*Computational Methods for Differential Equations*
http://cmde.tabrizu.ac.ir
**Vol. 6, No. 2, 2018, pp. 157-175**

# Numerical solution of nonlinear SPDEs using a multi-scale method

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**Abstract** In this paper we establish a new numerical method for solving a class of stochastic partial differential equations (SPDEs) based on B-splines wavelets. The method combines implicit collocation with the multi-scale method. Using the multi-scale method, SPDEs can be solved on a given subdomain with more accuracy and lower computational cost than the rest of the domain. The stability and consistency of the method are provided. Also numerical experiments illustrate the behavior of the proposed method.

**Keywords.** Multi-scale method, Cubic B-splines, Stochastic partial differential equations.

**2010 Mathematics Subject Classification.** 60H35, 65D07, 65T60.

## 1. Introduction

Many interesting problems in physics, science, engineering and finance can be modeled using stochastic partial differential equations (SPDEs). Numerical methods for evolution SPDEs have been studied extensively over the past two decades [1, 7, 9, 11-14, 17, 18, 32, 41-46]. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for SPDEs such as finite element methods [1, 4, 8-10, 19, 22, 24, 26, 42, 46], finite difference methods [13, 15, 31, 35, 36, 38, 39, 41, 43], spectral Galerkin methods [11, 18, 20, 21, 23, 28, 29, 32-34] and also some numerical methods that are based on the wavelet approximations [9, 16, 18, 25, 27, 40]. In this work, we intend to extend the multi-scale

Received: 1 May 2017 ; Accepted: 10 March 2018.
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method [2, 3, 30] to approximate the solutions of the following SPDEs,

$$\begin{aligned} du(x,t) &= (\mathcal{A}u(x,t) + f(u(x,t)))\,dt + g(u(x,t))dW(t), \\ u(x,0) &= h(x), \\ u(0,t) &= u(m,t) = 0, \end{aligned} \tag{1.1}$$

by wavelets where $\mathcal{A} \equiv \frac{\partial^2}{\partial x^2}$, $f$ is Lipshcitz function, $g$ is a function with bounded derivative and $\dot{W}(t)$ is a time white noise. The existence of strong solutions of this SPDE have been investigated in [44]. We are interested in applying the multi-scale method to equation (1.1) to approximate the solution based on the wavelet expansion. In this method, first we reshape wavelets in such a way that satistify the boundary conditions exactly, second, the implicit $\theta$-Euler-Maruyama method is employed to discretize time. Then we approximate the operators in matrix forms to obtain a system, which due to the multi-scale method will divide to two smaller systems with less computations. Finally, we combine the solutions of these systems to approximate the solution of equation (1.1). This method is suitable to approximate the solutions of stochastic equations because in every realization of the approximation less computation should be done.

The outline of the paper is as follows. In section 2, the Multi-resolution analysis and the operational matrices of wavelets are explained and we introduce main notation used throughout the paper. In section 3, we propose our stochastic the multi-scale method based on the wavelets. In section 4, consistency and stability criterions of the method are investigated. In the last section, numerical simulations are presented to illustrate the efficiency of the method.

## 2. Preliminary remarks

In this section we use the multi-resolution analysis (MRA), to represent derivatives of functions in matrix-form on a given subdomain, with respect to cubic b-splines. From [5], we know that MRA is a sequance of subspaces $V_j$ in $L_2(\mathbb{R})$, which satisfy the following properties:

(i) $V_j \subset V_{j-1}, j \in \mathbb{Z}$,
(ii) $\underset{j\in\mathbb{Z}}{\cup} V_j$ is dense in $L_2(\mathbb{R})$ and $\underset{j\in\mathbb{Z}}{\cap} V_j = \{0\}$,
(iii) If $f(\cdot) \in V_0$ then $f(2^{-j}\cdot) \in V_j$ and vice versa,
(iv) $\phi(\cdot - k)$, $k \in \mathbb{Z}$ is a Riesz basis of $V_0$.

It can be inferred that the family

$$\left\{\phi_{j,k}(x) = 2^{-\frac{j}{2}}\phi\left(2^{-j}x - k\right),\ k \in \mathbb{Z}\right\},$$

is a basis for $V_j$. One may construct wavelets by completing the spaces $V_j$ to the space $V_{j-1}$ by means of a space $W_j$ , i.e. $V_{j-1} = V_j \oplus W_j$, in such a way that there exists a function $\psi$ such that $W_j$ is spanned by $\psi(2^{-j}\,.-k)$. For each $j \in \mathbb{Z}$, the space $W_j$ serves as the orthogonal complement of $V_j$ in $V_{j-1}$. In the biorthogonal case [6], the space $W_j$ is orthogonal to the dual of $V_j$. The sequence $\left\{\widetilde{V}_j\right\}$ constructs another multi-resolution analysis of $L_2(\mathbb{R})$. Two functions $\psi, \widetilde{\psi} \in L_2(\mathbb{R})$ are called biorthogonal wavelets if each of the sets $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ and $\left\{\widetilde{\psi}_{jk} : j, k \in \mathbb{Z}\right\}$ are Riesz basis of $L_2(\mathbb{R})$ and they are biorthogonal to each other in the following sense

$$\langle \psi_{jk}, \widetilde{\psi}_{im} \rangle = \delta_{j,i}\delta_{k,m} \quad \forall\, j, k, l, m \in \mathbb{Z},$$

where $\langle ., . \rangle$ is the inner product on $L_2(\mathbb{R})$ and $\delta_{i,j}$ is Kroneker delta function. Designing biorthogonal wavelets allows more degrees of freedom than orthogonal wavelets. One additional degree of freedom is the possibility to construct symmetric wavelet functions. Since they define a multi-resolution analysis, the dual functions must satisfy,

$$\widetilde{\phi} = \sum_k \widetilde{h}_k \widetilde{\phi}(2x-k) \quad \textit{and} \quad \widetilde{\psi} = \sum_k \widetilde{g}_k \widetilde{\phi}(2x-k), \tag{2.1}$$

where $\widetilde{h}_k$ and $\widetilde{g}_k$ have been introduced in [5]. The B-splines which are symmetric and have finite support, are defined by the following recursively formula

$$\begin{aligned} &B_0 = 1_{[0,1)}, \\ &B_{k+1}(x) = \tfrac{1}{2^k} \sum_{i=0}^{k+1} \begin{pmatrix} k+1 \\ i \end{pmatrix} B_k(2x-i). \end{aligned} \tag{2.2}$$

Furthermore, we have

$$\frac{d}{dx} B_{i+1}(x) = B_i(x) - B_i(x+1), \tag{2.3}$$

$$\frac{d^2}{dx^2} B_{i+1}(x) = B_{i-1}(x) - 2B_{i-1}(x+1) + B_{i-1}(x+2). \tag{2.4}$$

In this work, we consider the cubic B-splines,

$$\phi(x) = B_3(x) = \frac{1}{6} \sum_{i=0}^{4} \begin{pmatrix} 4 \\ i \end{pmatrix} (-1)^i (x-i)_+^3, \tag{2.5}$$

where

$$x_+^k = \begin{cases} x^k, & x > 0, \\ 0, & x \le 0. \end{cases}$$

The cubic B-spline wavelet is given by

$$\begin{aligned} \psi(x) = \frac{1}{2^7}\Big(&5\phi(2x+5) + 20\phi(2x+4) + \phi(2x+3) \\ &- 96\phi(2x+2) - 70\phi(2x+1) + 280\phi(2x) \\ &- 70\phi(2x-1) - 96\phi(2x-2) + \phi(2x-3) \\ &+ 20\phi(2x-4) + 5\phi(2x-5)\Big). \end{aligned}$$

The cubic B-spline wavelet has four vanishing moments, that is,

$$\int_{-3}^{4} x^p \psi(x) dx = 0; \quad p = 0, 1, 2, 3. \tag{2.6}$$

Let $f_{|\ V_j}$ denote the projection $f \in L_2(\mathbb{R})$ onto $V_j$. $f_{|\ V_j}$ can be represented by cubic B-splines as

$$f_{|\ V_j}(x) = \sum_{i=0}^{N'-1} a_{j,\,i}\phi_{j,\,i}(x), \tag{2.7}$$

where $a_{j,\,i} = \left\langle f, \widetilde{\phi}_{j,\,i} \right\rangle$ and $N' = m2^{-j}$. Since $V_{j-1} = V_j \oplus W_j$, we have two representations of the function $f_{|V_{j-1}}$ , one as an element in $V_{j-1}$ associated with the sequence $\{a_{j-1,k}\}$, and another as a sum of elements in $V_j$ and $W_j$ associated with the sequences $\{a_{j,k}\}$ and $\{b_{j,k}\}$.

$$f_{|\ V_{j-1}}(x) = \sum_{i=0}^{2N'-1} a_{j-1\,i}\,\phi_{j-1\,i}(x) = \left[\ \mathbf{a}_{j-1}\ \right]^T \left[\Phi_{j-1}\right], \tag{2.8}$$

$$\begin{aligned} f_{|\ V_j \oplus W_j}(x) &= \sum_{i=0}^{N'} a_{j\,i}\,\phi_{j\,i}(x) + \sum_{i=0}^{N'-1} b_{j\,i}\psi_{j\,i}(x) \\ &= \begin{bmatrix} \mathbf{a}_j \\ \mathbf{b}_j \end{bmatrix}^T \begin{bmatrix} \Phi_j \\ \Psi_j \end{bmatrix}, \end{aligned} \tag{2.9}$$

where $\mathbf{a}_j^T = [a_{j0},\, a_{j1},\, \ldots, a_{N'}]$, $\mathbf{b}_j^T = [b_{j0},\, b_{j1},\, \ldots, b_{j\,N'-1}]$, $b_{j\,i} = \left\langle f, \widetilde{\psi}_{j\,i} \right\rangle$, $a_{ji} = \left\langle f, \widetilde{\phi}_{j\,i} \right\rangle$, $\Phi_j^T = [\phi_{j0},\, \phi_{j1},\, \ldots, \phi_{j\,N'}]$ and $\Psi_j^T = [\psi_{j0},\, \psi_{j1},\, \ldots, \psi_{j\,N'-1}]$. In general case we have $L_2(\mathbb{R}) = V_0 \bigoplus_{k=0}^{-\infty} W_k = V_0 \oplus W_0 \oplus W_{-1} \oplus W_{-2} \oplus \ldots$ and

$$f = \sum_{k=0}^{m} a_{0,k}\phi_{0,k} + \sum_{j=0}^{-\infty}\sum_{k=0}^{N'-1} b_{j,k}\psi_{j,k}. \tag{2.10}$$

The following relations show how to pass between the representations (2.8) and (2.9). Applying (2.1), derive

$$\begin{aligned} a_{j,k} &= \left\langle \sum_m a_{j,m}\phi_{j,m} + \sum_n a_{j,n}\psi_{j,m}\,,\ \widetilde{\phi}_{j,k} \right\rangle \\ &= \left\langle \sum_m a_{j-1,m}\phi_{j-1,m}\,,\ \sum_i \widetilde{h}_i \widetilde{\phi}_{j-1,2k+i} \right\rangle \\ &= \sum_i \widetilde{h}_i a_{j-1,2k+i}, \end{aligned}$$

and similarly,

$$b_{j,k} = \sum_i \widetilde{g}_i a_{j-1,2k+i}.$$

These formulas define the fast wavelet transform (FWT), $\mathcal{F}_{j-1}$, which converts the coefficients $a_{j-1,\,i}$ of $f_{|\ V_{j-1}}$ to coefficients $a_{ji}$ and $b_{ji}$ of $f_{|\ V_j \oplus W_j}$

$$\mathcal{F}_{j-1}\left[\mathbf{a}_{j-1}\right] = \begin{bmatrix} \mathbf{a}_j \\ \mathbf{b}_j \end{bmatrix}.$$

For more details refer to [5]. Let $\Lambda$ be the subdomain of $\Omega$, we denote a restricted vector $F$ to subdomain $\Lambda$ by $F_\Lambda = \begin{bmatrix} \mathbf{a}_\Lambda \\ \mathbf{b}_\Lambda \end{bmatrix}$ where $\mathbf{a}_\Lambda$ and $\mathbf{b}_\Lambda$ are member of $\mathbf{a}$ and $\mathbf{b}$ of $\mathcal{F}$ belong to $\Lambda$, respectively. For representing second derivative by operational matrix, at first using the collocation approach we construct the matrix $P_j$ which its inverse interpolate a given function $f$ by the cubic B-splines

$$P_j F_{coef} = F_{value}, \tag{2.11}$$

where

$$f_{|\ V_j}(x) = \sum_{k=0}^{N'-1} a_{jk}\phi_{jk}(x), F_{coef} = [a_{j\,0},\, a_{j\,1},\, \dots,\, a_{j\,N'-1}]^T,$$

$$F_{value} = \sum_{k=0}^{N'-1} a_{jk}[\phi_{jk}(x_0),\, \phi_{jk}(x_1),\, \dots,\, \phi_{jk}(x_{N'-1})]^T, P_j = [\phi_{j\,k}(x_i)]_{k,\,i},$$

and $x_i = i\,2^j$ for $0 \le i \le N'$ and $j \in \mathbb{Z}$. When wavelets derived by cubic B-splines as scaling functions, $P_j = (a_{i\,k})$ is a tridiagonal matrix with,

$$a_{i\,k} = \begin{cases} 2^{\frac{-j}{2}}\frac{2}{3}, & i = k, \\ 2^{\frac{-j}{2}}\frac{1}{6}, & i = k \pm 1, \\ 0, & o.w. \end{cases}$$

Let

$$\mathcal{D}_j = \left[\phi''_{j\,k}(x_i)\right]_{k,\,i}, F'' = \sum_{k=0}^{N'-1} a_{jk}\left[\phi''_{j\,0}(x_0),\, \phi''_{j\,1}(x_1),\, \dots,\, \phi''_{j\,N'-1}(x_{N'-1})\right],$$

then

$$\mathcal{D}_j F_{coef} = F''. \tag{2.12}$$

Now we can construct differentiation matrix on $V_j$ as the following,

$$M_j = \mathcal{F}_j \times (P_j)^{-1} \times \mathcal{D}_j \times \mathcal{F}_j^{-1}. \tag{2.13}$$

2.1. **Boundary conditions.** We want to maintain consistency the wavelets with boundary conditions of equation (1.1). To do this, we reshape every B-splines $\phi_{j,k}$ supported at $x = 0$ and $x = m$. Let,

$$\begin{aligned} &\phi_0(x) = a\phi_{j,-1}(x) + b\phi_{j,0}(x) + c\phi_{j,+1}(x), \\ &\phi_m(x) = a'\phi_{j,2^{-j}m-1}(x) + b'\phi_{j,2^{-j}m}(x) + c'\phi_{j,2^{-j}m+1}(x). \end{aligned}$$

Due to boundary conditions and (1.1), $\phi_0$ and $\phi_m$ must satistify

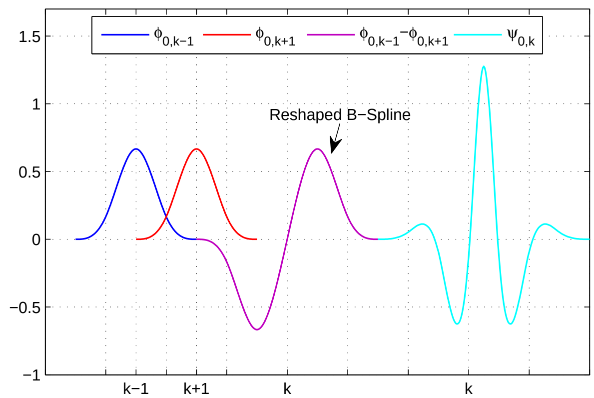
$$\phi_0(0) = 0, \quad \phi_0''(0) = 0, \quad \phi_m(m) = 0, \quad \phi_m''(m) = 0.$$

The coefficients of $\phi_0$ and $\phi_m$ can be derived from these equations, so we have,

$$\begin{aligned} &\phi_0(x) = \phi_{j,-1}(x) - \phi_{j,+1}(x), \\ &\phi_m(x) = \phi_{j,2^{-j}m-1}(x) - \phi_{j,2^{-j}m+1}(x). \end{aligned}$$

The process for the B-spline wavelets is the same. We also described more complicated boundary conditions via wavelets in [3].

FIGURE 1. The cubic B-spline functions $\phi_{0,\,k-1}$ and $\phi_{0,\,k+1}$, the cubic B-spline wavelet $\psi_{0,\,k}$ and the reshaped scaling function $(\phi_{0,\,k+1} - \phi_{0,\,k-1})$.




## 3. The approximation for the stochastic evolution equation

For the time discretization of equation (1.1) we use the implicit $\theta$-Euler-Maruyama scheme. Let $h = \frac{T}{N}$, be a time step, $t_n = nh$, $n = 0, \ldots, N$. We denote an approximate solution of $u(x,t)$ in the space $V_{j-1}$ by $u_j(x,t)$. Using time stepping $\theta$-Euler-Maruyama scheme, the SPDE (1.1) is written as

$$\begin{aligned} u_j^{n+1}(x) &= u_j^n(x) + h\mathcal{A}\left(\theta\, u_j^{n+1}(x) + (1-\theta)\; u_j^n(x)\right) \\ &\quad + hf(u_j^{n+1}(x)) + g(u_j^n(x))\Delta W_n. \end{aligned} \tag{3.1}$$

For the grid points $x_i = i2^{j-1}$, $i = 0, 1, \ldots, m2^{-j+1}$, let $\mathbf{u}_j^k = [u_j^k(x_0), \ldots, u_j^k(x_{N'})]$. Now we put the matrix form of the differential operator $\mathcal{A}$ in the numerical (3.1), then we have

$$\begin{aligned} \mathbf{u}_{j-1}^{n+1} &= \mathbf{u}_{j-1}^n + h\mathcal{D}_j P_{j-1}^{-1}\left(\theta\, \mathbf{u}_{j-1}^{n+1} + (1-\theta)\; \mathbf{u}_{j-1}^n\right) \\ &\quad + h\mathbf{F}_{j-1}^n + \mathbf{G}_{j-1}^n \Delta W_n, \end{aligned} \tag{3.2}$$

where $\mathbf{F}_j^n = [f(u_j^n(x_0), \ldots, f(u_j^n(x_{N'}))]^T$ and $\mathbf{G}_j^n = [g(u_j^n(x_0),\ \ldots,\ g(u_j^n(x_{N'}))]^T$. Now, multiplying (3.2) by $\left(\mathcal{F}_{j-1} \cdot P_{j-1}^{-1}\right)$, we derive

$$\begin{aligned} \begin{bmatrix} \mathbf{a}^{n+1} \\ \mathbf{b}^{n+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{a}^{n} \\ \mathbf{b}^{n} \end{bmatrix} + \theta M_{j-1} \begin{bmatrix} \mathbf{a}^{n+1} \\ \mathbf{b}^{n+1} \end{bmatrix} h + (1-\theta)\, M_{j-1} \begin{bmatrix} \mathbf{a}^{n} \\ \mathbf{b}^{n} \end{bmatrix} h \\ &\quad + h \begin{bmatrix} \mathbf{f}_a^n \\ \mathbf{f}_b^n \end{bmatrix} + \begin{bmatrix} \mathbf{g}_a^n \\ \mathbf{g}_b^n \end{bmatrix} \Delta W_n, \end{aligned} \tag{3.3}$$

where

$$\begin{bmatrix}\mathbf{f}_a^n\\ \mathbf{f}_b^n\end{bmatrix} = \mathcal{F}_{j-1}\cdot P_{j-1}^{-1}\mathbf{F}_{j-1}^n,\ \begin{bmatrix}\mathbf{g}_a^n\\ \mathbf{g}_b^n,\end{bmatrix} = \mathcal{F}_{j-1}\cdot P_{j-1}^{-1}\mathbf{G}_{j-1}^n,\ and\ M_{j-1} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}.$$

Therefore,

$$\begin{aligned}(I-\theta h M_{j-1})\begin{bmatrix}\mathbf{a}^{n+1}\\ \mathbf{b}^{n+1}\end{bmatrix} &= (I+(1-\theta)\,hM_{j-1})\begin{bmatrix}\mathbf{a}^{n}\\ \mathbf{b}^{n}\end{bmatrix}\\ &\quad + h\begin{bmatrix}\mathbf{f}_a^n\\ \mathbf{f}_b^n\end{bmatrix} + \begin{bmatrix}\mathbf{g}_a^n\\ \mathbf{g}_b^n\end{bmatrix}\Delta W_n. \end{aligned} \tag{3.4}$$

To increase the accuracy of the solution in some places of domain $\Omega$ and to avoid growing the calculations we use the multi-scale method. This means that we solve the system in a space $V_j$ and domain $\Omega$ which we call the large scale system (coarse resolution). Once again we solve the system in a finer space $V_{j-1}$ and subdomain $\Lambda$ that we call small scale system (fine resolution). Combination of the solutions of these two systems makes suitable accuracy and less computation than the solutions of the system achieved in the space $V_{j-1}$ on domain $\Omega$. One can consider several subdomains and solve the SPDE in different resolutions, but we consider a subdomain for simplicity. In fact, depending on the number of mentioned subdomains we have the same number of additional systems.

At first, for constructing large scale system, we don't use all of the elements in $M_{j-1}$. The elements of $M_{j-1}$ must be broken up into a block decomposition that is compatible with the block structure of the vector $\begin{bmatrix}\mathbf{a}^n\\ \mathbf{b}^n\end{bmatrix}$.

We only consider the block $A$ over all domain $\Omega$. Then using time stepping scheme (3.3), we find an approximation for $\mathbf{a}^{n+1}$ via (3.5) which we call $\mathbf{a}_\Lambda^{Tm}$

$$(I-\theta hA)\,\mathbf{a}^{n+1} = (I+(1-\theta)\,hA)\,\mathbf{a}^n + h\mathbf{f}_a^n + \mathbf{g}_a^n\Delta W_n. \tag{3.5}$$

In the next step, we solve the system on the subdomain $\Lambda$ at the small scale resolution $V_{j-1}$

$$\begin{aligned}\left(I-\theta h\begin{bmatrix}A & B\\ C & D\end{bmatrix}\right)_\Lambda \begin{bmatrix}\mathbf{a}^{n+1}\\ \mathbf{b}^{n+1}\end{bmatrix}_\Lambda &= \left(I+(1-\theta)\,h\begin{bmatrix}A & B\\ C & D\end{bmatrix}\right)_\Lambda \begin{bmatrix}\mathbf{a}^{n}\\ \mathbf{b}^{n}\end{bmatrix}_\Lambda\\ &\quad + h\begin{bmatrix}\mathbf{f}_a^n\\ \mathbf{f}_b^n\end{bmatrix}_\Lambda + \begin{bmatrix}\mathbf{g}_a^n\\ \mathbf{g}_b^n\end{bmatrix}_\Lambda \Delta W_n,\end{aligned}$$

where $A_\Lambda$, $B_\Lambda$, $C_\Lambda$ and $D_\Lambda$ are composed from the elements of $A$, $B$, $C$ and $D$ that are related to $\Lambda$.

Now, we are looking for the vector correction $\mathbf{a}^{Cr}$ where $\mathbf{a}^{n+1} = \mathbf{a}^{Tm} + \mathbf{a}^{Cr}$. Consider the $\theta$-Euler-Maruyama method for this case, since $\mathbf{a}_\Lambda^{n+1} = \mathbf{a}_\Lambda^{Tm} + \mathbf{a}_\Lambda^{Cr}$ thus

$$
\begin{aligned}
(I - h\theta M_{j-1})_\Lambda \begin{bmatrix} \mathbf{a}_\Lambda^{Cr} \\ \mathbf{b}_\Lambda^{n+1} \end{bmatrix} = (I + (1-\theta)\, hM_{j-1})_\Lambda \begin{bmatrix} 0 \\ \mathbf{b}^n \end{bmatrix}_\Lambda \\
+ h \begin{bmatrix} 0 \\ \mathbf{f}_a^n \end{bmatrix}_\Lambda + \begin{bmatrix} 0 \\ \mathbf{g}_a^n \end{bmatrix}_\Lambda \Delta W_n \\
+ \begin{bmatrix} 0 \\ C_\Lambda \end{bmatrix} \left(\theta h \mathbf{a}_\Lambda^{Tm} + (1-\theta)\, h\mathbf{a}_\Lambda^n\right).
\end{aligned} \tag{3.6}
$$

By solving system (3.6), we get the vector $\begin{bmatrix} \mathbf{a}^{n+1} \\ \mathbf{b}^{n+1} \end{bmatrix}_\Lambda$.

The last step is to construct the vector $\begin{bmatrix} \mathbf{a}^{n+1} \\ \mathbf{b}^{n+1} \end{bmatrix}_\Omega$ from vectors $\begin{bmatrix} \mathbf{a}^{Tm} \\ 0 \end{bmatrix}_\Omega$ and $\begin{bmatrix} \mathbf{a}^{n+1} \\ \mathbf{b}^{n+1} \end{bmatrix}_\Lambda$. In the subdomain $\Lambda$, the vector $\begin{bmatrix} \mathbf{a}^{n+1} \\ \mathbf{b}^{n+1} \end{bmatrix}_\Lambda$ is a better approximate solution than the vector $\begin{bmatrix} \mathbf{a}^{Tm} \\ 0 \end{bmatrix}_\Lambda$ for system (3.5). So to increase the accuracy of the approximate vector $\begin{bmatrix} \mathbf{a}^{Tm} \\ 0 \end{bmatrix}_\Omega$ we must replace elements of $\begin{bmatrix} \mathbf{a}^{Tm} \\ 0 \end{bmatrix}_\Omega$ by the elements of $\begin{bmatrix} \mathbf{a}^{n+1} \\ \mathbf{b}^{n+1} \end{bmatrix}_\Lambda$, indeed we substitude the only ones that are related to subdomain $\Lambda$

$$
\begin{bmatrix} \mathbf{a}^{T_m} \\ \hline 0 \end{bmatrix} = \begin{bmatrix} X_{j0}(T_m) \\ \vdots \\ a_\Lambda^{Tm} \\ \vdots \\ X_{jN'}(T_m) \\ \hline 0 \\ \vdots \\ 0_\Lambda \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftrightarrow \\ \\ \textit{and} \\ \\ \leftrightarrow \\ \\ \\ \end{matrix} \begin{bmatrix} \mathbf{a}_\Lambda^{n+1} \\ \\ \\ \\ \mathbf{b}_\Lambda^{n+1} \end{bmatrix}
$$

$$\xrightarrow{Replace} \begin{bmatrix} X_{j0}\left(T_m\right) \\ \vdots \\ \mathbf{a}_{\Lambda}^{n+1} \\ \vdots \\ X_{jN'}\left(T_m\right) \\ \hline 0 \\ \vdots \\ \mathbf{b}_{\Lambda}^{n+1} \\ \vdots \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \mathbf{a}^{n+1} \\ \hline \mathbf{b}^{n+1} \end{bmatrix}. \tag{3.7}$$

This completes the method.

As an example, let $\Omega = [0, 10]$. If one solves the problem in $V_{-7}$, $\mathbf{a}^n$ involves $2^7(10) = 1280$ elements . But in $V_{-6}$, the resolution of the large scale system has $2^6(10) = 640$ coefficients. If our problem requires high resolution, in $V_{-7}$, in a subdomain such as $\Lambda = [2.5, 4.5]$, then small scale system has $2^7(4.5-2.5) = 256$ coefficients. So solving the small scale system and the large scale system in $V_{-6}$ separately would involve 896 elements which has less computational cost than solving the problem in whole domain $\Omega$ in $V_{-7}$.

## 4. Stability and Consistency

In this section we consider the consistency and stability of the method, we recall the following definitions from [36].

**Definition 4.1.** (Consistency) A stochastic difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the SPDE $Lu = G$, if for any continuously twice differentiable function $\Phi$ in mean square

$$E \left\| (Lu - G)|_k^n - L_k^n u\left(k\Delta x, n\Delta t\right) - G_k^n \right\|_\infty^2 \to 0, \tag{4.1}$$

as $\Delta x \to 0$ and $\Delta t \to 0$.

**Definition 4.2.** (Stability) A stochastic difference scheme is said to be stable in mean square if there exist some positive constants $\overline{\Delta x_0}$ and $\overline{\Delta t_0}$ and non-negative constants $K$ and $\beta$ such that

$$E \left\| u^{n+1} \right\|_\infty^2 \leq K e^{\beta t} E \left\| u^0 \right\|_\infty^2, \tag{4.2}$$

for all $0 \leq t = (n+1)\,\Delta t$, $0 \leq \Delta x \leq \overline{\Delta x_0}$, and $0 \leq \Delta t \leq \overline{\Delta t_0}$.

**Theorem 4.3.** *Let $e_j(x)$ be the error of second derivative of approximation of $\Phi \in C^2[0, m]$ in $V_{j-1}$, we have*

$$|e_j(x)| = O\left(2^j\right), \tag{4.3}$$

*Proof.* Assume that $\Phi$ is represented by (2.10), then let Taylor expansion of $\Phi \in C^2[0,\, m]$ in $x_0 \in [0,\, m]$, can be written

$$\Phi(x) = \Phi(x_0) + (x - x_0)\Phi'(x_0) + \frac{(x-x_0)^2}{2}\Phi''(\zeta), \quad \zeta \in D_\Phi,$$

let $x_0 = 2^j k$ and $b_{j,\,k}$ be the coefficient of representation $\Phi$ by expansion of cubic B-spline wavelet as (2.10). Then from (2.6) we have

$$\begin{aligned} b_{j,k} &= \int_{2^{j-1}(k-3)}^{2^{j-1}(k+4)} \Phi(x)\widetilde{\psi}_{j,k}(x)dx \\ &= \int_{2^{j-1}(k-3)}^{2^{j-1}(k+4)} \Phi(2^j k)\widetilde{\psi}_{j,k}(x) + (x - 2^j k)\Phi'(2^j k)\widetilde{\psi}_{j,k}(x)dx \\ &+ \int_{2^{j-1}(k-3)}^{2^{j-1}(k+4)} \frac{(x-2^j k)^2}{2}\Phi''(\zeta)\widetilde{\psi}_{j,k}(x)dx. \end{aligned} \tag{4.4}$$

Using (2.6) and substituting $u = 2^{-j}x - k$ in the above equation the first integral vanishes

$$b_{j,k} = \int_{2^{j-1}(k-3)}^{2^{j-1}(k+4)} \frac{(x-2^j k)^2}{2}\Phi''(\zeta)\widetilde{\psi}_{j,k}(x)dx. \tag{4.5}$$

For all $x = 2^j l$, $0 \le l \le m2^{-j}$, we have

$$|e_j(x)| = \left| \sum_{i=j}^{-\infty} \sum_{k=0}^{m2^{-i}-1} b_{i,k}\psi''_{i,k}(x) \right|, \tag{4.6}$$

$$\begin{aligned} |e_j(x)| &= \left| \sum_{i=j}^{-\infty} \sum_{k=0}^{m2^{-i}-1} b_{i,k} \frac{\partial^2}{\partial x^2} \sqrt{2} \sum_r g_r \phi_{i-1,r+k}(x) \right| \\ &= \sqrt{2} \left| \sum_{i=j}^{-\infty} \sum_{k=0}^{m2^{-i}-1} b_{i,k} \sum_r g_r \frac{\partial^2}{\partial x^2} 2^{\frac{1-i}{2}} B_3(2^{1-i}x - r - k) \right|. \end{aligned}$$

From (2.4), we have

$$\begin{aligned} |e_j(x)| &= 8 \Bigg| \sum_{i=j}^{-\infty} \sum_{k=0}^{m2^{-i}-1} b_{i,k} 2^{\frac{-5i}{2}} \sum_r g_r \big(B_1(2^{1-i}x - r - k) \\ &\qquad - 2B_1(2^{1-i}x - r - k - 1) + B_1(2^{1-i}x - r - k - 2)\big) \Bigg| \\ &= 8 \Bigg| \sum_{i=j}^{-\infty} 2^{\frac{-5i}{2}} \sum_r g_r \sum_{k=0}^{m2^{-i}-1} b_{i,k} \big(B_1(2^{1-i}x - r - k) \\ &\qquad - 2B_1(2^{1-i}x - r - k - 1) + B_1(2^{1-i}x - r - k - 2)\big) \Bigg|, \end{aligned} \tag{4.7}$$

since $B_1(2^{1-i}2^j l - r - k)$ is nonzero only for $k = 2^{j-i+1}l - r$, so we have

$$\left|e_j(2^j l)\right| = 8\left|\sum_{i=j}^{-\infty} 2^{\frac{-5i}{2}} \sum_r g_r B\left(b_{i,s} - 2b_{i,s-1} + b_{i,s-2}\right)\right|,$$

where $s = 2^{j-i+1}l - r$ and $B = B_1(0)$. From (4.5) we get

$$\begin{aligned}\left|e_j(2^j l)\right| = 8B\Bigg|\sum_{i=j}^{-\infty} 2^{\frac{-5i}{2}} \sum_r g_r & \int_{2^{i-1}(s-3)}^{2^{i-1}(s+4)} \frac{(x-2^i s)^2}{2}\Phi''(\zeta)\widetilde{\psi}_{i,s}(x)dx \\ & - 2\int_{2^{i-1}(s-4)}^{2^{i-1}(s+3)} \frac{(x-2^i(s-1))^2}{2}\Phi''(\zeta)\widetilde{\psi}_{i,s-1}(x)dx \\ & + \int_{2^{i-1}(s-5)}^{2^{i-1}(s+2)} \frac{(x-2^i(s-2))^2}{2}\Phi''(\zeta)\widetilde{\psi}_{i,s-2}(x)dx\Bigg|. \end{aligned} \tag{4.8}$$

Thus

$$\begin{aligned}\left|e_j(2^j l)\right| \leq 8B\Bigg|\sum_{i=j}^{-\infty}\sum_r g_r & \left|\Phi''(\zeta_{i,s}) - 2\Phi''(\zeta_{i,s-1}) + \Phi''(\zeta_{i,s-2})\right| \\ & \times \int_{-3}^{4} \frac{u^2}{2}\left|\widetilde{\psi}(u)\right| du\Bigg|, \end{aligned} \tag{4.9}$$

where $\zeta_{i,s}$, $\zeta_{i,s-1}$ and $\zeta_{i,s-2}$ are in $[2^{i-1}(S-5), 2^{i-1}(S+4)]$. Since $\Phi$ is continuesly twice differentiable we can consider $j$ such that

$$|\Phi''(\zeta_{i,s}) - 2\Phi''(\zeta_{i,s-1}) + \Phi''(\zeta_{i,s-2})| < 2^i, \quad \forall i \leq j < 0,$$

finally from (4.9), we get

$$\left|e_j(2^j l)\right| \leq 8B2^{j+1}\left|\sum_r g_r\right| \int_{-3}^{4} \frac{u^2}{2}\left|\widetilde{\psi}(u)\right| du. \tag{4.10}$$

□

Now, we investigate the consistency of the proposed method.

**Theorem 4.4.** *The stochastic scheme (3.4) is consistent in mean square.*

*Proof.* Let $\varphi(x,t)$ be a smooth function, and

$$\begin{aligned} L(\varphi)|_k^n = {} & \varphi(k\Delta x, (n+1)\Delta t) - \varphi(k\Delta x, n\Delta t) \\ & - \int_{n\Delta t}^{(n+1)\Delta t} \mathcal{A}\varphi(k\Delta x, s)\,ds - \int_{n\Delta t}^{(n+1)\Delta t} f(\varphi(k\Delta x, s))\,ds \\ & - \int_{n\Delta t}^{(n+1)\Delta t} g(\varphi(k\Delta x, s))\,dW(s). \end{aligned} \tag{4.11}$$

We present (4.11) in the matrix form:

$$\begin{aligned} L(\varphi)|^{n} &= \Phi((n+1)\Delta t) - \Phi(n\Delta t) \\ &\quad - \int_{n\Delta t}^{(n+1)\Delta t} \mathcal{A}\Phi(s)ds - \int_{n\Delta t}^{(n+1)\Delta t} \bar{\mathbf{F}}(s)ds \\ &\quad - \int_{n\Delta t}^{(n+1)\Delta t} \bar{\mathbf{G}}(s)dW(s), \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} L(\varphi)|^{n} &= \left[ L(\varphi)|_{1}^{n}, L(\varphi)|_{2}^{n}, \ldots, L(\varphi)|_{2q}^{n} \right]^{T}, \\ \Phi^{n} &= \Phi(n\Delta t) = \left[\varphi(\Delta x, n\Delta t), \varphi(2\Delta x, n\Delta t), \ldots, \varphi(2q\Delta x, n\Delta t)\right]^{T} \\ \bar{\mathbf{F}}(s) &= \left[f(\varphi(\Delta x, s)), f(\varphi(2\Delta x, s)), \ldots, f(\varphi(2q\Delta x, s))\right]^{T} \\ \bar{\mathbf{F}}^{n} &= \left[f(\varphi(\Delta x, nh)), f(\varphi(2\Delta x, nh)), \ldots, f(\varphi(2q\Delta x, nh))\right]^{T}. \end{aligned} \tag{4.13}$$

and $\bar{\mathbf{G}}$, $\bar{\mathbf{G}}^{n}$ are the same as the defination of $\bar{\mathbf{F}}$ and $\bar{\mathbf{F}}^{n}$ for function $g$ instead of $f$. On the other hand, from (3.2),

$$\begin{aligned} L^{n}\varphi &= \Phi^{n+1} - \Phi^{n} - \theta h \mathcal{D}_{j}(P_{j-1})^{-1}\Phi^{n+1} \\ &\quad - (1-\theta) h \mathcal{D}_{j}(P_{j-1})^{-1}\Phi^{n} \\ &\quad - h\,\bar{\mathbf{F}}^{n} - \bar{\mathbf{G}}^{n}\Delta W_{n}. \end{aligned} \tag{4.14}$$

Then

$$\begin{aligned} E\,\| L\varphi|^{n} - L^{n}\varphi\|_{\infty}^{2} < E \Bigg\| \theta \int_{n\Delta t}^{(n+1)\Delta t} & \mathcal{D}_{j}(P_{j-1})^{-1}\Phi^{n+1} - \mathcal{A}\Phi(s)\,ds \\ & - (1-\theta) \int_{n\Delta t}^{(n+1)\Delta t} \mathcal{D}_{j}(P_{j-1})^{-1}\Phi^{n} - \mathcal{A}\Phi(s)\,ds \\ & - \int_{n\Delta t}^{(n+1)\Delta t} \bar{\mathbf{F}}^{n} - \bar{\mathbf{F}}(s)\,ds \\ & - \int_{n\Delta t}^{(n+1)\Delta t} \bar{\mathbf{G}}^{n} - \bar{\mathbf{G}}(s)\,dW(s) \Bigg\|_{\infty}. \end{aligned} \tag{4.15}$$

From Lipschitz property of $f$, boundedness of derivative of $g$, and Theorem 4.3, also from the square property of the Itô integral, we conclude

$$E\,\| L\Phi|^{n} - L^{n}\Phi\|_{\infty}^{2} \to 0, \quad n \to \infty. \tag{4.16}$$

□

In the following we want to investigate stability of the solution (3.4) .

**Lemma 4.5.** *The infinity norm of the matrix* $(P_j)^{-1}$ *is bounded. In fact*

$$\left\| (P_j)^{-1} \right\|_{\infty} \leq 2^{\frac{j}{2}} 3.$$

*Proof.* We rewrite $P_j = 2^{\frac{-j}{2}} \frac{2}{3} (I + Q)$ where $Q = (q_{i\,k})$ is a tridiagonal with

$$q_{i\,k} = \begin{cases} \frac{1}{4}, & i = k \pm 1, \\ 0, & o.w. \end{cases}$$

So $(P_j)^{-1} = 2^{\frac{j}{2}} \frac{3}{2} \sum_{k=0}^{\infty} (-1)^k Q^k$. Since $\|P_j\|_\infty = 2^{\frac{-j}{2}}$ and $\|Q\|_\infty = \frac{1}{2}$, we get

$$\left\|(P_j)^{-1}\right\|_\infty \leq 2^{\frac{j}{2}} \frac{3}{2} \sum_{k=0}^{\infty} \|Q^k\|_\infty = 2^{\frac{j}{2}} \frac{3}{2} \frac{1}{1 - \frac{1}{2}} = 2^{\frac{j}{2}} 3. \tag{4.17}$$

□

**Theorem 4.6.** *The stochastic scheme (3.2) approximating the solution of (1.1) is stable in mean square sense with respect to the $\|\cdot\|_\infty$-norm.*

*Proof.* From (3.2), we have

$$\left(I - \theta h \mathcal{D}_j (P_{j-1})^{-1}\right) \mathbf{u}^{n+1} = \left(I + (1 - \theta h)\, \mathcal{D}_j (P_{j-1})^{-1}\right) \mathbf{u}^n + h\bar{\mathbf{F}}^n + \bar{\mathbf{G}}^n \Delta W_n. \tag{4.18}$$

Choose $h$ small enough such that

$$\theta h \left\|\mathcal{D}_j (P_{j-1})^{-1}\right\|_\infty \leq \frac{1}{2} < 1, \tag{4.19}$$

Then the following matrix is invertible

$$I - \theta h \mathcal{D}_j (P_{j-1})^{-1}. \tag{4.20}$$

Also we obtain

$$\left\|\left(I - \theta h \mathcal{D}_j (P_{j-1})^{-1}\right)^{-1}\right\|_\infty \leq \frac{1}{1 - \left\|\theta h \mathcal{D}_j (P_{j-1})^{-1}\right\|_\infty} \leq 2, \tag{4.21}$$

Using (4.18) we have

$$E \left\|\mathbf{u}^{n+1}\right\|_\infty^2 = E \Bigg\| \left(I - \theta h \mathcal{D}_j (P_{j-1})^{-1}\right)^{-1} \times \left(\left(I + (1 - \theta)\, h \mathcal{D}_j (P_{j-1})^{-1}\right) \mathbf{u}^n + \bar{\mathbf{F}}^n + \bar{\mathbf{G}}^n \Delta W_n\right) \Bigg\|_\infty^2 . \tag{4.22}$$

Due to the properties of $f$ and $g$ we conclude

$$E \left\|\mathbf{u}^{n+1}\right\|_\infty^2 \leq \left\|\left(I - \theta h\, \mathcal{D}_j (P_{j-1})^{-1}\right)^{-1}\right\|_\infty^2 \times \left(1 + (1 - \theta)\, h \left\|\mathcal{D}_j (P_{j-1})^{-1}\right\|_\infty^2 + c\, h\right) E \left\|\mathbf{u}^n\right\|_\infty^2 . \tag{4.23}$$

Finally, using (4.23) and Lemma 4.5, there is a positive constant $C$ such that

$$E \left\|\mathbf{u}^{n+1}\right\|_\infty^2 \leq C\, E \left\|\mathbf{u}^n\right\|_\infty^2 . \tag{4.24}$$

□

According to the Theorems 4.4 and 4.6 and the stochastic version of the Lax-Richtmyer theorem [37, 38], the stochastic method (3.4) is convergent to the solution of the stochastic parabolic partial differential equation (1.1).

## 5. Numerical Results

In this section we consider some SPDEs, and apply our numerical method to approximate their solutions for different resolution levels. For each resulotion level $j$ and choice $\Delta t$, 10,000 runs are performed with different samples of the noise, and the averaged value $E\left|u_j^n(x) - u_{-7}^n(x)\right|$ is calculated. Here we assume the approximate solution in $V_{-7}$ as the exact solution. The examples illustrate the convergence of the numerical method and also they show that the approximate solutions satisfy in the boundary points exactly.

**Example 5.1.** Consider the following SPDE

$$\begin{aligned} du(x,t) &= \left(\frac{\partial^2}{\partial x^2}u(x,t) - u^3(x,t)\right)dt + u(x,t)dW(t), \qquad (5.1)\\ u(x,0) &= 10x(1-x),\\ u(0,t) &= u(1,t) = 0. \end{aligned}$$

From Figure 2 and Table 1 it is obvious that the method is convergence. Furthermore, from Figure 2 we can see that the approximate solutions satisfy boundary conditions exactly.

Figure 2. Approximate solution of Example 5.1 in different resolutions at t=0.3.

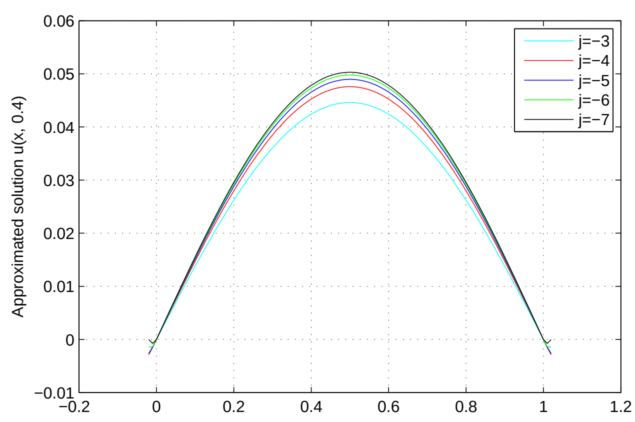
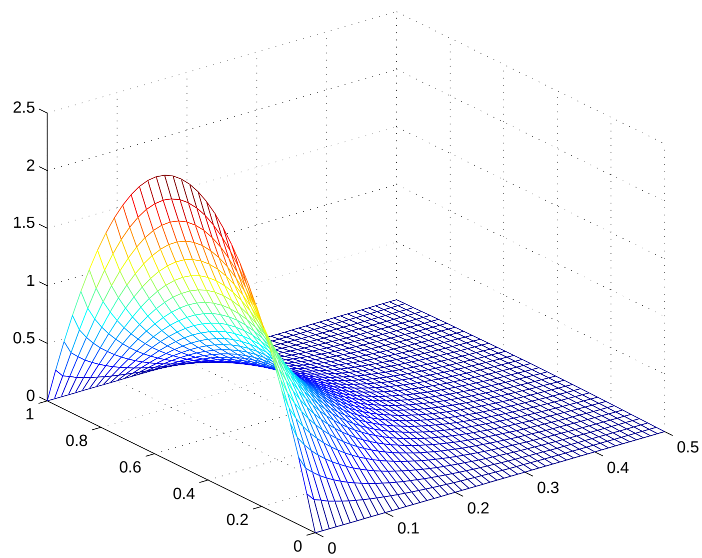


TABLE 1. The errors are the difference between the results in $V_j$ and the results in $V_{-7}$ (denoted by $V_j/V_{-7}$) with $h = 0.01$ at $t = 0.4$, $\theta = 0.6$, for Example 5.1.

| | $V_{-3}/V_{-7}$ | $V_{-4}/V_{-7}$ | $V_{-5}/V_{-7}$ | $V_{-6}/V_{-7}$ |
|---|---|---|---|---|
| Error $L_\infty$ | 0.005706 | 0.002870 | 0.002639 | 0.001446 |
| Error $L_2$ | 0.004047 | 0.001949 | 0.000987 | 0.000413 |

FIGURE 3. Approximate solution of Example 5.1, in $V_{-5}$ with $h = 0.01$.




**Example 5.2.** Consider the following SPDE

$$\begin{aligned} du(x,t) &= \left(\frac{\partial^2}{\partial x^2}u(x,t) + \sin(u(x,t))\right)dt + u^2(x,t)dW(t), \qquad (5.2)\\ u(x,0) &= \sin(x),\\ u(0,t) &= u(1,t) = 0. \end{aligned}$$

From Figure 5 and Table 2 it is obvious that the method is convergent. In Table 3, the numerically approximate errors of the different the multi-scale systems are compared with the full domain $V_{-3}$ system and the full domain $V_{-4}$ system. Here, the multi-scale method is only used at two resolutions but one can apply the method at different resolutions, in different domains.

FIGURE 4. Approximate solution of Example 5.2, in $V_{-5}$ with $h = 0.01$.

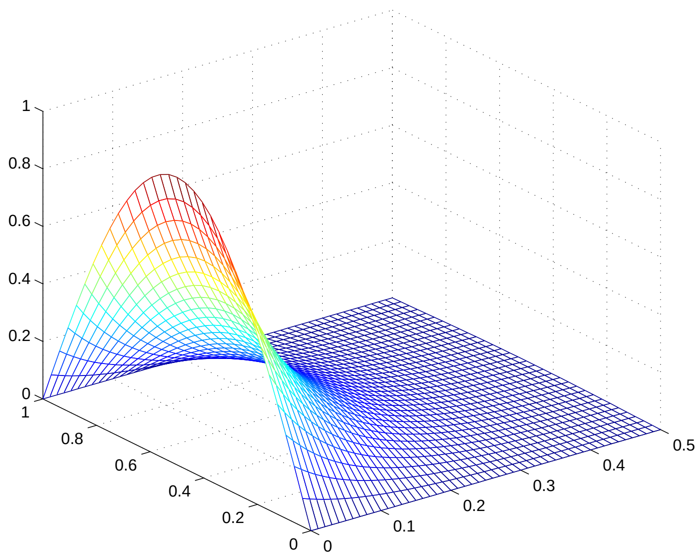
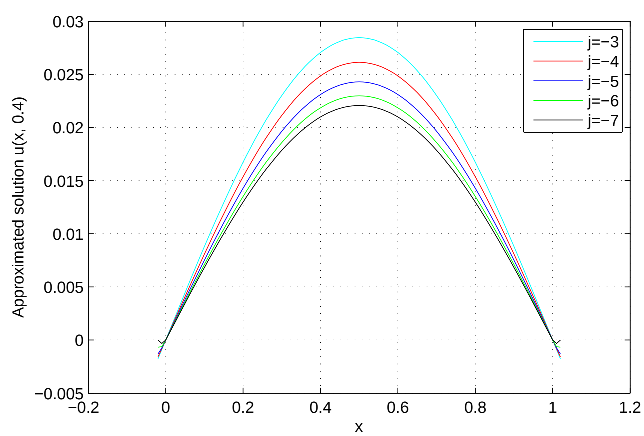



FIGURE 5. Approximate solution of Example 5.2 in different resolutions at $t = 0.3$.




## 6. Conclusion

The multi-scale method based on the B-spline wavelets to solve the SPDE (1.1) was investigated. Our approach has the ability to approximate the solution of stochastic

Table 2. The errors are the difference between the results in $V_j$ and the results in $V_{-7}$ (denoted by $V_j/V_{-7}$ ) with $h = 0.01$ at $t = 0.4$, $\theta = 0.6$, for Example 5.2.

| | $V_{-3}/V_{-7}$ | $V_{-4}/V_{-7}$ | $V_{-5}/V_{-7}$ | $V_{-6}/V_{-7}$ |
|---|---|---|---|---|
| Error $L_\infty$ | 0.006385 | 0.004064 | 0.002219 | 0.000908 |
| Error $L_2$ | 0.004518 | 0.002879 | 0.001576 | 0.000647 |

Table 3. The errors and computational time of the multi-scale systems for Example 5.2, compared with the results of a $V_{-3}$ resolution single system and results of a $V_{-4}$ resolution single system with $h = 0.01$ at $t = 0.4$ and $\theta = 0.6$. The computational time is the average computing time (in seconds) per time run.

| Resulotion | (sub)Domain | Average time | $L_2 Error$ | $L_\infty Error$ |
|---|---|---|---|---|
| $V_{-3\|\Omega}$ | $\Omega = [0, 1]$ | 0.0997 | 0.004518 | 0.006385 |
| $V_{-3\|\Omega}$ , $V_{-4\|\Lambda}$ | $\Lambda = [0.4, 0.6]$ | 0.1073 | 0.004223 | 0.005932 |
| $V_{-3\|\Omega}$ , $V_{-4\|\Lambda}$ | $\Lambda = [0.2, 0.8]$ | 0.1163 | 0.003618 | 0.005033 |
| $V_{-4\|\Omega}$ | $\Omega = [0, 1]$ | 0.1246 | 0.002879 | 0.004064 |

evolution equations at different resolutions, in different domains. In this method, wavlets are reshaped to satisfy boundary conditions (1.1) exactly. Then by reducing the equation into two systems, it was found that the method requires less computational cost than the high resolution results. The convergence of the method has been shown in this article. At last, numerical experiments have confirmed the efficiency of this method.

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