) \\ hf(x_{n-1}) \\ h^2g(x_{n-1}) \\ hf(x_{n-2}) \\ h^2g(x_{n-2}) \end{bmatrix}, \tag{2.5}$$

  respectively. Now we define the vector Z_n^*: The first component of it is y_n^* obtained by predictor equation (2.4) and the next two components are related to the evaluate step, and the last four components are $hf(x_{n-1})$, $h^2g(x_{n-1})$, $hf(x_{n-2})$, and $h^2g(x_{n-2})$. So it can be written in the form

$$Z_n^* = H\, Z_{n-1}, \tag{2.6}$$

  where

$$H = \begin{bmatrix} 1 & -\frac{949}{240} & \frac{637}{240} & \frac{38}{15} & \frac{9}{2} & \frac{581}{240} & \frac{173}{240} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

  We note that Z_n^* means the vector of predicted values and Z_{n-1} is the vector of values at the previous step.
- Second step
  The second step is to evaluate functions f and g at the point (x_n, y_n^*) and add them into the formula (2.6). To do this, we define the vectors

$$E_1 = \begin{bmatrix} 0 & hf(x_n, y_n^*) & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

  and

$$E_2 = \begin{bmatrix} 0 & 0 & h^2g(x_n, y_n^*) & 0 & 0 & 0 & 0 \end{bmatrix},$$

  then we define a new vector of two stages predict-evaluate in the form

$$Y_n^* = H\, Z_{n-1} + E_1 + E_2. \tag{2.7}$$

To write (2.7) in terms of Nordsieck vectors, at first we note that by Taylor expansion, we have

$$E_1 = e_2 \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} N_{6,n-1} + O(h^7),$$

and

$$E_2 = e_3 \begin{bmatrix} 0 & 0 & 1 & 3 & 6 & 10 & 15 \end{bmatrix} N_{6,n-1} + O(h^7),$$

where the vectors e_2 and e_3 are the second and third columns of the identity matrix of dimension seven, respectively. Now, the Nordsieck representation of (2.7) is

$$N^*_{6,n} = P\, N_{6,n-1},$$

with

$$\begin{aligned} P = T^{-1}HT + T^{-1}e_2 &\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \\ &+ 2T^{-1}e_3 \begin{bmatrix} 0 & 0 & 1 & 3 & 6 & 10 & 15 \end{bmatrix}, \end{aligned}$$

which is the upper triangular Pascal matrix.

- Third step
  In this step, we include the correction equation. The Nordsieck representation for PEC Adams method of order six is

$$N_{6,n} = PN_{6,n-1} + \delta_1 T^{-1}\alpha + \delta_2 T^{-1}\beta, \tag{2.8}$$

  where α and β are equal to vectors $a_0\, e_1 + e_2$ and $b_0\, e_1 + e_3$, that are the corresponding coefficients $a_0 = \frac{101}{240}$ and $b_0 = -\frac{13}{240}$ from the terms $\frac{101}{240}hf_n$ and $-\frac{13}{240}h^2 g_n$, respectively, in the correction equation (1.5). Here, the error estimations δ_1 and δ_2 with

$$\delta_1 = hf(y^*_n) - [0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6]N_{6,n-1},$$

$$\delta_2 = h^2 g(y^*_n) - 2[0 \quad 0 \quad 1 \quad 3 \quad 6 \quad 10 \quad 15]N_{6,n-1}.$$

  represent the correction of the first and second derivatives, respectively.
- Fourth step
  Finally, we include the final step which is related to the functions evaluations at (x_n, y_n)

$$f_n = f(x_n, y_n), \qquad g_n = g(x_n, y_n),$$

  to be used in the next time step.

3. Practical implementation of the method

In this section, we concentrate on the implementation issues for our method. Nordsieck PC method (2.8) in the variable stepsize mode is in the form

$$N_{6,n} = (PD(\theta_n))N_{6,n-1} + \delta_1 T_1 + \delta_2 T_2 \tag{3.1}$$

where

$$T_1 = \Big[\frac{101}{240} \quad 1 \quad 0 \quad -\frac{23}{12} \quad -\frac{33}{16} \quad -\frac{17}{20} \quad -\frac{1}{8}\Big],$$

$$T_2 = \Big[-\frac{13}{240} \quad 0 \quad \frac{1}{2} \quad 1 \quad \frac{13}{16} \quad \frac{3}{10} \quad \frac{1}{24}\Big].$$

and $D(\theta_n)$ is the rescaling matrix defined by

$$D(\theta_n) := \mathrm{diag}(1, \theta_n, \theta_n^2, \ldots, \theta_n^p),$$

with θ_n as the ratio of consecutive stepsizes, $\theta_n = h_n/h_{n-1}$, and $h_n = x_n - x_{n-1}$.

3.1. **Starting procedure.** For the method of order $p = 6$ (2.8), a starting procedure of order six is required to approximate the initial Nordsieck vector $N_{6,0}$ which is an approximation to the vector

$$\Big[y(x_0) \quad hy'(x_0) \quad \frac{h^2}{2!}y''(x_0) \quad \cdots \quad \frac{h^6}{6!}y^{(6)}(x_0)\Big]^T.$$

Since the first three components of this vector is known, we need to approximate only the last four components of that. To do that, we carry out one step of a Runge–Kutta method with abscissa vector $\widetilde{c}$ which gives sufficient output information, $\widetilde{y}_1 \approx y(x_0 + h_0)$ and $\widetilde{Y}_i \approx y(x_0 + \widetilde{c}_i h_0)$, $i = 1, 2, \ldots, s$. We can obtain a reliable approximations by using some linear combination of these information as

$$\begin{aligned}
h^3y^{(3)}(x_0) &= a_1y(x_0) + a_2hy'(x_0) + a_3h^2y''(x_0) + a_4h^2g(\widetilde{Y}_1) + a_5h^2g(\widetilde{Y}_2)\\
&\quad + a_6h^2g(\widetilde{Y}_3) + a_7h^2g(\widetilde{Y}_4) + O(h^7),\\
h^4y^{(4)}(x_0) &= b_1y(x_0) + b_2hy'(x_0) + b_3h^2y''(x_0) + b_4h^2g(\widetilde{Y}_1) + b_5h^2g(\widetilde{Y}_2)\\
&\quad + b_6h^2g(\widetilde{Y}_3) + b_7h^2g(\widetilde{Y}_4) + O(h^7),\\
h^5y^{(5)}(x_0) &= c_1y(x_0) + c_2hy'(x_0) + c_3h^2y''(x_0) + c_4h^2g(\widetilde{Y}_1) + c_5h^2g(\widetilde{Y}_2)\\
&\quad + c_6h^2g(\widetilde{Y}_3) + c_7h^2g(\widetilde{Y}_4) + O(h^7),\\
h^6y^{(6)}(x_0) &= d_1y(x_0) + d_2hy'(x_0) + d_3h^2y''(x_0) + d_4h^2g(\widetilde{Y}_1) + d_5h^2g(\widetilde{Y}_2)\\
&\quad + d_6h^2g(\widetilde{Y}_3) + d_7h^2g(\widetilde{Y}_4) + O(h^7).
\end{aligned}$$

3.2. **Local error estimation.** The local truncation error of a method of order p, in the step number n, is defined by

$$\mathrm{LTE}(x_n) = C_p h^{p+1} y^{(p+1)}(x_n) + O(h^{p+2}),$$

where C_p is the error constant of the method. In order to control the stepsize, we need to estimate the local truncation error for each step. Here, the estimation of LTE is produced by using the Milne device which is based on the difference of the predictor and corrector approximations: Denoting the error constants for the Adams-Bashforth and Adams-Moulton of order p by C_p^* and C_p, respectively, we have

$$y_n^* = y(x_n) - C_p^* h^{p+1} y^{(p+1)}(x_n) + O(h^{p+2}),$$

$$y_n = y(x_n) - C_p h^{p+1} y^{(p+1)}(x_n) + O(h^{p+2}),$$

which using the Milne device implies

$$y(x_n) - y_n = \frac{C_p}{C_p - C_p^*}(y^* - y_n) + O(h^{p+2}).$$

For $p = 6$, we have $C_6^* = \frac{53}{4725}$, $C_6 = \frac{1}{9450}$, and so

$$y(x_n) = y_n + \frac{1}{105}(y_n - y_n^*) + O(h^8).$$

Hence $\|\mathrm{LTE}(x_n)\|$ can be estimated as

$$\|\mathrm{LTE}(x_n)\| = \frac{1}{105}\|(y_n - y_n^*)\|.$$

3.3. **Stepsize control.** After estimating the local truncation error, we can control the stepsize by monitoring this estimation. For the given absolute and relative tolerances, *Atol* and *Rtol* respectively, we use the following control

$$\|\mathrm{LTE}(x_n)\| \leq Rtol \cdot \max\{\|y_n\|, \|y_{n+1}\|\} + Atol, \tag{3.2}$$

to control the stepsize in the proceeding from x_n to x_{n+1}. If the control (3.2) is not satisfied, the current step is repeated with the halved stepsize. Otherwise, the current step is accepted and we carry our the next step with the new stepsize as

$$h_{n+1} = \theta_{n+1} h_n,$$

where

$$\theta_{n+1} = \min\Big\{2, \alpha\Big(\frac{tol}{\|\mathrm{LTE}(x_n)\|}\Big)^{\frac{1}{7}}\Big\}.$$

In our numerical experiments we have used $Atol = Rtol = tol$, and the safety factor $\alpha = 0.9$ to guard against unnecessary step failures.

4. Numerical experiments

In this section we present the results of numerical experiments to show efficiency of the constructed method of order six in the variable stepsize mode.

Computational experiments are done by applying our method on the following problems.

P1. The modified Kepler problem

$$\begin{cases} y_1' = \frac{\partial H(y)}{\partial y_3}, \\ y_2' = \frac{\partial H(y)}{\partial y_4}, \\ y_3' = -\frac{\partial H(y)}{\partial y_1}, \\ y_4' = -\frac{\partial H(y)}{\partial y_2}, \end{cases}$$

with $H(y) = \dfrac{y_3^2 + y_4^2}{2} - \dfrac{1}{r} - \dfrac{\epsilon}{2r^3}$ and $r = \sqrt{y_1^2 + y_2^2}$ where ϵ is a positive or negative small number. The initial conditions are

$$\begin{cases} y_1 = 1 - e, \\ y_2 = 0, \\ y_3 = 0, \\ y_4 = \sqrt{\dfrac{1+e}{1-e}}. \end{cases}$$

In the numerical results, we take $\epsilon = 0.01$ and $e = 0.6$.

P2. The famous Lorenz equations provide a simple example of a chaotic system. They are given by

$$\begin{cases} y_1' = \sigma(y_2 - y_1), \\ y_2' = ry_1 - y_2 - y_1 y_3, \\ y_3' = y_1 y_2 - by_3, \end{cases}$$

where σ, r and b are positive parameters. Following Lorenz, we set $\sigma = 10$, $b = 8/3$, $r = 28$, $y(0) = [0, 1, 0]^T$ and $x \in [0, 50]$.

P3. The third problem is the Kepler's problem also known as the one-body problem which describes the motion of a single planet moving around a heavy sun. The problem is given by

$$\begin{cases} y_1' = y_3, \\ y_2' = y_4, \\ y_3' = \dfrac{-y_1}{(y_1^2 + y_2^2)^{3/2}}, \\ y_4' = \dfrac{-y_2}{(y_1^2 + y_2^2)^{3/2}}, \end{cases}$$

and the initial values are prescribed to be

$$\begin{cases} y_1 = 1 - e, \\ y_2 = 0, \\ y_3 = 0, \\ y_4 = \sqrt{\dfrac{1+e}{1-e}}, \end{cases}$$

where e is the eccentricity of an ellipse on which the orbit lies. With these initial values, all points in the orbit lie on the ellipse

$$(y_1 + e)^2 + \frac{y_2^2}{1 - e^2} = 1.$$

The results of numerical experiments for these problems are presented in Tables 1 – 4. In these tables *ns*, *nrs*, $h_{\min}$, $h_{\max}$, and *ge* are the number of steps, the number of rejected steps, the smallest used stepsize, the largest used stepsize, and the global error, respectively.

TABLE 1. Numerical results for problem P1 solved by the method (3.1) with $h_0 = 10^{-2}$ and $x \in [0, 500]$.

tol	*ns*	*nrs*	h_{min}	h_{max}	*ge*
10^{-10}	9787	185	$\approx 6.92 \times 10^{-4}$	$\approx 2.11 \times 10^{-1}$	6.8136×10^{-12}
10^{-11}	13365	5	$\approx 3.27 \times 10^{-4}$	$\approx 1.48 \times 10^{-1}$	6.4037×10^{-12}
10^{-12}	18583	6	$\approx 1.56 \times 10^{-4}$	$\approx 1.04 \times 10^{-1}$	5.8367×10^{-13}
10^{-14}	35896	8	$\approx 3.64 \times 10^{-5}$	$\approx 5.33 \times 10^{-2}$	5.2873×10^{-15}

TABLE 2. Numerical results for problem P2 solved by the method (3.1) with $h_0 = 10^{-2}$ and $x \in [0, 50]$.

tol	*ns*	*nrs*	h_{min}	h_{max}	*ge*
10^{-7}	4952	592	$\approx 1.32 \times 10^{-3}$	$\approx 2.65 \times 10^{-2}$	3.7237×10^{-8}
10^{-8}	5289	278	$\approx 6.47 \times 10^{-4}$	$\approx 2.05 \times 10^{-2}$	4.5766×10^{-9}
10^{-9}	6474	48	$\approx 3.05 \times 10^{-4}$	$\approx 1.71 \times 10^{-2}$	6.0246×10^{-10}
10^{-10}	8923	9	$\approx 1.46 \times 10^{-4}$	$\approx 1.26 \times 10^{-2}$	4.7838×10^{-11}

TABLE 3. Numerical results for problem P3 solved by the method (3.1) with $e = 0.5$, $h_0 = 10^{-3}$ and $x \in [0, 10\pi]$.

tol	*ns*	*nrs*	h_{min}	h_{max}	*ge*
10^{-10}	759	331	$\approx 4.98 \times 10^{-4}$	$\approx 1.85 \times 10^{-1}$	1.6253×10^{-7}
10^{-11}	1050	488	$\approx 1.79 \times 10^{-4}$	$\approx 1.14 \times 10^{-1}$	1.0812×10^{-8}
10^{-12}	1448	677	$\approx 2.51 \times 10^{-4}$	$\approx 8.95 \times 10^{-2}$	1.3658×10^{-9}
10^{-14}	2778	1313	$\approx 4.87 \times 10^{-5}$	$\approx 5.07 \times 10^{-2}$	1.3166×10^{-11}

TABLE 4. Numerical results for problem P3 solved by the method (3.1) with $e = 0.75$, $h_0 = 10^{-3}$ and $x \in [0, 10\pi]$.

tol	*ns*	*nrs*	h_{min}	h_{max}	*ge*
10^{-10}	1074	580	$\approx 1.58 \times 10^{-4}$	$\approx 1.79 \times 10^{-1}$	1.7627×10^{-7}
10^{-11}	1482	766	$\approx 1.19 \times 10^{-4}$	$\approx 1.30 \times 10^{-1}$	3.5347×10^{-8}
10^{-12}	2045	1083	$\approx 6.19 \times 10^{-5}$	$\approx 9.38 \times 10^{-2}$	1.8575×10^{-9}
10^{-14}	3942	2159	$\approx 1.29 \times 10^{-5}$	$\approx 5.17 \times 10^{-2}$	1.3269×10^{-11}

It is known that Kepler's problem is a Hamiltonian problem and by Figures 1 and 2, we can see that the method does not preserve the structure for the tolerance equals to $tol = 10^{-4}$. While by decreasing the tolerance to $tol = 10^{-14}$, Figures 3 and 4 present the solution almost without errors. Also, in Figures 5 and 6, errors for the

FIGURE 1. Solution of Kepler's problem with $e = 0.5$, $tol = 10^{-4}$, and $xmax = 50\pi$.

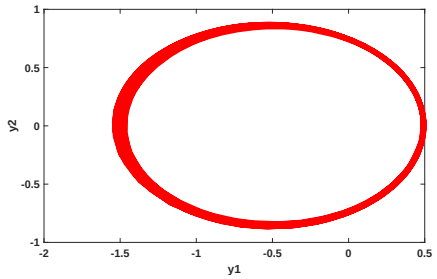


FIGURE 2. Solution of Kepler's problem with $e = 0.75$, $tol = 10^{-4}$, and $xmax = 50\pi$.

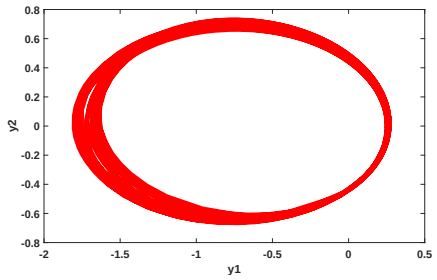
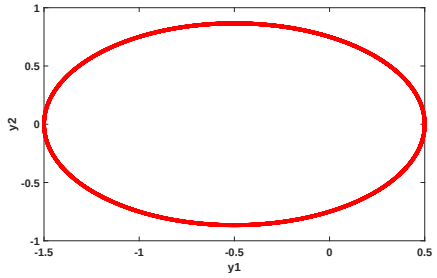


FIGURE 3. Solution of Kepler's problem with $e = 0.5$, $tol = 10^{-14}$, and $xmax = 10\pi$.



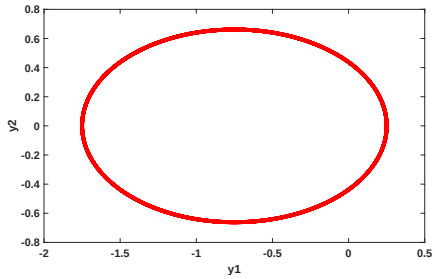
quantities

$$H(y) = \frac{1}{2}(y_3^2 + y_4^2) - (y_1^2 + y_2^2)^{-1/2},$$

as the "Hamiltonian" of system and

$$A(y) = y_1 y_4 - y_2 y_3,$$

FIGURE 4. Solution of Kepler's problem with $e = 0.75$, $tol = 10^{-14}$, and $xmax = 10\pi$.



as the "angular momentum" of system which are constant over the integration, have been plotted. This figures confirm that the method is capable to conserve these quantities of the system.

FIGURE 5. Errors in the Hamiltonian H for the Kepler's problem.

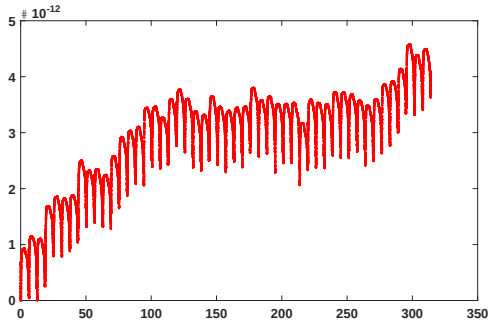
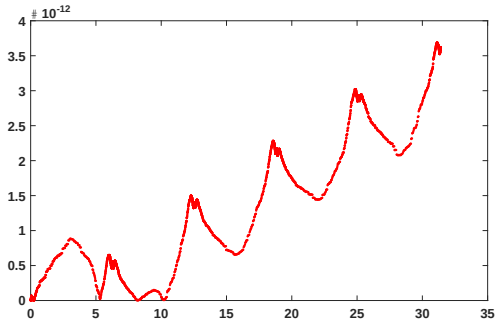


FIGURE 6. Errors in the angular momentum for the Kepler's problem.



5. Conclusion

A predictor-corrector method of order six based on the three-step second derivative Adams-Bashforth and two-step second derivative Adams-Moulton was analyzed. Using Nordsieck technique, variable stepsize mode of the method was introduced and its practical implementation was discussed by preparing a starting procedure, an estimation for the local truncation error, and changing stepsize strategy. Some system of differential equations were successfully tested.

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