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Symmetry group, Hamiltonian equations and conservation laws of general three-dimensional anisotropic non-linear sourceless heat transfer equation

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Abstract In this paper Lie point symmetries, Hamiltonian equations and conservation laws of general three-dimensional anisotropic non-linear sourceless heat transfer equation are investigated. First of all Lie symmetries are obtained by using the general method based on invariance condition of a system of differential equations under a prolonged vector field. Then the structure of symmetry operators as a Lie algebra are clarified and the classification of subalgebras under adjoint transformation is given. Hamiltonian equations including Hamiltonian symmetry are obtained. Finally a modified version of Noether's method including the direct method are applied in order to find local conservation laws of the equation.

Keywords. Heat transfer equation; Lie symmetry; Partial differential equation; Hamiltonian equations; Conservation laws.

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1. Introduction

Lie group theory plays a very important role in geometric analysis of differential equations and there are lots of papers and books have been presented about this subject, [5, 6, 7, 8, 9, 11, 12, 14, 16].

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For a smooth function $u(x, y, z, t)$ of three spatial variables (x, y, z) and the time variable t, the general three-dimensional anisotropic non-linear sourceless heat transfer equation is

$$u_t = [f(u)u_x]_x + [g(u)u_y]_y + [h(z)u_z]_z\,, \tag{1.1}$$

where f, g and h are smooth functions of u and they are non-negative according to their physical meaning [4, 10]. More generally the Eq. (1.1) is rectified to the equation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u,$$

where ∇ denotes the Laplace operator. In the physical problem of temperature variation, $u(x, y, z, t)$ is the temperature and α is the thermal diffusivity. For the mathematical treatment, we suppose $\alpha = 1$. The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic PDE. In probability theory, the heat equation is corresponded to the study of Brownian motion via the Fokker-Planck equation. In financial mathematics, it is used to solve the Black-Scholes PDE and the modeling of option pricing. Many of the extensions to the simple option models do not have closed form solutions and thus must be solved numerically to obtain a modeled option price. The equation describing pressure diffusion in a porous medium is identical in form with the heat equation. Diffusion problems dealing with Dirichlet, Neumann and Robin boundary conditions have closed form analytic solutions (Thambynayagam 2011). The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

Suppose one has a function u that describes the temperature at a given location (x, y, z, t). This function will change over time as heat spreads throughout space. The heat equation is used to determine the change in the function u over time. The rate of change of u is proportional to the "curvature" of u. Over time, the tendency is for peaks to be eroded, and valleys filled in. If u is linear in space (or has a constant gradient) at a given point, then u has reached steady-state and is unchanging at this point. The heat equation in probability also describes the random walks. It is also important in Riemannian geometry and topology: it was adapted by Richard Hamilton when he defined the Ricci flow that was later used by Grigori Perelman to solve the topological Poincaré conjecture.

The heat equation is also widely used in image analysis (Perona & Malik 1990) and in machine-learning as the driving theory behind scale-space or graph Laplacian methods. The heat equation can be efficiently solved numerically using the Crank-Nicolson method of (Crank & Nicolson 1947). This method can be extended to many of the models with no closed form solution, see for instance (Wilmott, Howison & Dewynne 1995). An abstract form of heat equation on manifolds provides a major approach to the Atiyah-Singer index theorem, and has led to much further works on heat equations in Riemannian geometry.

In the first section we begin by reminding the symmetry method of differential equations to obtain symmetries of the equation as a real Lie algebra. The structure of this Lie algebra is analyzed. The next section is devoted to study Hamiltonian equations and Hamiltonian symmetry of the heat equation. As we know there are several methods for finding conservation laws, but direct method and Noether's method are more practical. In the fourth section conservation laws of the heat equation via Noether's method is obtained. Then, direct method is applied to find some new conservation laws for some special cases of the considered equation.

2. Lie Symmetries of three-dimensional heat transfer equation

In this article we focus on the Eq. (1.1) for finding Lie symmetries of the equation. Let us consider a one-parameter Lie group of infinitesimal transformations in (x, y, z, t, u) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi^1(x, y, z, t, u) + \mathcal{O}(\varepsilon^2), \\ y^* &= y + \varepsilon\xi^2(x, y, z, t, u) + \mathcal{O}(\varepsilon^2), \\ z^* &= z + \varepsilon\xi^3(x, y, z, t, u) + \mathcal{O}(\varepsilon^2), \\ t^* &= t + \varepsilon\xi^4(x, y, z, t, u) + \mathcal{O}(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x, y, z, t, u) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{2.1}$$

including the transformations:

$$\begin{aligned} f^* &= f + \varepsilon\mu_1(x, y, z, t, u, f, g, h) + \mathcal{O}(\varepsilon^2), \\ g^* &= g + \varepsilon\mu_2(x, y, z, t, u, f, g, h) + \mathcal{O}(\varepsilon^2), \\ h^* &= h + \varepsilon\mu_3(x, y, z, t, u, f, g, h) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{2.2}$$

for the arbitrary functions f, g and h, where ε is the group parameter. These transformations leave invariant the set of solutions of Eq. (1.1). This yields to the overdetermined linear system of eleven equations for the coefficients of infinitesimals $\xi^1, \xi^2, \xi^3, \xi^4, \phi, \mu_1, \mu_2$ and μ_3. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$X = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial z} + \xi^4\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \mu_1\frac{\partial}{\partial f} + \mu_2\frac{\partial}{\partial g} + \mu_3\frac{\partial}{\partial h}. \tag{2.3}$$

Having determined the infinitesimals, the symmetry coefficients are found by solving the invariant surface condition

$$\Phi \equiv X^{[2]}\left(u_t - [f(u)u_x]_x - [f(u)u_y]_y - [f(u)u_z]_z\right) = 0, \tag{2.4}$$

where $X^{[2]}$ is the second prolongation of the insinitesimal (2.3) defined by

$$\begin{aligned} X^{[2]} = X &+ \phi^{(1)}(x, y, z, t, u, f, g, h, u^{(1)})\frac{\partial}{\partial u^{(1)}} \\ &+ \phi^{(2)}(x, y, z, t, u, f, g, h, u^{(1)}, u^{(2)})\frac{\partial}{\partial u^{(2)}}, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \phi^{(1)} &= \mathrm{D}_{(1)}(\phi - \xi^i u) + \xi^i u_i^{(1)}, \\ \phi^{(2)} &= \mathrm{D}_{(2)}(\phi^{(1)} - \xi^i u) + \xi^i u_i^{(2)}, \end{aligned} \tag{2.6}$$

and D is the total derivative with respect to independent variables.

Table 1. Commutators Table of $\mathcal{G}$

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	X_1
X_2	0	0	0	0	X_2
X_3	0	0	0	0	X_3
X_4	0	0	0	0	$2X_4$
X_5	$-X_1$	$-X_2$	$-X_3$	$-2X_4$	0

Inserting Eqs. (2.6) into (2.5) yields the following defining equations for the eqution (1.1):

$$\begin{aligned}
&\xi^1_x = \xi^1_y = \xi^1_z = \xi^1_u = \xi^2_u = \xi^3_u = \xi^4_u = 0,\\
&\xi^2_{uu} = \xi^3_{uu} = \xi^4_{uu} = 0, \quad \eta_{uu} = 2\xi^2_{xu},\\
&\xi^i_t = f\xi^i_{xx} + g\xi^i_{yy} + h\xi^i_{zz}, \quad i = 2,3,4,\\
&f\xi^3_x + g\xi^2_y = f\xi^4_x + h\xi^2_z = g\xi^4_y + h\xi^3_z = 0,\\
&\mu_1 = f\left(-2\eta_u + 2\xi^2_x + \xi^1_t\right) - f\xi^1_{xx} - g\xi^1_{yy} - h\xi^3_z,\\
&\mu_2 = g\left(-2\eta_u + 2\xi^3_y + \xi^1_t\right) - f\xi^1_{xx} - g\xi^1_{yy} - h\xi^3_z,\\
&\mu_3 = f\left(-2\eta_u + 2\xi^4_z + \xi^1_t\right) - f\xi^1_{xx} - g\xi^1_{yy} - h\xi^3_z.
\end{aligned} \tag{2.7}$$

The solutions of the system (2.7) give coefficients function for infinitesimal generators of the one-parameter Lie group of the point symmetries for the Eq. (1.1) as follows:

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial t},\\
&X_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t}.
\end{aligned} \tag{2.8}$$

The brackets between these operators are shown in Table 1 [10].

The one-parameter groups g_i, generated by the X_i are given as follows:

$$\begin{aligned}
&g_1(x,y,z,t,u) = (x+\varepsilon, y, z, t, u), && g_2(x,y,z,t,u) = (x, y+\varepsilon, z, t, u),\\
&g_3(x,y,z,t,u) = (x, y, z+\varepsilon, t, u), && g_4(x,y,z,t,u) = (x, y, z, t+\varepsilon, u),\\
&g_5(x,y,z,t,u) = (e^{\varepsilon}x, e^{\varepsilon}y, e^{\varepsilon}z, e^{2\varepsilon}t, u).
\end{aligned}$$

As each agroup, g_i is a symmetry. Then if $u = f(x,y,z,t)$ is a solution of the Eq. (1.1), we have [4],

$$\begin{aligned}
&u_1 = f(x-\varepsilon, y, z, t), && u_2 = f(x, y-\varepsilon, z, t),\\
&u_3 = f(x, y, z-\varepsilon, t), && u_4 = f(x, y, z, t-\varepsilon),\\
&u_5 = f(e^{-\varepsilon}x, e^{-\varepsilon}y, e^{-\varepsilon}z, e^{-2\varepsilon}),
\end{aligned}$$

where ε is a real parameter. The groups g_1, g_2, g_3 and g_4 demonstrate the (x,y,z) space-and time-invariance of the equation. The last group g_5 represents scaling local group of transformations.

The most general one-parameter group of symmetries obtained by considering a general linear combination $c_1X_1 + \cdots c_5X_5$ of the given vector fields; the explicit formula for the group transformations is a bit complicated. Alternatively, an arbitrary group transformation g is as the composition of transformations in the various one-parameter subgroups $g_1, \ldots, g_5$. In particular, if g is near identity, it can be represented uniquely in the form $g = \exp(\varepsilon_5 X_5) \circ \cdots \circ \exp(\varepsilon_1 X_1)$. Thus the most general solution would be obtained from a given solution $u = f(x, y, z, t)$ where the group transformation is of the form

$$u = f\left(e^{-\varepsilon_1}(x - \varepsilon_1), e^{-\varepsilon_2}(y - \varepsilon_2), e^{-\varepsilon_3}(z - \varepsilon_3), e^{2\varepsilon_4}(t - \varepsilon_4)\right),$$

where $\varepsilon_1, \ldots, \varepsilon_5$ are real constants.

2.1. **Structure of $\mathcal{G}$.** $\mathcal{G}$ has no non-trivial Levi decomposition in the form of $\mathcal{G} = \mathcal{R} \ltimes \mathcal{G}_1$, because $\mathcal{G}$ has no any non-trivial radical, i.e., if $\mathcal{R}$ be the radical of $\mathcal{G}$, then $\mathcal{G} = \mathcal{R}$. This Lie algebra is solvable and non-semisimple. It is solvable because if $\mathcal{G}^{(1)} = \langle X_i, [X_i, X_j]$, we have $\mathcal{G}^{(1)} = \mathcal{G}$, and $\mathcal{G}^{(2)} = [\mathcal{G}^{(1)}, \mathcal{G}^{(1)}] = \langle X_1, X_2, X_3, 2X_4 \rangle$. As a result we have a chain of ideals $\mathcal{G}^{(1)} \supset \mathcal{G}^{(2)} \supset \{0\}$, it is also non-semisimple because its Killing form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix},$$

is degenerate.

Finally the table of commutator shows that the Lie algebra $\mathcal{G}$ is isomorphic to $4A_1 \oplus A_1$.

3. Hamiltonian Equations and Hamiltonian Symmetry Group

Consider the five-dimensional Lie algebra $\mathcal{G}$ of the symmetry group for Eq. (1.1). Let $\omega_1, \ldots, \omega_5$ be a dual basis for $\mathcal{G}^*$ and $u = u^1\omega_1 + \cdots + u^5\omega_5$ be a typical point therein. If $F : \mathcal{G}^* \to \mathbb{R}$, then its gradient is the vector

$$\nabla F = \frac{\partial F}{\partial u^1} X_1 + \cdots + \frac{\partial F}{\partial u^5} X_5. \tag{3.1}$$

Thus the Lie-Poisson bracket on $\mathcal{G}^*$ is

$$\begin{aligned} \{F, H\} = u^1 \left(\frac{\partial F}{\partial u^1} \frac{\partial H}{\partial u^5} - \frac{\partial F}{\partial u^5} \frac{\partial H}{\partial u^1} \right) + u^2 \left(\frac{\partial F}{\partial u^2} \frac{\partial H}{\partial u^5} - \frac{\partial F}{\partial u^5} \frac{\partial H}{\partial u^2} \right) \\ + u^3 \left(\frac{\partial F}{\partial u^3} \frac{\partial H}{\partial u^5} - \frac{\partial F}{\partial u^5} \frac{\partial H}{\partial u^3} \right) + 2u^4 \left(\frac{\partial F}{\partial u^4} \frac{\partial H}{\partial u^5} - \frac{\partial F}{\partial u^5} \frac{\partial H}{\partial u^4} \right). \end{aligned} \tag{3.2}$$

The structure matrix $\mathbf{J}(u) = \big(J^{ij}(u)\big)$ where $J^{ij} = \{u^i, u^j\}$ with respect to (3.2) is given by

$$\mathbf{J}(u) = \begin{pmatrix} 0 & 0 & 0 & 0 & u^1 \\ 0 & 0 & 0 & 0 & u^2 \\ 0 & 0 & 0 & 0 & u^3 \\ 0 & 0 & 0 & 0 & 2u^4 \\ -u^1 & -u^2 & -u^3 & -2u^4 & 0 \end{pmatrix}. \tag{3.3}$$

Hamilton's equations corresponding to the Hamiltonian function $H(u)$ are in the form of

$$\frac{du^i}{dt} = \mathbf{J}(u) \cdot \nabla H(u). \tag{3.4}$$

For example, if

$$H(u) = \sum_{i=1}^{5} \frac{(u^i)^2}{2I_i}, \tag{3.5}$$

where I_i's are certain constants, then, Hamilton's equations become the equations of a rigid body

$$\frac{du^1}{dt} = u^1\frac{u^5}{I_5}, \qquad \frac{du^2}{dt} = u^2\frac{u^5}{I_5}, \qquad \frac{du^3}{dt} = u^3\frac{u^5}{I_5},$$
$$\frac{du^4}{dt} = 2u^4\frac{u^5}{I_5}, \quad \frac{du^5}{dt} = -u^1\frac{u^1}{I_1} - u^2\frac{u^2}{I_2} - u^3\frac{u^3}{I_3} - 2u^4\frac{u^4}{I_4},$$

in which $(I_1, ..., I_5)$ are the moments of inertia about the coordinate axes and $u^1, ..., u^5$ are the corresponding body angular momenta. (The angular velocities are $\omega^i = u^i/I_i$.) This Hamiltonian function is called the kinetic energy of the body.

Lemma 3.1. *The Hamiltonian vector field associated with $H(u)$ has the form*

$$\mathbf{v}_H = \sum_{i,j=1}^{m} J^{ij}(u)\frac{\partial H}{\partial u^j}\frac{\partial}{\partial u^i}, \tag{3.6}$$

where m is the dimension of the associated manifold.

Consider a system of ODEs in Hamiltonian form

$$\frac{du}{dt} = \mathbf{J}(u) \cdot \nabla H(u,t), \tag{3.7}$$

where $H(u,t)$ is a Hamiltonian function and $\mathbf{J}(u)$ is the structure matrix determining the Poisson bracket.

As we know in Hamiltonian mechanics [13], a time-independent function $P(u)$ is a first integral of the Hamiltonian system if $\{P, H\} = 0$ holds for every Hamiltonian function H. So we have the following corollary.

Corollary 3.2. *If $u_t = \mathbf{J} \cdot \nabla H$ is any Hamiltonian system with time-independent Hamiltonian function $H(u)$, then $H(u)$ itself is automatically a first integral.*

For a Hamiltonian system, symmetry groups are one-parameter Hamiltonian symmetry groups such that the infinitesimal generators (in evolutionary form) are Hamiltonian vector fields. The coming lemma shows that any first integral leads to such a symmetry group.

Lemma 3.3. *Let $P(u,t)$ be a first integral of a Hamiltonian system. Then the Hamiltonian vector field $\mathbf{v}_P$ determined by P generates a one-parameter symmetry group of the system.*

Generally there is not a one-to-one correspondence between Hamiltonian vector fields and their corresponding Hamiltonian function [12, 15]. More generally, we can add any time-dependent function $C(u,t)$ (meaning that for each fixed t, C is a time-independent function) to a given function P without changing the form of its Hamiltonian vector field. Once we recognize the possibility of modifying the function determining a Hamiltonian symmetry group, we can readily prove a converse to the preceding proposition. This forms the Hamiltonian version of Noether's theorem.

Theorem 3.4. *A vector $\mathbf{w}$ generates a Hamiltonian symmetry group of a Hamiltonian system of ODEs if and only if there exists a first integral $P(u,t)$ so that $\mathbf{w}=\mathbf{v}_P$ is the corresponding Hamiltonian vector field. A second function $\tilde{P}(u,t)$ determines the same Hamiltonian symmetry if and only if $\tilde{P}=P+C$ for some time-dependent function $C(u,t)$.*

Using this theorem and Eq. (3.6) we conclude that the Hamiltonian symmetry group for the Eq. (1.1) corresponding to the Hamiltonian function (3.5) is given by

$$\mathbf{v}_H=\frac{u^5}{I_5}\left(\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^2}+\frac{\partial}{\partial u^3}+2\frac{\partial}{\partial u^4}\right)-\left(u^1\frac{u^1}{I_1}+u^2\frac{u^2}{I_2}+u^3\frac{u^3}{I_3}+2u^4\frac{u^4}{I_4}\right)\frac{\partial}{\partial u^5}.$$

4. Conservation Laws

As we know a conservation law of a non-degenerate system of differential equation is a divergence expression that vanishes on all solutions of the given system. In general, any such non-trivial expression that yields a local conservation law of the system arises from a linear combination formed local multipliers (characteristic) with each differential equation in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of the dependent variables of the given system of differential equations. There are many different methods for finding conservation laws. But direct method and Noether's method are more practical. The direct method could be applied to any arbitrary system of differential equations, but Noether's method is so much limited with simpler calculations. In this section we find the local conservation laws of the Eq. (1.1) via Noether's method since the conditions are satisfied. Then, we use the direct method for finding local conservation laws of some applicable kinds of Eq. (1.1) which are useful in physics and fluid mechanics [1, 2, 3, 10, 15].

4.1. **Conservation Laws via Noether's Method.** Consider the operator identity

$$X^{[\infty]}+\mathrm{D}_i(\xi^i)=Q^\alpha\frac{\delta}{\delta u^\alpha}+\mathrm{D}_i\mathrm{N}^i, \tag{4.1}$$

where

$$X^{[\infty]} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \eta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{4.2}$$

is the prolongation of any differential operator such as (2.3),

$$Q^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \tag{4.3}$$

is the corresponding characteristic,

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}, \tag{4.4}$$

is the variational derivative operator and

$$\mathrm{N}^i = \xi^i + Q^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s=1}^{\infty} \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_s} (Q^\alpha) \frac{\delta}{\delta_{i i_1 \cdots i_s}}. \tag{4.5}$$

Recall that Noether's theorem, associating conservation laws with symmetries of differential equations obtained from variational principles, was originally proved by calculus of variations. Consider the Euler- Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \qquad \alpha = 1, ..., q, \tag{4.6}$$

written by the operator (4.4), thus, the variational integral

$$\int \mathcal{L}(x, u, u_{(1)}, ...) dx,$$

is invariant under the one-parameter group of the generator $X^{[\infty]}$ and so

$$X^{[\infty]}(\mathcal{L}) + \mathrm{D}_i(\xi^i)\mathcal{L} = 0. \tag{4.7}$$

By acting the both side of the identity (4.1) on $\mathcal{L}$ we have

$$X^{[\infty]}(\mathcal{L}) + \mathrm{D}_i(\xi^i)\mathcal{L} = Q^\alpha \frac{\delta \mathcal{L}}{\delta u^\alpha} + \mathrm{D}_i \left[\mathrm{N}^i(\mathcal{L}) \right],$$

and take it to account equations (4.6) and (4.7), we see that the vector with the components

$$C^i = \mathrm{N}^i(\mathcal{L}), \qquad i = 1, ..., p, \tag{4.8}$$

satisfies the conservation equation

$$\mathrm{D}_i(C^i) = 0, \qquad i = 1, ..., p. \tag{4.9}$$

For practical applications, when we deal with low order Lagrangians $\mathcal{L}$, it is convenient to restrict the operator (4.5) on the derivatives involved in $\mathcal{L}$ and write the expressions (4.8) in the expanded form

$$\begin{aligned} C^i = \xi^i \mathcal{L} + Q^\alpha &\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - \mathrm{D}_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + \mathrm{D}_j \mathrm{D}_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \cdots \right] \\ + \mathrm{D}_j(Q^\alpha) &\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - \mathrm{D}_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] + \mathrm{D}_j \mathrm{D}_k(Q^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots \right], \end{aligned} \tag{4.10}$$

thus, Noether's theorem can be formulated as the following theorem.

Theorem 4.1. *If the operator $X^{[\infty]}$ is admitted by Eq. (4.6) and satisfies the condition (4.7) of the invariance of the variational integral, the vector (36) constructed by equations (4.8) satisfies the conservation law (4.9).*

According to the discussed theory the operator

$$\tilde{X} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial z} + \xi^4 \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u},$$

admitted by Eq. (1.1) provides a conserved vector given by

$$C^i = Q\left[\frac{\partial \mathcal{L}}{\partial u_i} - \mathrm{D}_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}}\right)\right] + \mathrm{D}_j(Q)\frac{\partial \mathcal{L}}{\partial u_{ij}}, \tag{4.11}$$

where $Q = \eta - \xi^1 u_x - \xi^2 u_y - \xi^3 u_z - \xi^4 u_t$ is the corresponding characteristic and the vectors (4.11) satisfy the conservation equation

$$\mathrm{D}_x(C^1) + \mathrm{D}_y(C^2) + \mathrm{D}_z(C^3) + \mathrm{D}_t(C^4) = 0, \tag{4.12}$$

on the set of all solutions of the Eq. (1.1). If we write the Eq. (1.1) in the form

$$L \equiv -u_t + f u_{xx} + g u_{yy} + h u_{zz} + f' u_x^2 + g' u_y^2 + h' u_z^2 = 0, \tag{4.13}$$

then the formal Lagrangian is obtained by the equation

$$\mathcal{L} = \psi L = \psi(-u_t + f u_{xx} + g u_{yy} + h u_{zz} + f' u_x^2 + g' u_y^2 + h' u_z^2), \tag{4.14}$$

then, solving the Eq. (4.14) leads to

$$\psi = a_1 xyz + a_2 xy + a_3 xz + a_4 yz + a_5 x + a_6 y + a_7 z + a_8. \tag{4.15}$$

Consequently, the conserved vectors (4.11) are:

$$
\begin{aligned}
C^1 &= Q\left[\frac{\partial \mathcal{L}}{\partial u_x} - \mathrm{D}_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) - \mathrm{D}_y\left(\frac{\partial \mathcal{L}}{\partial u_{xy}}\right) - \mathrm{D}_z\left(\frac{\partial \mathcal{L}}{\partial u_{xz}}\right) - \mathrm{D}_t\left(\frac{\partial \mathcal{L}}{\partial u_{xt}}\right)\right] \\
&\quad + \mathrm{D}_x(Q)\frac{\partial \mathcal{L}}{\partial u_{xx}} + \mathrm{D}_y(Q)\frac{\partial \mathcal{L}}{\partial u_{xy}} + \mathrm{D}_z(Q)\frac{\partial \mathcal{L}}{\partial u_{xz}} + \mathrm{D}_t(Q)\frac{\partial \mathcal{L}}{\partial u_{xt}} \\
&= Q[2f' u_x \psi - f\psi_x] + f\psi \mathrm{D}_x(Q), \qquad (4.16) \\
C^2 &= Q\left[\frac{\partial \mathcal{L}}{\partial u_y} - \mathrm{D}_x\left(\frac{\partial \mathcal{L}}{\partial u_{xy}}\right) - \mathrm{D}_y\left(\frac{\partial \mathcal{L}}{\partial u_{yy}}\right) - \mathrm{D}_z\left(\frac{\partial \mathcal{L}}{\partial u_{yz}}\right) - \mathrm{D}_t\left(\frac{\partial \mathcal{L}}{\partial u_{yt}}\right)\right] \\
&\quad + \mathrm{D}_x(Q)\frac{\partial \mathcal{L}}{\partial u_{xy}} + \mathrm{D}_y(Q)\frac{\partial \mathcal{L}}{\partial u_{yy}} + \mathrm{D}_z(Q)\frac{\partial \mathcal{L}}{\partial u_{yz}} + \mathrm{D}_t(Q)\frac{\partial \mathcal{L}}{\partial u_{yt}} \\
&= Q[2g' u_y \psi - f\psi_y] + g\psi \mathrm{D}_y(Q), \qquad (4.17) \\
C^3 &= Q\left[\frac{\partial \mathcal{L}}{\partial u_z} - \mathrm{D}_x\left(\frac{\partial \mathcal{L}}{\partial u_{xz}}\right) - \mathrm{D}_y\left(\frac{\partial \mathcal{L}}{\partial u_{yz}}\right) - \mathrm{D}_z\left(\frac{\partial \mathcal{L}}{\partial u_{zz}}\right) - \mathrm{D}_t\left(\frac{\partial \mathcal{L}}{\partial u_{zt}}\right)\right] \\
&\quad + \mathrm{D}_x(Q)\frac{\partial \mathcal{L}}{\partial u_{xz}} + \mathrm{D}_y(Q)\frac{\partial \mathcal{L}}{\partial u_{yz}} + \mathrm{D}_z(Q)\frac{\partial \mathcal{L}}{\partial u_{zz}} + \mathrm{D}_t(Q)\frac{\partial \mathcal{L}}{\partial u_{zt}} \\
&= Q[2h' u_z \psi - h\psi_z] + h\psi \mathrm{D}_z(Q), \qquad (4.18) \\
C^4 &= Q\left[\frac{\partial \mathcal{L}}{\partial u_t} - \mathrm{D}_x\left(\frac{\partial \mathcal{L}}{\partial u_{xt}}\right) - \mathrm{D}_y\left(\frac{\partial \mathcal{L}}{\partial u_{yt}}\right) - \mathrm{D}_z\left(\frac{\partial \mathcal{L}}{\partial u_{zt}}\right) - \mathrm{D}_t\left(\frac{\partial \mathcal{L}}{\partial u_{tt}}\right)\right] \\
&\quad + \mathrm{D}_x(Q)\frac{\partial \mathcal{L}}{\partial u_{xt}} + \mathrm{D}_y(Q)\frac{\partial \mathcal{L}}{\partial u_{yt}} + \mathrm{D}_z(Q)\frac{\partial \mathcal{L}}{\partial u_{zt}} + \mathrm{D}_t(Q)\frac{\partial \mathcal{L}}{\partial u_{tt}} \\
&= -Q\psi. \qquad (4.19)
\end{aligned}
$$

4.1.1. *Conservation Laws of the Symmetry X_1.* In this case the corresponding characteristic is $Q = -u_x$. Inserting $-u_x$ in the conserved vector (4.16)-(4.19) leads to

$$
\begin{aligned}
C^1 &= f(u)u_x\psi_x - g(u)u_y\psi_y - h(u)u_z\psi_z, \\
C^2 &= g(u)(u_x\psi_y + u_y\psi_x), \\
C^3 &= h(u)(u_x\psi_z + u_z\psi_x), \\
C^4 &= -u\psi_x.
\end{aligned} \qquad (4.20)
$$

4.1.2. *Conservation Laws of the Symmetry X_2.* In this case the corresponding characteristic is $Q = -u_y$. Now applying the symmetry X_2 to the expanded conserved vectors (4.16)-(4.19) yields the following conservation laws:

$$
\begin{aligned}
C^1 &= f(u)(u_y\psi_x + u_x\psi_y), \\
C^2 &= g(u)u_y\psi_y - f(u)u_x\psi_x - h(u)u_z\psi_z, \\
C^3 &= h(u)(u_y\psi_z + u_z\psi_y), \\
C^4 &= -u\psi_y.
\end{aligned} \qquad (4.21)
$$

4.2. **Conservation Laws of the Symmetry X_3.** In this case the corresponding characteristic is $Q = -u_z$. Inserting $-u_x$ in the conserved vector (4.16)-(4.19) yields

$$\begin{aligned} C^1 &= f(u)(u_z\psi_x + u_x\psi_z), \\ C^2 &= g(u)(u_z\psi_y + u_y\psi_z), \\ C^3 &= h(u)u_z\psi_z - f(u)u_x\psi_x - g(u)u_y\psi_y, \\ C^4 &= -u\psi_z. \end{aligned} \tag{4.22}$$

4.3. **Conservation Laws of the Symmetry X_4.** In this case the corresponding characteristic is $Q = -u_t$. Inserting the right hand side of the Eq. (1.1) into the Eq. (4.19) we obtain

$$C^4 = \psi\left[(f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z\right]. \tag{4.23}$$

Using a derivative chain rule we have:

$$\psi(f(u)u_x)_x = (\psi f(u)u_x)_x - \psi_x f(u)u_x = (\psi f(u)u_x)_x - (\psi_x F(u))_x + \psi_{xx}F(u),$$

where $F(u) = \displaystyle\int f(u)du$. According to the Eq. (4.15) we can see that, $\psi_{xx} = 0$. Thus,

$$\psi(f(u)u_x)_x = (\psi f(u)u_x - \psi_x F(u))_x.$$

Similarly, we conclude

$$\psi(f(u)u_y)_y = (\psi f(u)u_y - \psi_y G(u))_y, \qquad \psi(f(u)u_z)_z = (\psi f(u)u_z - \psi_z H(u))_z.$$

Thus, the right hand side of the Eq. (4.23) is:

$$\begin{aligned} C^4 = \psi[&(\psi f(u)u_x - \psi_x F(u))_x + (\psi f(u)u_y - \psi_y G(u))_y \\ &+(\psi f(u)u_z - \psi_z H(u))_z]. \end{aligned} \tag{4.24}$$

Substituting C^4 into fourth term of Eq. (4.12) and the commutativity of the differentiation we have

$$\begin{aligned} \mathrm{D}_t(C^4) &= \mathrm{D}_t([(\psi f(u)u_x - \psi_x F(u))_x + (\psi f(u)u_y - \psi_y G(u))_y \\ &\quad +(\psi f(u)u_z - \psi_z H(u))_z]) \\ &= \mathrm{D}_x[(\psi f(u)u_x - \psi_x F(u))_x]_t + \mathrm{D}_y[(\psi f(u)u_y - \psi_y G(u))_y] \\ &\quad +\mathrm{D}_z[(\psi f(u)u_z - \psi_z H(u))_z]. \end{aligned}$$

An straightforward calculation shows that:

$$C^1 + [(\psi f(u)u_x - \psi_x F(u))_x]_t = 0, \tag{4.25}$$

$$C^2 + [(\psi f(u)u_y - \psi_y G(u))_y]_t = 0, \tag{4.26}$$

$$C^3 + [(\psi f(u)u_z - \psi_z H(u))_z]_t = 0, \tag{4.27}$$

thus,

$$\begin{aligned} &\mathrm{D}_x(C^1 + [(\psi f(u)u_x - \psi_x F(u))_x]_t) + \mathrm{D}_y(C^2 + [(\psi f(u)u_y - \psi_y G(u))_y]_t) \\ &\quad +\mathrm{D}_z(C^3 + [(\psi f(u)u_z - \psi_z H(u))_z]_t) = 0. \end{aligned} \tag{4.28}$$

Eq. (4.28) including Eqs. (4.26)-(4.27) shows that in this case the canservation laws are all trivial.

4.4. **Conservation Laws of the Symmetry X_5.** For the last case the corresponding characteristic is $Q = -xu_x - yu_y - zu_z - 2tu_t$. Now applying the symmetry X_5 to the expanded conserved vectors (4.16)-(4.19) yields the following conservation laws:

$$\begin{aligned}
C^1 &= [2\psi f'(u)u_x - f(u)\psi_x](xu_x + yu_y + zu_z + 2tu_t)\\
&\quad -f(u)\psi(u_x + xu_{xx} + yu_{xy} + zu_{xz} + 2tu_{xt}),\\
C^2 &= [2\psi g'(u)u_y - f(u)\psi_y](xu_x + yu_y + zu_z + 2tu_t)\\
&\quad -g(u)\psi(u_y + xu_{xy} + yu_{yy} + zu_{yz} + 2tu_{yt}),\\
C^3 &= [2\psi h'(u)u_z - f(u)\psi_z](xu_x + yu_y + zu_z + 2tu_t)\\
&\quad -h(u)\psi(u_z + xu_{xz} + yu_{yz} + zu_{zz} + 2tu_{zt}),\\
C^4 &= \psi(xu_x + yu_y + zu_z + 2tu_t).
\end{aligned} \tag{4.29}$$

4.5. **Conservation Laws via Direct Method.** The problem of finding local conservation laws of a system of differential equations reduces to the problem of finding local multipliers. The product of these multipliers with each differential equation in the system is annihilated by the Euler operators. These operators are associated with each dependent and their derivatives. In the given set of local conservation laws multipliers, there is an integral formula to obtain the fluxes and densities of the local conservation laws [1, 2]. Often it is straightforward to obtain the conservation law by direct calculation, after its multipliers are known [3]. What has been outlined here, is the direct method for obtaining local conservation laws of Eq. (1.1) in some special cases.

4.5.1. $f(u) = 1, g(u) = h(u) = 0$. In this case the equation reduces to linear heat transfer equation in one-dimensional rod. The local conservation laws are shown in Table (2).

Table 2. Fluxes and densities of $u_t = u_{xx}$.

Flux	Density
1	$-u_x$
t^3	x
x^2	$-2tx$
$t^2 + xt$	$-\frac{1}{2}t^2$
$-xu_t$	xu_x
$-xu_x + u$	$x_x^2 u$
$-u - tu_t$	tu_x

4.5.2. $f(u) = u, g(u) = h(u) = 0$. In this situation the equation becomes to the nonlinear one-dimensional heat transfer equation called *Burger*'s equation. The local conservation laws are presented in Table (3).

4.5.3. $f(u) = g(u) = 1, h(u) = 0$. This status leads to the linear two-dimensional heat transfer equation. The results for conservation laws are shown in Table (4).

TABLE 3. Fluxes and densities of Burger's equation.

Flux	Density
0	1
1	0
0	x
$-x$	t
x^2+t^3	$-2xt$
t^2+xt	$-\frac{1}{2}t^2$
$t+u_t$	$-uu_x$
$-u_t-uu_x$	$u+u_x$
$-u-tu_t$	tu_x
$-xu_t+uu_x$	xu_x
$-xuu_x+\frac{1}{2}u^2$	xu

TABLE 4. Fluxes and densities of $u_t = u_{xx} + u_{yy}$.

$x-$ Flux	$y-$ Flux	Density
0	0	1
u_t	0	$-u_x$
t^2	0	0
yt	0	0
xy	0	$-y$
xt	0	$-\frac{1}{2}t^2$
$x+t+u_x$	u_y	$-t-u$
uu_t	0	$-uu_x$
$u+(y+t)u_t$	0	$-(x+y)u_x$
xu_t+uu_y	$-uu_x$	$y+u-xu_x$
$-tu_x+tu_y$	0	0
$u+yu_y$	$-yu_x$	0
$-u_xu_x$	xu_y	$-xu_x$
yu_x+xu_y	$-2u-xu_x+yu_y$	$-yu$

4.5.4. $f(u) = g(u) = h(u) = 1$. For this condition the equation reduces to the linear three-dimensional heat transfer equation. Thus, fluxes and densities are outlined in Table (5).

5. Conclusion

It is concluded that, Lie symmetries are used for finding general form of similarity solutions and Hamiltonian equations of (1.1). Only, the local conservation laws via Noether's method are obtained and further investigation of conservation laws is to be carried out in some practical form of the Eq. (1.1). Finally some local conservation laws for special cases of the equation are given.

TABLE 5. Fluxes and densities of $u_t = u_{xx} + u_{yy} + u_{zz}$.

$x-$ Flux	$y-$ Flux	$z-$ Flux	Density
0	$-y$	0	t
0	0	$-z$	t
0	0	$-z^2+y^2$	
$-uu_t$	0	0	uu_x
$-zu_t$	0	0	zu_x
$-u_t$	0	0	u_x
$u+(t-y)u_t$	0	0	$(y-t)u_x$
xu_t	0	0	$-u-xu_x$
u_y	$-u_x$	0	0
$(t+u)u_y$	$-(t+u)u_x$	0	0
$(-x-y+z)u_y$	$u+(x+y-z)u_x$	0	0
0	0	uu_t	$-uu_z$
0	0	yu_t	$-yu_z$
$u+zu_z$	0	xu_t-zu_x	$-xu_z$
$(x-u)u_z$	0	0	$-u+uu_x$

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