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An improved pseudospectral approximation of generalized Burger-Huxley and Fitzhugh-Nagumo equations

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Abstract In this paper, an improved Chebyshev-Gauss-Lobatto pseudospectral approximation of nonlinear Burger-Huxley and Fitzhugh-Nagumo (FN) equations have been presented. The spectral method has been employed in time and space based upon Chebyshev Gauss-Labatto points and achieved spectral accuracy. A mapping has used to transform the initial-boundary value non-homogeneous problems to homogeneous problems and finally it reduced to a system of algebraic equations, which has solved by standard numerical method. Numerical results for various cases of generalized Burger-Huxley equation and other examples of Fitzhugh-Nagumo equation have presented to demonstrate the performance and effectiveness of the method. Comparison of the method with existing other methods, available in literature, are also given.

Keywords. Generalized Burger-Huxley equation, Fitzhugh-Nagumo (FN) equation, Pseudospectral method, Chebyshev-Gauss-Lobbato points.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Our contribution in this paper is to propose the improved Chebyshev pseudospectral method in both space and time for nonlinear generalized one dimensional partial differential equation

$$U_t + q_1(t)U_x - q_2(t)U_{xx} + q_3(t)(\phi(U))_x - q_4(t)\varphi(U) = 0, \quad x \in [\xi, \eta] \text{ and } t \in [0, T], \tag{1.1}$$

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where $\phi(U)$ and $\varphi(U)$ are some nonlinear functions and $q_1(t), q_2(t), q_3(t)$ and $q_4(t)$ are arbitrary real valued functions of t. Initial condition and boundary conditions are

$$U(x,0) = \tilde{f}(x), \qquad x \in [\xi, \eta],$$

and

$$\begin{aligned} U(\xi,t) &= \tilde{g}_1(t), \qquad t \in [0,T], \\ U(\eta,t) &= \tilde{g}_2(t), \qquad t \in [0,T], \end{aligned}$$

respectively.

1.1. **Generalized Burger-Huxley equation.** Let us take $q_1(t) = 0, q_2(t) = q_3(t) = 1, q_4(t) = \beta$, $\phi(U) = \frac{\gamma}{\delta+1}U^{\delta+1}$ and $\varphi(U) = U(1-U^\delta)(U^\delta - \sigma)$ then equation (1.1) is called nonlinear generalized Burger-Huxley equation which describe the relation of reaction, convection and diffusion term. Here γ is advection coefficient, β is a reaction coefficient, $\delta \geq 1$ is a positive number and $\sigma \in (0,1)$. These equations have been investigated by numerous authors, some numerical approximation methods are listed as follows, Adomian-decomposition method [15], iterative differential quadrature method [27], compact finite difference method [25], Haar wavelet approach [7] and varitional iteration method [2]. Numerical solutions for generalized Burger's-Huxley equation using mixed collocation and finite difference scheme have been presented in [10]. A meshless method for numerical solutions of the generalized Burger's-Huxley equation have been discussed in [13, 19]. In this work, they explained method based on scattered nodes instead of mesh in the domain. Mohammadi [22] proposed collocation B-spline method to obtained the numerical solutions of the generalized Burger's-Huxley equation. If $\phi(U) = 0$, the equation (1.1) will reduce to Huxley equation [31] and describes wall motion in liquid crystal and nerve fibers [32, 16].

1.2. **Fitzhugh-Nagumo equation.** When $\phi(U) = 0$, and $\varphi(U) = U(U-\sigma)(1-U)$ then equation (1.1) is called nonlinear Fitzhugh-Nagumo (FN) partial differential equation. This is a special case of the Burgers-Huxley equation and introduced by Fitzhugh [12] and Nagumo *et al.* [23]. Many mathematicians and physicists have studied this equation and used in various applications such as logistic population growth, branching brownian motion process, neurophysiology, nuclear reactor theory and chemical reaction [5]. In past, the number of methods for the numerical solution of Fitzhugh-Nagumo equation have presented, such as conditional symmetry method [26], homotopy analysis method [30], Jacobi-Gauss-Lobatto collocation method [3], Semi-explicit finite-difference method [8] and Haar wavelet method [14]. Li and Guo [20] presented new exact solution for FN equation using integral method for obtaining exact solution of FN equation. Triki and Wazwaz [29] have introduced specific solitary wave ansatz and the tanh method for generalized Fitzhugh-Nagumo equation and presented new type of existence and uniqueness of soliton solutions.

There are number of practical problems exist in engineering and mathematical physics which do not have exact solution. Therefore, high order approximation methods are always attraction among the researchers. Spectral method [6, 24, 28, 4] is a high order method to obtain numerical approximations for linear and nonlinear partial differential equations. Broadly the method has classified into two categories, interpolation and non-interpolation. The collocation method is depends upon interpolation approach, which is called pseudospectral method also. To the best of authors knowledge, pseudospectral method in both time and space for generalized Burger-Huxley and Fitzhugh-Nagumo equations have not reported by any other researcher.

This research paper is organized as follows. In section 2, discretization of the improved pseudospectral method based on CGL points in both space and time is given. In section 3, some numerical examples have been presented and also accuracy and efficiency of the proposed method is given. Finally, the conclusion of our scheme is given in section 4.

2. Discretization

In this section, we present the pseduospectral method based on CGL points in both space and time to approximate the solutions. We use linear transformations $x \longrightarrow \frac{\eta-\xi}{2}x+\frac{\eta+\xi}{2}$ and $t \longrightarrow \frac{T}{2}t+\frac{T}{2}$ to transform the given space interval $x \in [\xi, \eta]$ and time interval $t \in [0, T]$ into new interval $[-1, 1]$, respectively. Accordingly Equation (1.1) is transformed as follows

$$\begin{aligned} U_t + \frac{T}{\eta-\xi} q_1(t) U_x - \frac{2T}{(\eta-\xi)^2} q_2(t) U_{xx} + \frac{T}{\eta-\xi} q_3(t)(\phi(U))_x - \\ \frac{T}{2} q_4(t)\varphi(U) = 0, \qquad x \in [-1,1] \text{ and } t \in [-1,1], \end{aligned} \tag{2.1}$$

with initial condition

$$U(x,-1) = f(x), \qquad x \in [-1,1],$$

and boundary conditions

$$U(-1,t) = g_1(t), \qquad U(1,t) = g_2(t), \qquad t \in [-1,1].$$

The above initial and boundary conditions of the resulting problem are non-homogeneous, which will be reduced to homogeneous using the following mapping

$$\begin{aligned} \Upsilon(x,t) = \frac{1-t}{2}\ f(x) + \frac{1-x}{2}\ g_1(t) + \frac{1+x}{2}\ g_2(t) - \frac{(1-t)\ (1-x)}{4}\ g_1(-1) \\ - \frac{(1-t)\ (1+x)}{4}\ g_2(-1). \end{aligned} \tag{2.2}$$

Further we define a new variable $V(x,t)$ to represent the equation (2.1),

$$V(x,t) = U(x,t) - \Upsilon(x,t). \tag{2.3}$$

Using Equation (2.3) in (2.1), the homogeneous initial and boundary value problem are as follows

$$\begin{aligned} (V+\Upsilon)_t + \frac{T}{\eta-\xi}(q_1(t)(V+\Upsilon)_x + q_3(t)(\phi(V+\Upsilon))_x) \\ - \frac{2T}{(\eta-\xi)^2} q_2(t)(V+\Upsilon)_{xx} - \frac{T}{2} q_4(t)\varphi(V+\Upsilon) = 0. \end{aligned} \tag{2.4}$$

We seek a spectral approximation V^a, which can be used to approximate the solution of nonlinear differential equations, of the form:

$$V^a(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{M}\Psi_i(x)\Psi_j(t)\ V^a_{ij}, \tag{2.5}$$

here the trial functions $\Psi_i(x)$ and $\Psi_j(t)$ are the M^{th} degree Lagrange polynomials with space and time variables respectively, where CGL points are defined by

$$(x_i) = -\cos(i\pi/M)_{i=0}^{M}, \qquad (t_j) = -\cos(j\pi/M)_{j=0}^{M}. \tag{2.6}$$

Partial derivatives of the approximate solution at the collocation points (x_n, t_m) are computed as

$$\begin{aligned}\frac{\partial^r}{\partial x^r}(V^a(x_n,t_m)) &= \sum_{i=0}^{M}\sum_{j=0}^{M}\Psi_i^{(r)}(x_n)\Psi_j(t_m)\ V^a_{ij} = \sum_{i=0}^{M}\sum_{j=0}^{M}H^{(r)}_{in}\Psi_j(t_m)\ V^a_{ij},\\ &= \left(\left[H^{(r)}_{[0:M,n]}\otimes\Psi(t_m)\right]\right)^T V^a,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\frac{\partial^s}{\partial t^s}(V^a(x_n,t_m)) &= \sum_{i=0}^{M}\sum_{j=0}^{M}\Psi_i(x_n)\Psi_j^{(s)}(t_m)\ V^a_{ij} = \sum_{i=0}^{M}\sum_{j=0}^{M}\Psi_i(x_n)H^{(s)}_{jm}\ V^a_{ij},\\ &= \left(\left[\Psi(x_n)\otimes H^{(s)}_{[0:M,m]}\right]\right)^T V^a,\end{aligned} \tag{2.8}$$

where

$$\Psi(x_n) = [\Psi_0(x_n),\Psi_1(x_n),...,\Psi_M(x_n)]^T,\ \ \Psi(t_m) = [\Psi_0(t_m),\Psi_1(t_m),...,\Psi_M(t_m)]^T,$$

and V^a is a vector defined by

$$V^a \quad = \quad [V^a_{00},...,V^a_{0M},V^a_{10},...,V^a_{1M},...,V^a_{M0},...,V^a_{MM}]^T.$$

$H^{(s)}$ [9] represents as s^{th}-order Chebyshev differentiation matrix and $\otimes$ represent the Kronecker product of two vectors.
From Equation (2.7) and (2.8), Equation (2.4) at the CGL points, we obtain

$$\begin{aligned}&\left(\left[\Psi(x_n)\otimes H^{(1)}_{[0:M,m]}\right]^T\right)V^a + (\Upsilon_{nm})_t - \frac{2T}{(\eta-\xi)^2}q_2(t_m)\left(\left[H^{(2)}_{[0:M,n]}\otimes\Psi(t_m)\right]^T V^a + (\Upsilon_{nm})_{xx}\right)\\ &\quad+\frac{T}{\eta-\xi}\left[\left[H^{(1)}_{[0:M,n]}\otimes\Psi(t_m)\right]^T(q_1(t_m)V^a + q_3(t_m)\phi((V^a_{nm}+\Upsilon_{nm}))_x V^a) + q_1(t_m)(\Upsilon_{nm})_x\right]\\ &\qquad-\frac{T}{2}q_4(t_m)\left[\varphi((V^a_{nm}+\Upsilon_{nm}))\right] = 0,\end{aligned}$$

where $\Upsilon_{nm} = \Upsilon(x_n, t_m)$.

Apply the method at all CGL points in both time and space direction and using initial and boundary conditions $i.e.\Psi_0(x_i) = \Psi_M(x_i) = \Psi_0(t_j) = 0\ \forall\ (i,j)$, the above equation reduced in system of $M \times (M-1)$ nonlinear algebraic equations.

$$\left(\left[I_{M-1} \otimes H^{(1)}_{[1:M,1:M]}\right]^T\right) V^a + P_1 - \frac{2T}{(\eta-\xi)^2}\left(I_{M-1} \otimes I_{q_2}\right)\left(\left[H^{(2)}_{[1:M-1,1:M-1]} \otimes I_M\right]^T V^a + Q\right)$$
$$+\frac{T}{\eta-\xi}\left[\left[H^{(1)}_{[1:M-1,1:M-1]} \otimes I_M\right]^T \left(\left(I_{M-1} \otimes I_{q_1}\right) V^a + \left(I_{M-1} \otimes I_{q_3}\right) I_{R_1} V^a\right) + \left(I_{M-1} \otimes I_{q_1}\right) P_2\right]$$
$$-\frac{T}{2}\left(I_{M-1} \otimes I_{q_1}\right) R_2 = 0,$$

where

$$V^a \quad = \quad [V^a_{11}, ..., V^a_{1M}, V^a_{21}, ..., V^a_{2M}, ..., V^a_{(M-1)1}, ..., V^a_{(M-1)M}]^T,$$

$$P_1 = \left[\Upsilon_t(x_1,\ t_1), \ldots, \Upsilon_t(x_1,\ t_M) \mid \ldots \mid \Upsilon_t(x_{M-1},\ t_1), \ldots, \Upsilon_t(x_{M-1},\ t_M)\right]^T,$$
$$P_2 = \left[\Upsilon_x(x_1,\ t_1), \ldots, \Upsilon_x(x_1,\ t_M) \mid \ldots \mid \Upsilon_x(x_{M-1},\ t_1), \ldots, \Upsilon_x(x_{M-1},\ t_M)\right]^T,$$
$$Q = \left[\Upsilon_{xx}(x_1,\ t_1), \ldots, \Upsilon_{xx}(x_1,\ t_M) \mid \ldots \mid \Upsilon_{xx}(x_{M-1},\ t_1), \ldots, \Upsilon_{xx}(x_{M-1},\ t_M)\right]^T,$$
$$q_1 = [q_1(t_1), q_1(t_2), ..., q_1(t_M)]^T,\ q_2 = [q_2(t_1), q_2(t_2), ..., q_2(t_M)]^T,$$
$$q_3 = [q_3(t_1), q_3(t_2), ..., q_3(t_M)]^T,\ q_4 = [q_4(t_1), q_4(t_2), ..., q_4(t_M)]^T,$$

$$R_1 = \left[\phi((V^a_{11}+\Upsilon_{11}))_x, \ldots, \phi((V^a_{1M}+\Upsilon_{1M}))_x \mid \ldots \mid \phi((V^a_{(M-1)1}+\Upsilon_{(M-1)1}))_x, \ldots, \phi((V^a_{(M-1)M}+\Upsilon_{(M-1)M}))_x\right]^T,$$
$$R_2 = \left[\varphi((V^a_{11}+\Upsilon_{11})), \ldots, \varphi((V^a_{1M}+\Upsilon_{1M})) \mid \ldots \mid \varphi((V^a_{(M-1)1}+\Upsilon_{(M-1)1})), \ldots, \varphi((V^a_{(M-1)M}+\Upsilon_{(M-1)M}))\right]^T,$$

and $I_{q_i},\ \forall\ i = 1, 2, 3, 4$ represents the values of q_i in form of diagonal matrix.

3. Numerical results and discussion

In this section, we consider examples to obtain approximate solutions of generalized Burger-Huxley and time dependent Fitzhugh-Nagumo equations and demonstrate the L_∞-norm and relative errors in L_2- norm, which are defined by

$$L_\infty = \quad \|U^a - U\|_\infty = \quad \max_{n,m} |U^a(x_n, t_m) - U(x_n, t_m)|,$$

and

$$L_2 = \left(\frac{\sum_{n=0}^M \sum_{m=0}^M [U^a(x_n, t_m) - U(x_n, t_m)]^2}{\sum_{n=0}^M \sum_{m=0}^M [U(x_n, t_m)]^2}\right)^{\frac{1}{2}},$$

where U^a and U are spectral approximations and the analytical solutions, respectively.

Example 1. Let us take generalized Burger-Huxley equation *i.e.* $q_1(t) = 0, q_2(t) = q_3(t) = 1, q_4(t) = \beta, \phi(U) = \frac{\gamma}{\delta+1}U^{\delta+1}$ and $\varphi(U) = U(1-U^\delta)(U^\delta - \sigma)$.
Here $\beta, \gamma, \sigma \geq 0$ are given real parameters and δ is a positive integer.
Initial condition and boundary conditions are

$$U(x, 0) = [0.5\sigma + 0.5\sigma \tanh(\tau_1 x)]^{\frac{1}{\delta}}, \qquad x \in [\xi, \eta], \tag{3.1}$$

and

$$U(\xi, t) \quad = \quad [0.5\sigma + 0.5\sigma \tanh(\tau_1(\xi - \tau_2 t))]^{\frac{1}{\delta}}, \tag{3.2}$$
$$U(\eta, t) \quad = \quad [0.5\sigma + 0.5\sigma \tanh(\tau_1(\eta - \tau_2 t))]^{\frac{1}{\delta}}, \tag{3.3}$$

respectively. The exact solution of this problem is [19]

$$U(x,t) = \left[0.5\sigma + 0.5\sigma \tanh\left(\tau_1\left(x - \tau_2 t\right)\right)\right]^{\frac{1}{\delta}}, \tag{3.4}$$

where $\tau_1 = \frac{-\gamma\delta+\delta\sqrt{\gamma^2+4\beta(1+\delta)}}{4(1+\delta)}\sigma$ and $\tau_2 = \frac{\gamma\sigma}{1+\delta} - \frac{(1+\delta-\sigma)(-\gamma+\sqrt{\gamma^2+4\beta(1+\delta)})}{2(1+\delta)}$.
We now consider different value of the parameters $(\gamma, \beta, \delta, \sigma)$ for numerical experiments.

Case 1.1: Let us take $\gamma = \beta = \delta = 1$ and $\sigma = 0.5$. For this case, numerical results have computed for different time intervals $T = 15, 30, 60$ and 120 and shown in Figure 1. Also obtained tabulated results for the proposed method and compairison with [21] have shown in Table 1. For the values $\gamma = \beta = \delta = 1$ and $\sigma = 2$. Figure 2 contains 3D surface plots, one for numerical solution and other for analytical solution, at time $T = 10$ and space domain $[-10, 20]$. These 3D plots also depict the exactness of method.

Case 1.2: Let us take $\gamma = \beta = 1$ and $\sigma = 0.001$. Numrical results have computed with different value of δ, namely $\delta = 1, 4, 8$ for time interval $T = 0.2$ and $T = 1$. The comparison of the proposed method with existing methods available in the literature have shown in Table 2. Figure 3 has shown plot of numerical solution and exact solution at different times intervals, namely $T = 0.01, 1.0, 5.0, 10.0$ and space domain $[-10, 20]$.

Case 1.3: Let us take $\gamma = 0.1, \beta = 0.001$ and $\sigma = 0.0001$. In this problem, numerical solutions have computed with different value of δ, namely $\delta = 1, 4, 8$ and time interval $T = 0.2$ and $T = 1$ in Table 3 and can be seen clearly the accuracy of the proposed method with existing methods. The numerical solutions for $\delta = 4, 8$ at $T = 1$ are plotted in Figure 4.

Case 1.4: Let us consider $\gamma = 5, \delta = 1, \sigma = 0.00001$ and different values of β, namely $\beta = 1, 10, 100$. In this problem, numerical solutions have obtained for time interval $T = 0.3$ and $T = 0.9$ and the comparison of the proposed method with existing methods have shown in Table 4. In Figure 5, the 3D curves of numerical solutions are plotted for time interval $T = 0.3, 0.9$ and $\beta = 100$. Numerical results by proposed method obtained spectral accuracy and better as compare to existing methods.

Table 1. Comparison of proposed method with existing method for case 1.1.

		Proposed Method		Mittal [21]	
M	t	L_∞	L_2	L_∞	L_2
16	15	4.474e-08	1.291e-07	2.832e-07	4.079e-07
	30	7.739e-08	1.248e-07	1.635e-07	3.216e-07
	60	3.453e-08	3.288e-07	1.589e-07	2.953e-07
	120	2.128e-08	7.229e-08	6.943e-08	1.157e-07

FIGURE 1. Numerical solutions of case 1.1 for (a) $T = 15$, (b) $T = 30$, (c) $T = 60$ and (d) $T = 120$ at $\gamma = \beta = \delta = 1$ and $\sigma = 0.5$.

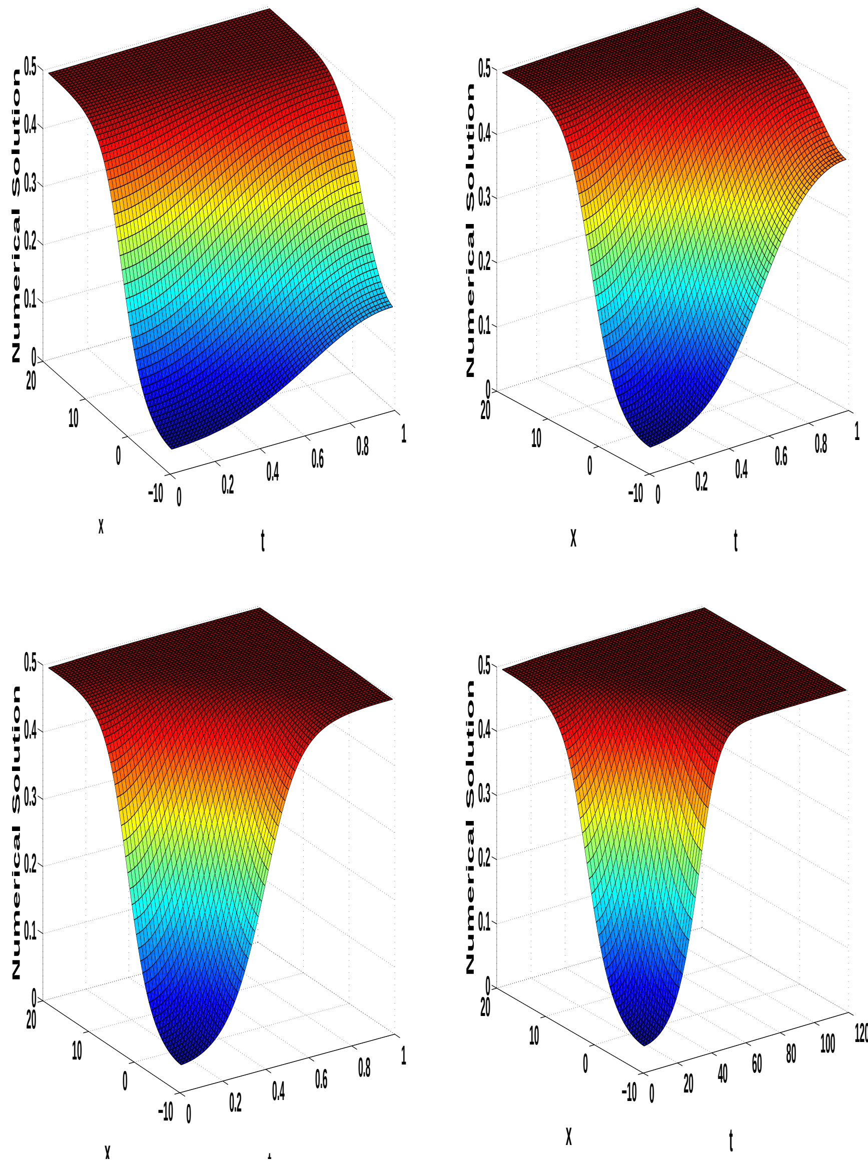

FIGURE 2. Numerical and exact solutions of case 1.1 for $\gamma = \beta = \delta = 1, \sigma = 2$ at $T = 10$.

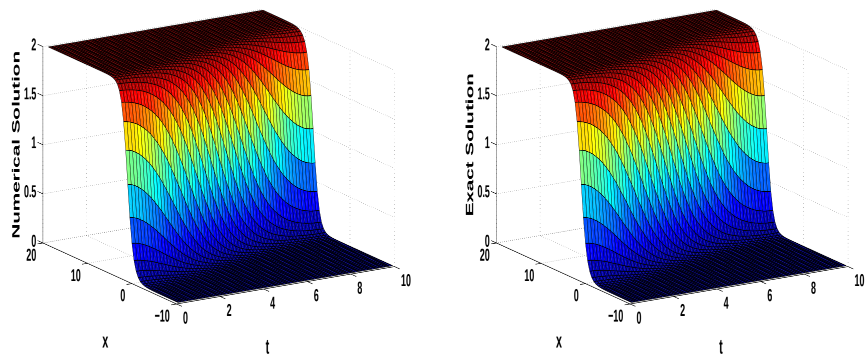

FIGURE 3. Numerical and exact solutions of case 1.2 for $\gamma = \beta = \delta = 1,\ \sigma = 0.001$ at different T.

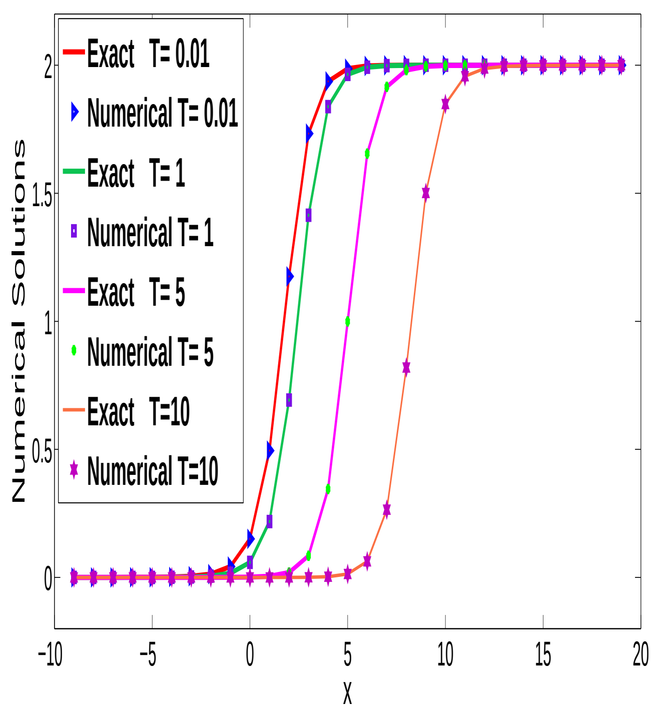

FIGURE 4. Numerical solutions of case 1.3 for $\gamma = 0.1, \beta = 0.001, \sigma = 0.0001$ and time $T = 1$ at (a) $\delta = 4$ and (b) $\delta = 8$.

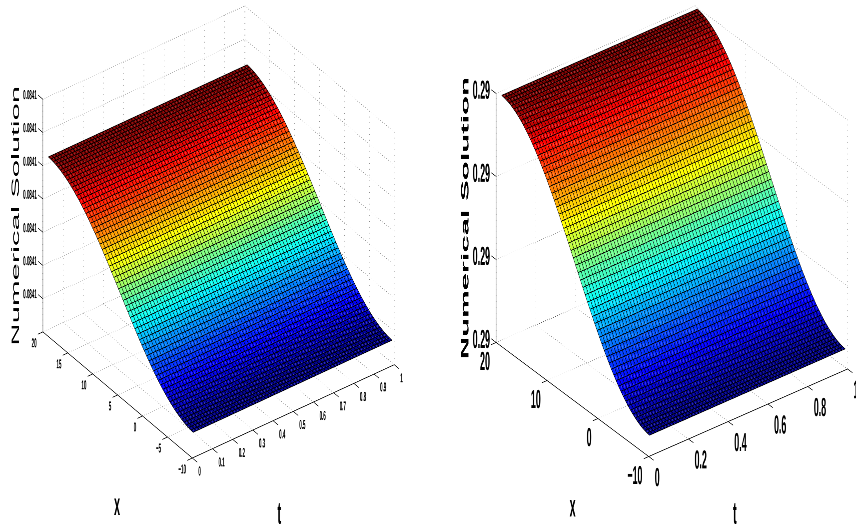

FIGURE 5. Numerical solutions of case 1.4 for $\gamma = 5, \beta = 100, \delta = 1$ and $\sigma = 0.00001$ at (a) $T = 0.3$ and (b) $T = 0.9$.

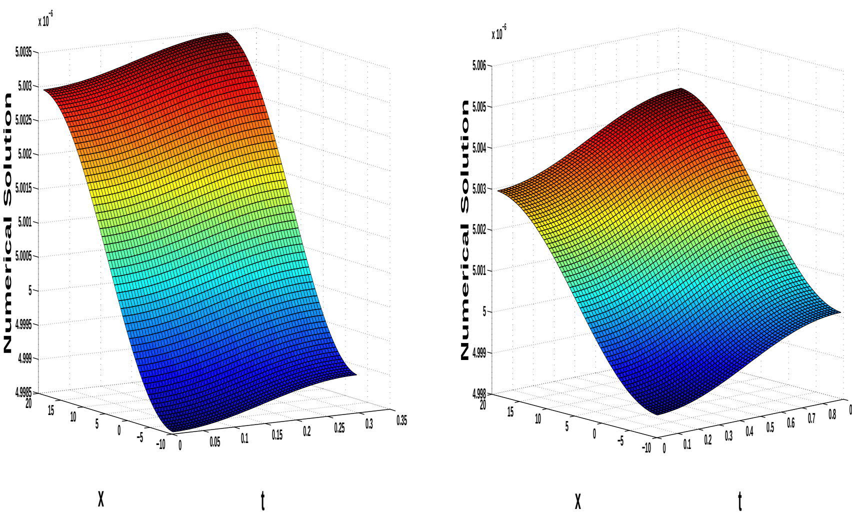

Table 2. Comparison of proposed method with existing methods for case 1.2.

		Proposed Method	Javidi and Golbabai [17]	Zhang *et al.* [33]	Mittal[21]
T	δ	L_∞	L_∞	L_∞	L_∞
0.2	1	6.6183e-09	4.0138e-08	3.7715e-08	3.7487e-08
	4	3.6398e-06	1.3139e-05	1.2346e-05	1.2270e-05
	8	8.7319e-06	3.5540e-05	3.3394e-05	3.3191e-05
1	1	9.2500e-09	4.6849e-08	4.3912e-08	4.2939e-08
	4	2.3396e-06	1.5325e-05	1.4366e-05	1.4045e-05
	8	1.2209e-06	4.1407e-05	3.8818e-05	3.7949e-05

Table 3. Comparison of proposed method with existing methods for case 1.3.

		Proposed Method	Javidi and Golbabai [17]	Duan *et al.* [11]	Mittal [21]
T	δ	L_∞	L_∞	L_∞	L_∞
0.2	1	5.5845e-14	2.9927e-13	2.7503e-13	5.7293e-13
	4	9.2422e-11	5.5795e-10	5.1183e-10	1.0662e-10
	8	5.7721e-11	2.0759e-09	1.8988e-09	3.9555e-09
1	1	1.5885e-13	3.1427e-13	3.4050e-13	2.8645e-13
	4	3.3183e-11	6.0028e-10	6.3366e-10	5.3309e-10
	8	3.4687e-10	2.1951e-09	2.3507e-09	1.9776e-09

Table 4. Comparison of proposed method with existing methods for case 1.4.

		Proposed Method	Javidi and Golbabai [17]	Duan *et al.* [11]	Mittal [21]
T	β	L_∞	L_∞	L_∞	L_∞
0.3	1	1.5366e-12	3.1632e-12	2.8782e-12	8.0197e-12
	10	2.1259e-11	3.9762e-11	3.6179e-11	1.0080e-11
	100	5.5659e-10	5.0392e-10	4.5851e-10	1.2775e-10
0.9	1	4.4230e-12	3.3411e-12	2.8819e-12	2.4059e-12
	10	2.2682e-11	4.1998e-11	3.6226e-11	3.0242e-11
	100	1.8593e-10	5.3225e-10	4.5911e-10	3.8327e-10

TABLE 5. Comparison of the proposed method with differential quadrature method for example 2 at different time T with $\sigma = 0.75$.

	Proposed	Method	Jiwari *et al.*	[18]
T	L_∞	L_2	L_∞	L_2
0.2	7.488e-07	7.547e-07	1.2350e-05	4.5670e-05
0.5	5.584e-07	5.489e-07	5.1986e-04	5.6423e-05
1.0	4.897e-07	8.691e-06	6.3283e-04	8.1671e-05
1.5	9.242e-06	7.278e-06	8.5383e-04	2.3681e-04
2.0	5.772e-06	9.471e-06	9.9123e-04	3.0123e-04
5.0	3.469e-06	7.666e-05	1.7904e-04	5.3420e-04
10.0	2.590e-06	5.165e-05	3.3551e-03	7.8900e-04

Example 2. Let us take time dependent Fitzhugh-Nagumo equation *i.e.* $\phi(U) = 0, q_1(t) = q_2(t) = \cos(t), q_4(t) = 2\cos(t)$ and $\varphi(U) = U(U-\sigma)(1-U)$. Here σ is a given parameter. Initial condition and boundary conditions are

$$U(x,0) = \frac{\sigma}{2} + \frac{\sigma}{2}\tanh\left(\frac{\sigma x}{2}\right), \; x \in [\xi, \eta], \tag{3.5}$$

and

$$U(\xi,t) = \frac{\sigma}{2} + \frac{\sigma}{2}\tanh\left(\frac{\sigma}{2}\left(\xi - (3-\sigma)\sin(t)\right)\right), \tag{3.6}$$

$$U(\eta,t) = \frac{\sigma}{2} + \frac{\sigma}{2}\tanh\left(\frac{\sigma}{2}\left(\eta - (3-\sigma)\sin(t)\right)\right), \tag{3.7}$$

respectively. The exact solution of this problem is [3, 29]

$$U(x,t) = \frac{\sigma}{2} + \frac{\sigma}{2}\tanh\left(\frac{\sigma}{2}\left(x - (3-\sigma)\sin(t)\right)\right). \tag{3.8}$$

In this example, if parameter $\sigma = 3$ then the analytical solutions moving only space direction and independent of time direction which has shown in Figure 6. Figures 7 and 8 have shown numerical solution and analytical solution for $\sigma = 0.5$ and $\sigma = 0.75$, respectively in 3D surface plots at time $T = 1$. Figure 9 has shown 2D plot of numerical and exact solution for $\sigma = 0.75$ at different times $T = 0.2, 0.5, 1.0, 2.0, 3.0, 5.0$. Table 5 shows numerical results of proposed method and comparison with differential quadrature method for $\sigma = 0.75$ at different time intervals.

4. CONCLUSION

In this research paper, we have proposed improved time-space pseudospectral method for numerical solution of generalized Burger-Huxley and time dependent Fitzhugh-Nagumo equations. The proposed scheme is based on pointwise discretzation of delta function based on lagrange polynomial. The performance of the method has shown using numerical examples and studied various cases. All computed results have shown very good agreement with the exact solutions and comparison of our results with existing results have shown in the paper.

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FIGURE 6. Numerical solution of example 2 for $\sigma = 3$ at $T = 5$.

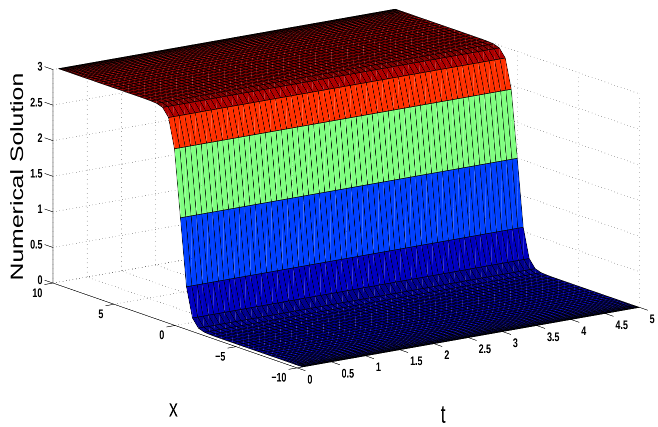


FIGURE 7. Numerical and exact solutions of example 2 for $\sigma = 0.5$ at $T = 1$.

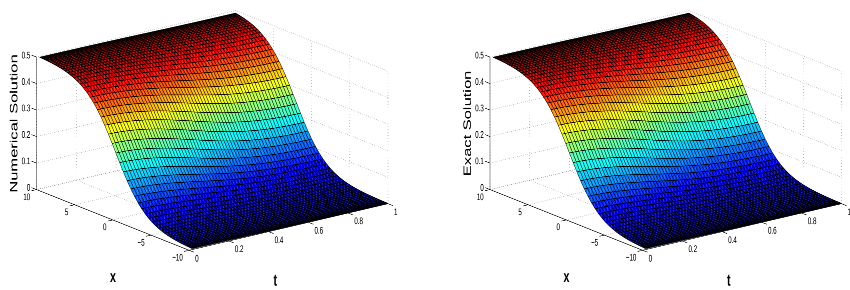


FIGURE 8. Numerical and exact solutions of example 2 for $\sigma = 0.75$ at $T = 1$.

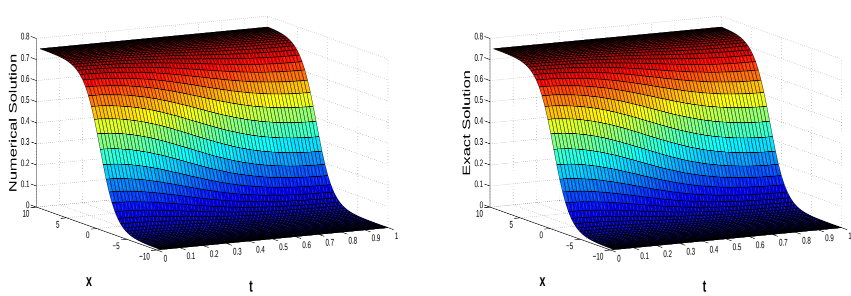
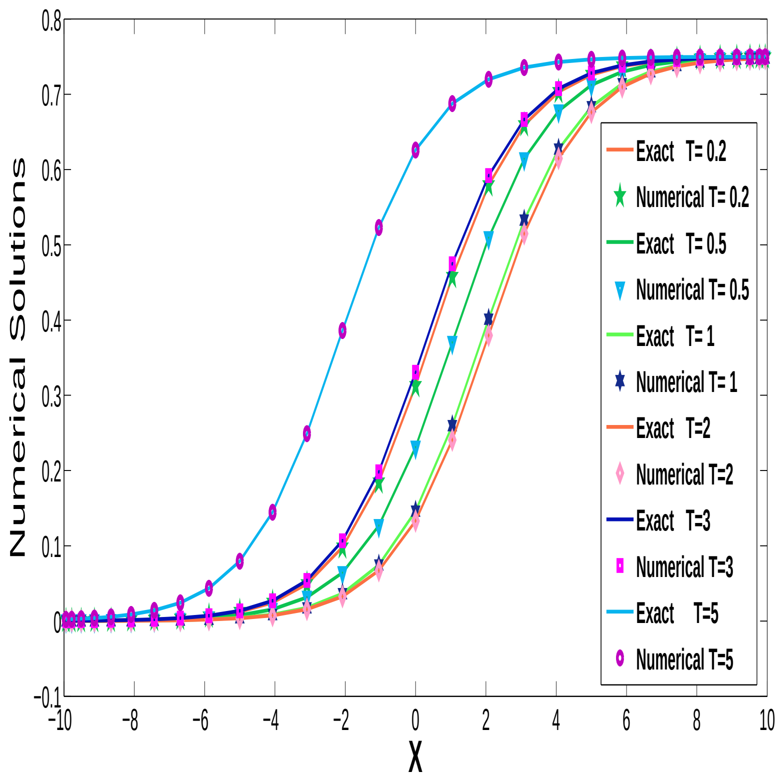


FIGURE 9. Numerical and exact solutions of example 2 for $\sigma = 0.75$ at different T.



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