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Discretization of a fractional order ratio-dependent functional response predator-prey model, bifurcation and chaos

Razie Shafeii Lashkarian*
Department of Basic science, Hashtgerd Branch,
Islamic Azad University, Alborz, Iran.
E-mail: r.shafei@alzahra.ac.ir

Dariush Behmardi Sharifabad
Department of Mathematics,
Alzahra university, Tehran, Iran.
E-mail: behmardi@alzahra.ac.ir

Abstract This paper deals with a ratio-dependent functional response predator-prey model with a fractional order derivative. The ratio-dependent models are very interesting, since they expose neither the paradox of enrichment nor the biological control paradox. We study the local stability of equilibria of the original system and its discretized counterpart. We show that the discretized system, which is not more of fractional order, exhibits much richer dynamical behavior than its corresponding fractional order model. Specially, in the discretized system, many types of bifurcations (transcritical, flip, Neimark-Sacker) and chaos may happen, however, the local analysis of the fractional-order counterpart, only deals with the stability (unstability) of the equilibria. Finally, some numerical simulations are performed by MATLAB, to support our analytic results.

Keywords. Ratio-dependent functional response model, Fractional derivative, Discretization, Bifurcation, Chaos.

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1. Introduction

Mathematical model for predator-prey interaction is studied originally by Lotka [26] and Volterra [41] as

$$\begin{cases} \dot{x} &= \gamma x - \alpha xy, \\ \dot{y} &= \beta xy - \delta y, \end{cases} \tag{1.1}$$

where x and y are the numbers of prey and predator, respectively. In this classical model the positive parameters γ, α, β, and δ stand for growth rate of prey, predation rate, conversion rate to change prey biomass into predator reproduction and death rate of predator, respectively.

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* Corresponding author.

More generally the predator-prey model is the following system

$$\begin{cases} \dot{x} &=& rx(1-\frac{x}{k}) - F(x,y), \\ \dot{y} &=& \beta\, F(x,y) - \delta y. \end{cases} \tag{1.2}$$

The positive parameters r, k, β and δ represent the prey intrinsic growth rate, the environmental carrying capacity, conversion rate to change prey biomass into predator reproduction and predator's death rate, respectively. The function $F(x, y)$ describes predation and is called the *functional response*.

Traditionally, $F(x, y)$ is assumed to be a function of the prey population x, that is, $F(x, y) = F(x)$, where $F(x)$ is a Holling type (II) function [34]. It is shown that a predator-prey model with the prey-dependent functional response, may expose the so-called *paradox of enrichment* or the *biological control paradox* [3, 11, 13, 37].

The following ratio-dependent functional response predator-prey model has been suggested by Arditi and Ginzburg in [4]

$$\begin{cases} \dot{x} &=& rx(1-\frac{x}{k}) - \frac{axy}{abx+y}, \\ \\ \dot{y} &=& y\left(-d + \frac{\eta a x}{abx+y}\right). \end{cases} \tag{1.3}$$

Here $a > 0$ and $b > 0$ are predator's attack rate and handling time, respectively.

System (1.3) exposes neither the paradox of enrichment nor the biological control paradox [5, 17, 18, 39]. One can simplify (1.3), by rescaling

$$t \to rt, \quad x \to x/k \quad y \to y/abk.$$

Therefore the ratio-dependent functional response predator-prey model is written as

$$\begin{cases} \dot{x} &=& x(1-x) - \frac{\alpha xy}{x+y}, \\ \\ \dot{y} &=& -\delta y + \frac{\beta xy}{x+y}, \end{cases} \tag{1.4}$$

where

$$\alpha = \frac{a}{r}, \beta = \frac{\eta}{br}, \delta = \frac{d}{r}. \tag{1.5}$$

Note that once r varies in system (1.3), then the values of α, β and δ change in system (1.4). So in numerical simulations we plot the bifurcation diagrams in (r, x) plane.

On the other hand, the fractional derivative provides an excellent tool for describing memory and the hereditary properties of various materials and processes [14, 30]. In other words, the fractional derivative can adequately represent some long-term memory and non-local effects [27]. In this calculus, a Caputo derivative implies a memory effect via convolution between an integer-order derivative and power of time [28]. Fractional differential equations (FDE) also help to reduce errors that arise from neglected parameters in modeling of real-life phenomena [36]. The relation between memory and fractional mathematics is pointed out in [2]. Fractional calculus arose originally from the generalization of the ordinary integrals and derivatives. For example, in [21] the authors show that the fractional calculus is a combination of stochastic processes, probability, integro-differential equations, integral transforms, special functions, numerical analysis, etc. Consequently, considerable attention has been given to the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest, see for example [7, 19, 20, 29, 33, 35, 38].

Furthermore, studying chaos in fractional order dynamical systems is an interesting topic as well [15, 16, 32]. Chaos is found in some autonomous fractional-order systems with orders less than three, unlike their integer-order counterparts (according to the Poincaré-Bendixon theorem).

However, at our best knowledge, the bifurcations of fractional order systems, is not defined yet and to study it, one should use a discretization algorithm. In this method one should interchange the fractional order system of order θ, with a parametric ordinary discrete dynamical system in the parameter θ, and then one study the bifurcations of the new system.

In this paper we consider a fractional order ratio-dependent functional response predator- prey model and its discretization. We show that the discretized fractional-order system produces a much richer dynamic (bifurcations and chaos) than the system's counterpart. At our best knowledge, this is the first study on the dynamic of a discretized fractional order ratio-dependent response predator- prey model.

The organization of the paper is as follows. In Section 2, after some fractional calculus preliminaries, we review some results for relations between properties of equilibria and possibility of existence of chaos in a fractional order system. In Section 3, we determine the equilibria of the model and then the discretization process of the system is given. In Section 4, we study the local stability of the equilibria, and we investigate the dynamics of the discretized model. Section 5 is devoted to some numerical simulations and bifurcation diagrams, to support the analytic results.

2. Preliminaries

Definition 2.1. Let function $f \in L^1(\mathbb{R}^+)$. The Riemann-Liouville fractional integral of order $\theta \in \mathbb{R}^+$, is defined as

$$I^\theta f(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-\tau)^{\theta-1} f(\tau) d\tau, \tag{2.1}$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. The Caputo fractional derivative of order $\theta \in (n-1, n), n \in \mathbb{N}$, of $f(t), t > 0$, is defined by

$$\begin{aligned} D^\theta f(t) &= \tfrac{1}{\Gamma(n-\theta)} \left(\tfrac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\theta-1} f(\tau) d\tau \\ &= \tfrac{1}{\Gamma(n-\theta)} \int_0^t \tfrac{f^{(n)}(\tau)}{(t-\tau)^{n-\theta-1}} d\tau, \end{aligned} \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Fractional order differential equations are, as stable as their integer order counterpart, because systems with memory are typically more stable than their memoryless counterpart [2]. Consider the nonlinear autonomous fractional order system

$$D^\theta x(t) = f(x), \tag{2.3}$$

where $\theta \in [0, 1)$ and $x \in \mathbb{R}^n$. The equilibria of system (2.3)are given by solving equation $f(x) = 0$. These points are locally asymptotically stable if all the eigenvalues of the Jacobian matrix $A = \frac{\partial f}{\partial x}$ at the equilibrium points satisfy the following condition [31]:

$$|\arg(\lambda_i)| > \theta\pi/2, \ i = 1, 2, \ldots, \tag{2.4}$$

where $\lambda_i, i = 1, \ldots, n$ are the eigenvalues of the Jacobian matrix A, through the equilibrium point of system (2.3).

The Lyapunov exponent of a dynamical system, is a quantity that characterizes the rate of separation of infinitesimally close trajectories. Quantitatively, two trajectories in phase space with initial separation δX_0, diverge (provided that the divergence can be treated within the linearized approximation) at a rate given by

$$|\delta X(t)| \approx e^{\lambda t}|\delta X_0|,$$

where λ is the Lyapunov exponent.

Let $X(t)$ be the trajectory of the following n-dimensional linear ordinary differential equation with constant coefficients

$$\dot{X} = AX + f^t, \tag{2.5}$$

and if the constant coefficient matrix A has n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the real parts of the n different eigenvalues are naturally the Lyapunov exponents. However, if the dynamical system is not given by (2.5), for instance, if the dynamical system is a nonlinear polynomial autonomous system, the conception on Lyapunov exponents becomes complicated [12].

Also the rate of separation can be different for different orientations of initial separation vector. Thus, there is a spectrum of Lyapunov exponents, equal in number to the dimensionality of the phase space. It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. A positive MLE is usually taken as an indication that the system is chaotic.

The Maximal Lyapunov exponent can be defined as follows:

$$\lambda = \lim_{t\to\infty} \lim_{\delta X_0 \to 0} \frac{1}{t} \ln \frac{|\delta X(t)|}{|\delta X_0|},$$

The limit $\delta X_0 \to 0$ ensures the validity of the linear approximation at any time [8].

For discrete time system $x_{n+1} = f(x_n)$, for an orbit starting with x_0 this translates into:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f^{'}(x_i)|.$$

For a dynamical system with evolution equation f^t in an n- dimensional phase space, the spectrum of Lyapunov exponents

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\},$$

in general, depends on the starting point x_0. However, we will usually be interested in the attractor (or attractors) of a dynamical system, and there will normally be one set of exponents associated with each attractor. The choice of starting point may determine which attractor the system ends up on, if there is more than one. The Lyapunov exponents describe the behavior of vectors in the tangent space of the phase space and are defined from the Jacobian matrix

$$J^t(x_0) = \left.\frac{df^t(x)}{dx}\right|_{x_0},$$

The J^t matrix describes how a small change at the point x_0 propagates to the final point $f^t(x_0)$. The limit

$$L(x_0) = \lim_{t\to\infty}(J^t \cdot \text{Transpose}(J^t))^{1/2t}$$

defines a matrix $L(x_0)$ (the conditions for the existence of the limit are given by the Oseledec theorem). If $\Lambda_i(x_0)$ are the eigenvalues of $L(x_0)$, then the Lyapunov exponents λ_i are defined by

$$\lambda_i(x_0) = \ln \Lambda_i(x_0),$$

Based on the experience of the linear system (2.5) and some plausible thinking, for a dissipative system, as criterions, it is proposed in the reference [8] that, if the attractor reduces to

- **a:** stable fixed point, all the exponents are negative;
- **b:** limit cycle, an exponent is zero and the remaining ones are all negative;
- **c:** k- dimensional stable torus, the first k Lyapunov exponents vanish and the remaining ones are negative;
- **d:** for strange attractor generated by a chaotic dynamics at least one exponent is positive.

The above mentioned definition on Lyapunov exponents and proposed criterions about the relations between the characteristic of LE and the properties of the attractors are widely used.

The set of Lyapunov exponents will be the same for almost all starting points of an ergodic component of the dynamical system.

Lyapunov exponent for time varying linearization: To introduce Lyapunov exponent let us consider a fundamental matrix $X(t)$ (e.g., for linearization along stationary solution x_0 in continuous system the fundamental matrix is $\exp\left(\left.\frac{df^t(x)}{dx}\right|_{x_0} t\right)$ consisting of the linear independent solutions of the first approximation system. The singular values $\{\alpha_j\big(X(t)\big)\}_1^n$ of the matrix $X(t)$ are the square roots of the eigenvalues of the matrix $X(t)^*X(t)$. The largest Lyapunov exponent λ_{max} is as follows [40]

$$\lambda_{max} = \max_j \limsup_{t\to\infty} \frac{1}{t} \ln \alpha_j\big(X(t)\big).$$

Lyapunov proved that if the system of the first approximation is regular (e.g., all systems with constant and periodic coefficients are regular) and its largest Lyapunov exponent is negative, then the solution of the original system is asymptotically Lyapunov stable. Later, it was stated by Perron that the requirement of regularity of the first approximation is substantial.

In 1930 Perron constructed an example of a second order system, where the first approximation has negative Lyapunov exponents along a zero solution of the original system but, at the same time, this zero solution of the original nonlinear system is Lyapunov unstable. Furthermore, in a certain neighborhood of this zero solution almost all solutions of original system have positive Lyapunov exponents. Also, it is possible to construct a reverse example in which the first approximation has positive Lyapunov exponents along a zero solution of the original system but, at the same time, this zero solution of original nonlinear system is Lyapunov stable [24, 25]. The effect of sign inversion of Lyapunov exponents of solutions of the original system and the system of first approximation with the same initial data was subsequently called the Perron effect [24, 25].

Perron's counterexample shows that a negative largest Lyapunov exponent does not, in general, indicate stability, and that a positive largest Lyapunov exponent does not, in general, indicate chaos.

Therefore, time varying linearization requires additional justification [25].

In [9] the stability of a linear fractional differential equation is characterized by its fractional Lyapunov spectrum.

In this paper using MATLAB, the Maximal Lyapunov exponents diagram of the discretized system, which is an ordinary discrete dynamical system is plotted and we show that the discrete system undergoes chaos.

3. Equilibria of the ratio dependent functional response model

Consider the following fractional order ratio-dependent functional response predator prey model

$$\begin{cases} D^{\theta}x(t) &= x(1-x) - \frac{\alpha x y}{x+y}, \\ D^{\theta}y(t) &= y\left(-\delta + \frac{\beta x}{x+y}\right), \end{cases} \tag{3.1}$$

with the initial conditions

$$x(0) > 0\,, y(0) > 0.$$

Denote N_x, N_y respectively, the prey and predator nullclines. That is

$$N_x = \{(x,y):\ x = 0\} \cup \left\{(x,y) : y = \frac{x(x-1)}{1-\alpha-x}\right\}, \tag{3.2}$$

$$N_y = \{(x,y):\ y = 0\} \cup \left\{(x,y) : x = \frac{\delta y}{\beta-\delta}\right\}. \tag{3.3}$$

As we are interested in biologically feasible equilibria, we only consider the points in $N_x \cap N_y \cap \mathbb{R}^2_+$, where $\mathbb{R}^2_+$ is the first quadrant. The system (3.1) has two boundary equilibria $O = (0,0)$, $E = (1,0)$. Furthermore $N_x \cap N_y$ has another common element $E^* = (x^*, y^*)$ given by

$$\begin{aligned} x^* &= \tfrac{\beta-\alpha\beta+\alpha\delta}{\beta}, \\ y^* = \tfrac{\beta-\delta}{\delta}x^* &= \tfrac{x^*(x^*-1)}{1-\alpha-x^*} = \tfrac{\beta^2-\alpha\beta^2+2\alpha\delta\beta-\beta\delta-\alpha\delta^2}{\beta\delta}. \end{aligned} \tag{3.4}$$

Thus the system has an interior equilibrium E^* in the first quadrant when

$$\beta - \alpha\beta + \alpha\delta > 0, \ \beta > \delta. \tag{3.5}$$

Note that if $0 < \alpha < 1$, then the condition $\beta > \delta$ implies the condition $\beta-\alpha\beta+\alpha\delta > 0$.

Remark 3.1. The condition $\beta > \delta$ grants a coexistence equilibrium, the predator growth parameter β must be sufficiently larger than the predator death parameter δ.

The above details are summarized in the following theorem.

Theorem 3.2. *The boundary equilibria of the system (3.1) in the first quadrant are the coextinction point $E_0 = (0,0)$ and the predator free point $E_1 = (1,0)$. If $\beta > \delta$ and $\beta - \alpha\beta + \alpha\delta > 0$, then the model has a coexistence equilibrium $E^* = (x^*, y^*)$ defined by (3.4).*

Now, the discretization process of the fractional order ratio dependent functional response predator prey system is as follows.

Let $x(0) = x_0, y(0) = y_0$ be the initial conditions of system (3.1). So, the discretization of system (3.1) with piecewise constant arguments is given by

$$\begin{cases} D^\theta x(t) &= x\left(\left[\frac{t}{s}\right]s\right)\left(1 - x\left(\left[\frac{t}{s}\right]s\right)\right) - \frac{\alpha x\left(\left[\frac{t}{s}\right]s\right)y\left(\left[\frac{t}{s}\right]s\right)}{x\left(\left[\frac{t}{s}\right]s\right)+y\left(\left[\frac{t}{s}\right]s\right)}, \\ \\ D^\theta y(t) &= y\left(\left[\frac{t}{s}\right]s\right)\left(-\delta + \frac{\beta x\left(\left[\frac{t}{s}\right]s\right)}{x\left(\left[\frac{t}{s}\right]s\right)+y\left(\left[\frac{t}{s}\right]s\right)}\right), \end{cases} \tag{3.6}$$

where $\left[\frac{t}{s}\right]$ is the value of $\frac{t}{s}$ rounded down to the nearest integer.

First, Assume $t \in [0, s)$ so $\left[\frac{t}{s}\right]s = 0$, hence we have

$$\begin{cases} D^\theta x(t) &= x_0(1 - x_0) - \frac{\alpha x_0 y_0}{x_0+y_0}, \\ \\ D^\theta y(t) &= y_0\left(-\delta + \frac{\beta x_0}{x_0+y_0}\right), \end{cases} \tag{3.7}$$

and the solutions of (3.7) are

$$\begin{cases} x_1(t) &= x_0 + I^\theta\left(x_0(1 - x_0) - \frac{\alpha x_0 y_0}{x_0+y_0}\right), \\ \\ y_1(t) &= y_0 + I^\theta\left(y_0\left(-\delta + \frac{\beta x_0}{x_0+y_0}\right)\right), \end{cases} \tag{3.8}$$

that is equivalent to

$$\begin{cases} x_1(t) &= x_0 + \frac{t^\theta}{\theta\Gamma(\theta)} \left(x_0(1-x_0) - \frac{\alpha x_0 y_0}{x_0+y_0} \right), \\ y_1(t) &= y_0 + \frac{t^\theta}{\theta\Gamma(\theta)} \left(y_0 \left(-\delta + \frac{\beta x_0}{x_0+y_0} \right) \right). \end{cases} \tag{3.9}$$

Second, let $t \in [s, 2s)$ so $\left[\frac{t}{s}\right] s = s$, hence we have

$$\begin{cases} D^\theta x(t) &= x_1(1-x_1) - \frac{\alpha x_1 y_1}{x_1+y_1}, \\ D^\theta y(t) &= y_1 \left(-\delta + \frac{\beta x_1}{x_1+y_1} \right), \end{cases} \tag{3.10}$$

with the following solution

$$\begin{cases} x_2(t) &= x_1(s) + I_s^\theta \left(x_1(s)(1-x_1(s)) - \frac{\alpha x_1(s) y_1(s)}{x_1(s)+y_1(s)} \right), \\ y_2(t) &= y_1(s) + I_s^\theta \left(y_1(s) \left(-\delta + \frac{\beta x_1(s)}{x_1(s)+y_1(s)} \right) \right), \end{cases} \tag{3.11}$$

where $I_s^\theta \equiv \frac{1}{\Gamma(\theta)} \int_s^t (t-\tau)^{\theta-1} d\tau, \theta > 0$. So the solution of (3.10) is as follows

$$\begin{cases} x_2(t) &= x_1(s) + \frac{(t-s)^\theta}{\theta\Gamma(\theta)} \left(x_1(s)(1-x_1(s)) - \frac{\alpha x_1(s) y_1(s)}{x_1(s)+y_1(s)} \right), \\ y_2(t) &= y_1(s) + \frac{(t-s)^\theta}{\theta\Gamma(\theta)} \left(y_1(s) \left(-\delta + \frac{\beta x_1(s)}{x_1(s)+y_1(s)} \right) \right). \end{cases} \tag{3.12}$$

Thus, by repeating the discretization process n times, we obtain

$$\begin{cases} x_{n+1}(t) &= x_n(ns) + \\ & \quad \frac{(t-ns)^\theta}{\theta\Gamma(\theta)} \left(x_n(ns)(1-x_n(ns)) - \frac{\alpha x_n(ns) y_n(ns)}{x_n(ns)+y_n(ns)} \right), \\ y_{n+1}(t) &= y_n(ns) + \\ & \quad \frac{(t-ns)^\theta}{\theta\Gamma(\theta)} \left(y_n(ns) \left(-\delta + \frac{\beta x_n(ns)}{x_n(ns)+y_n(ns)} \right) \right). \end{cases} \tag{3.13}$$

where $t \in [ns, (n+1)s)$. For $t \to (n+1)s$, system (3.13) is reduced to

$$\begin{cases} x_{n+1}(t) &= x_n + \frac{s^\theta}{\theta\Gamma(\theta)} \left(x_n(1-x_n) - \frac{\alpha x_n y_n}{x_n+y_n} \right), \\ y_{n+1}(t) &= y_n + \frac{s^\theta}{\theta\Gamma(\theta)} \left(y_n \left(-\delta + \frac{\beta x_n}{x_n+y_n} \right) \right). \end{cases} \tag{3.14}$$

Remark 3.3. Note that if $\alpha \to 1$ in (3.14), we obtain the Euler discretization of predator prey model.

4. Local stability of the equilibria and dynamic of discretiztion

In this section we study the local behavior of the model around its equilibria. The general Jacobian matrix of system (3.1) evaluated at an arbitrary point (x, y) equals to

$$J = \begin{pmatrix} 1 - 2x - \frac{\alpha y^2}{(x+y)^2} & -\frac{\alpha x^2}{(x+y)^2} \\ \frac{\beta y^2}{(x+y)^2} & -\delta + \frac{\beta x^2}{(x+y)^2} \end{pmatrix}. \tag{4.1}$$

We shall point out here that although $(0, 0)$ is defined for system (3.1), it cannot be linearized at. So, local stability of $(0, 0)$ can not be studied. Indeed, this singularity at the origin, while causes much difficulty in our analysis of the system, contributes significantly to the richness of dynamics of the model. A complete parametric analysis of stability properties and dynamic around the complicated equilibrium $(0, 0)$ for the model is done in [6].

The following theorem is an immediate result of expression (4.1).

Theorem 4.1. *The equilibrium point E_1 is locally asymptotically stable if $\beta < \delta$ and saddle point if $\beta > \delta$.*

Proof. The characteristic equation corresponding to the equilibrium E_1 has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \beta - \delta$. If $\beta < \delta$, then both of eigenvalues $\lambda_{1,2}$ are negative. Hence, the equilibrium point E_1 is locally asymptotically stable if $\beta < \delta$ and saddle point if $\beta > \delta$. □

Now we study the linearized system at the interior equilibrium $E^* = (x^*, y^*)$.

Theorem 4.2. *Let*

$$M = \frac{(\beta - \delta)(-\alpha\delta\beta^2 + \alpha\delta^2\beta + \beta^2\delta)}{\beta^3},$$

$$N = \frac{-\beta^2 + \alpha(\beta^2 - \delta^2) - \beta\delta(\beta - \delta)}{\beta^2}.$$

If $y^* \leq T$, then we have:

(1) *if $N^2 - 4M \geq 0$, $N < 0$ then E^* is asymptotically stable;*
(2) *if $N^2 - 4M > 0$ and $N > 0$, then E^* is unstable;*
(3) *if $N^2 - 4M < 0$ and $0 < \theta < 1$, then E^* is asymptotically stable.*

Proof. The Jacobian matrix of system (3.1) at E^* is

$$J = \begin{pmatrix} \frac{-\beta^2 + \alpha(\beta^2 - \delta^2)}{\beta^2} & -\frac{\alpha\delta^2}{\beta^2} \\ \frac{(\beta-\delta)^2}{\beta} & \frac{\delta(\delta-\beta)}{\beta} \end{pmatrix}.$$

The associated characteristic equation is

$$\lambda^2 - N\lambda + M = 0.$$

Thus the eigenvalues of the Jacobian matrix are

$$\lambda_{1,2} = \frac{N \pm \sqrt{N^2 - 4M}}{2},$$

and the result is immediately obtained. Note that if $N^2 - 4M < 0$, the relation $|arg(\lambda_{1,2})| = \arctan(\frac{\sqrt{4M-N^2}}{N}) > \frac{\theta\pi}{2}$ is satisfied for all $0 < \theta < 1$. □

In the following, we investigate the dynamics of the discretized fractional-order model (3.14), which is a discrete ordinary dynamical system . The dynamical behaviors of model (3.14) is determined by five parameters $\alpha, \beta, \delta, s, \theta$. Then we discuss the stability and bifurcation of fixed points of model (3.14). The Jacobian matrix J of model (3.14) at any equilibrium (x, y) is

$$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = 1 + \frac{s^\theta}{\theta\Gamma(\theta)}\left(1 - 2x - \frac{\alpha y^2}{(x+y)^2}\right),$$

$$B = -\frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{\alpha x^2}{(x+y)^2}\right),$$

$$C = \frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{\beta y^2}{(x+y)^2}\right),$$

$$D = 1 + \frac{s^\theta}{\theta\Gamma(\theta)}\left(-\delta + \frac{\beta x^2}{(x+y)^2}\right).$$

The characteristic equation of the Jacobian matrix can be written as

$$\lambda^2 - Tr\lambda + Det = 0, \tag{4.2}$$

where Tr is the trace and Det is the determinant of the Jacobian matrix J. Hence, we can consider one of the following cases for the system (3.14):

(1) dissipative dynamical system, if $Det < 1$;
(2) conservative dynamical system, if $Det = 1$;
(3) otherwise un-dissipated dynamical system.

In order to study stability analysis of the fixed points of the model (3.14), we recall the following Lemma that can be easily proved by using the relation between roots and coefficients of the characteristic Eq. (4.2).

Lemma 4.3. [1] *Let $F(\lambda) = \lambda^2 - Tr\lambda + Det$. Suppose that $F(1) > 0$, λ_1 and λ_2 are the two roots of $F(\lambda)$. Then*

(i): $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $Det < 1$.
(ii): $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2 < 1|$*) if and only if* $F(-1) < 0$.
(iii): $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $Det > 1$.
(iv): $\lambda_1 = -1$ *and* $\lambda_2 = 1$ *if and only if* $F(-1) = 0$ *and* $Tr = 0, 2$.
(v): λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2|$ *if and only if* $Tr^2 - 4Det < 0$ *and* $Det = 1$.

Let λ_1 and λ_2 be the two roots of Eq.(4.2), which are called eigenvalues of equilibrium (x, y), we recall that

(i): An equilibrium (x, y) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.
(ii): An equilibrium (x, y) is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$.
(iii): An equilibrium (x, y) is called a saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$.
(iv): An equilibrium (x, y) is called a non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma 4.4. [23] *A sink is locally asymptotically stable and a source is locally unstable.*

The necessary and sufficient conditions ensuring that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ are [10]

(i): $1 - TrJ + DetJ > 0$;
(ii): $1 + TrJ + DetJ > 0$;
(iii): $DetJ < 1$.

Once only one of these conditions doesn't satisfied, while the other being simultaneously fulfilled, bifurcations occur as follows:

If condition (i) is not satisfied and conditions (ii) and (iii) satisfied, that is a real eigenvalue passes through $+1$, a fold or transcritical bifurcation occurs. This local bifurcation leads to change the stability between two different equilibria; If condition (ii) is not satisfied and conditions (i) and (iii) satisfied, that is a real eigenvalue passes through -1, system exposes a flip bifurcation. This local bifurcation implicates the appearance of a limit cycle; If condition (iii) is not satisfied and conditions (i) and (ii) satisfied, that is the modulus of a complex eigenvalue pair that passes through 1, a Neimark-Sacker bifurcation arises. This local bifurcation implies the birth of an invariant curve in the phase plane. Neimark-Sacker bifurcation is considered to be equivalent to the Hopf bifurcation in continuous time and is indeed the major instrument to prove the existence of quasi-periodic orbits for the map. In [10] the author presents a complete study of the three main types of bifurcations, for two-dimensional maps.

Theorem 4.5. *If* $0 < s < \sqrt[\theta]{2\theta\Gamma(\theta)}$, *then the equilibrium* $E_0 = (0, 0)$ *is a saddle point. If* $s > \sqrt[\theta]{2\theta\Gamma(\theta)}$, *then* E_0 *is a source and if* $s = \sqrt[\theta]{2\theta\Gamma(\theta)}$, *then* E_0 *is non-hyperbolic.*

Proof. The Jacobian matrix J at E_0 is

$$J(E_0) = \begin{pmatrix} 1 + \frac{s^\theta}{\theta\Gamma(\theta)} & 0 \\ 0 & 1 - \delta\frac{s^\theta}{\theta\Gamma(\theta)} \end{pmatrix}.$$

Hence, the eigenvalues are $\lambda_1 = 1 + \frac{s^\theta}{\theta\Gamma(\theta)}$ and $\lambda_2 = 1 - \delta\frac{s^\theta}{\theta\Gamma(\theta)}$. Since $\frac{s^\theta}{\theta\Gamma(\theta)} > 0$, then $|\lambda_1| > 1$. Thus the equilibrium E_0 is saddle point if $0 < s < \sqrt[\theta]{2\theta\Gamma(\theta)}$, source if $s > \sqrt[\theta]{2\theta\Gamma(\theta)}$ and non-hyperbolic if $s = \sqrt[\theta]{2\theta\Gamma(\theta)}$. □

Theorem 4.6. *There are at least four different topological types of* $E_1 = (1, 0)$ *for all parameters values*

(i): *E_1 is a sink if and only if $\beta-\delta<0$ and $0<s<\min\{\sqrt[\theta]{2\theta\Gamma(\theta)}, \sqrt[\theta]{\frac{2\theta\Gamma(\theta)}{\delta-\beta}}\}$.*
(ii): *E_1 is a source if and only if $\beta-\delta>0$ and $s>\sqrt[\theta]{2\theta\Gamma(\theta)}$.*
(iii): *E_1 is non-hyperbolic if and only if $\beta-\delta<0$ and $s=\sqrt[\theta]{2\theta\Gamma(\theta)}$ or $s=\sqrt[\theta]{\frac{2\theta\Gamma(\theta)}{\delta-\beta}}$.*
(iv): *E_1 is a saddle for the other values of parameters except those values in (i)-(iii).*

Proof. The Jacobian matrix J at E_1 is given by

$$J(E_1)=\begin{pmatrix} 1-\frac{s^\theta}{\theta\Gamma(\theta)} & -\frac{\alpha s^\theta}{\theta\Gamma(\theta)} \\ 0 & 1+(\beta-\delta)\frac{s^\theta}{\theta\Gamma(\theta)} \end{pmatrix},$$

Hence, the eigenvalues are $\lambda_1=1-\frac{s^\theta}{\theta\Gamma(\theta)}$ and $\lambda_2=1+(\beta-\delta)\frac{s^\theta}{\theta\Gamma(\theta)}$. Note that

$$\begin{aligned} |\lambda_1|<1 &\iff \\ -1<1-\frac{s^\theta}{\theta\Gamma(\theta)}<1 &\iff \\ 0<\frac{s^\theta}{\theta\Gamma(\theta)}<2 &\iff \\ 0<s<\sqrt[\theta]{2\theta\Gamma(\theta)}. \end{aligned}$$

If $\beta-\delta<0$ then we have

$$\begin{aligned} |\lambda_2|<1 &\iff \\ -1<1+(\beta-\delta)\frac{s^\theta}{\theta\Gamma(\theta)}<1 &\iff \\ 0<(\delta-\beta)\frac{s^\theta}{\theta\Gamma(\theta)}<2 &\iff \\ 0<s<\sqrt[\theta]{\frac{2\theta\Gamma(\theta)}{\delta-\beta}}. \end{aligned}$$

Note that if $\beta-\delta>0$, then $|\lambda_2|>1$. □

In order to discuss the stability of the fixed point E^* of system (3.14), the following Lemma would be useful,

Lemma 4.7. [22] *The necessary and sufficient conditions for both eigenvalues of Jacobian matrix to have magnitude less than 1 are the following conditions:*

$$|TrJ|<1+DetJ<2. \tag{4.3}$$

By Lemma 4.7, instead of calculating the eigenvalues of J, one can check the so-called Jury conditions (4.3).

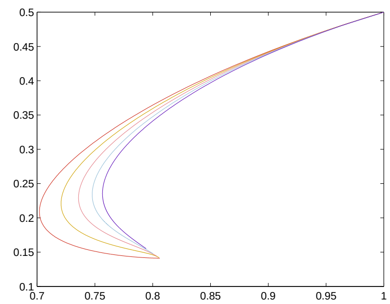
Theorem 4.8. *The positive fixed point $E^*=(x^*,y^*)$ of the system (3.14) is asymptotically stable if*

$$\frac{(-\beta^2+\alpha(\beta^2-\delta^2))(\delta(\delta-\beta))+(\beta-\delta^2)(\alpha\delta^2)}{\beta^3}>0,$$

and

$$\begin{aligned} &\frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{-\beta^2+\alpha\beta^2-\alpha\delta^2}{\beta^2}\right) \\ +\ &\left(\frac{s^\theta}{\theta\Gamma(\theta)}\right)^2\left(\frac{(-\beta^2+\alpha(\beta^2-\delta^2))(\delta(\delta-\beta))+(\beta-\delta^2)(\alpha\delta^2)}{\beta^3}\right)<3. \end{aligned}$$

FIGURE 1. $\alpha = 1.3, \beta = 0.8, \delta = 0.4, \theta = 0.6, 0.7, 0.8, 0.9, 1.$



Proof. The Jacobian matrix evaluated at the equilibrium E^* is of the form

$$J(E^*) = \begin{pmatrix} 1 + \frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{-\beta^2+\alpha(\beta^2-\delta^2)}{\beta^2}\right) & -\frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{\alpha\delta^2}{\beta^2}\right) \\ \frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{(\beta-\delta)^2}{\beta}\right) & 1 + \frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{\delta(\delta-\beta)}{\beta}\right) \end{pmatrix}.$$

The trace and determinant of the Jacobian matrix $J(E^*)$ are given by

$$Tr(J(E^*)) = 2 + \frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{-\beta^2+\alpha\beta^2-\alpha\delta^2}{\beta^2}\right), \tag{4.4}$$

$$\begin{aligned} Det(J(E^*)) &= 1 + \frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{-\beta^2+\alpha\beta^2-\alpha\delta^2}{\beta^2}\right) \\ &+ \left(\frac{s^\theta}{\theta\Gamma(\theta)}\right)^2\left(\frac{(-\beta^2+\alpha(\beta^2-\delta^2))(\delta(\delta-\beta))+(\beta-\delta^2)(\alpha\delta^2)}{\beta^3}\right). \end{aligned} \tag{4.5}$$

By Lemma 3, the coexistence equilibrium $E^*$is locally asymptotically stable whenever

$$\frac{(-\beta^2+\alpha(\beta^2-\delta^2))(\delta(\delta-\beta))+(\beta-\delta^2)(\alpha\delta^2)}{\beta^3} > 0,$$

and

$$\begin{aligned} &\frac{s^\theta}{\theta\Gamma(\theta)}\left(\frac{-\beta^2+\alpha\beta^2-\alpha\delta^2}{\beta^2}\right) \\ + &\left(\frac{s^\theta}{\theta\Gamma(\theta)}\right)^2\left(\frac{(-\beta^2+\alpha(\beta^2-\delta^2))(\delta(\delta-\beta))+(\beta-\delta^2)(\alpha\delta^2)}{\beta^3}\right) < 3. \end{aligned}$$

□

5. NUMERICAL SIMULATIONS

In this section we give some numerical simulations of the model (3.1), by use of MATLAB. In Figure 1, the phase portrait of the system is shown with the parameter values $\alpha = 1.3$, $\beta = 0.8$, $\delta = 0.4$. The initial state of the system (1.3) is $(0.1, 0.1)$.

FIGURE 2. Bifurcation diagram $a = 1.3, \eta = 0.8, d = 0.4, b = 1, \theta = 0.9, s = 0.01$.

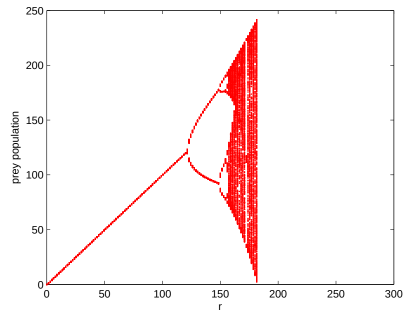
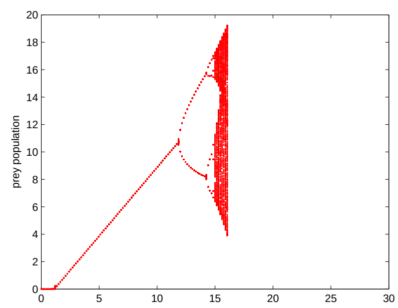


FIGURE 3. Bifurcation diagram $a = 1.3, \eta = 0.8, d = 0.4, b = 1, \theta = 0.9, s = 0.15$.



Then the bifurcation diagram of model (1.3) in (r, x) plane is shown in Figures 2-4 for $\theta = 0.9$ and $s = 0.01, s = 0.15, s = 0.2$ respectively. The Maximal Lyapunov exponent (MLE) corresponding to Figure 2 is given in Figure 5. One knows that the Maximal Lyapunov exponent determines the predictability for the dynamical system. A positive MLE usually implies that the system is chaotic. Figure 3 shows that for r less than roughly 1.3, all points are assembled at zero. Zero is an attractor for r less than 1.3. for r between 1.3 and 11.5, we still have one point attractors, but the attracted value of x increases as r increases, at least to $r = 11.5$. Bifurcations occur at $r = 11.5, r = 14.5$ (approximately), etc., until just beyond 15, where the system is chaotic. It is observed from Figures 2-4 that increasing the parameter s and fixing the fractional order parameter θ destabilize the system (1.3) and periodic behavior occurs. We study the bifurcation in (r, x) plane because by relations (1.5), α, β, δ that are present in Theorems 4.6,4.8, depends on r.

FIGURE 4. Bifurcation diagram $a = 1.3, \eta = 0.8, d = 0.4, b = 1, \theta = 0.9, s = 0.2$.

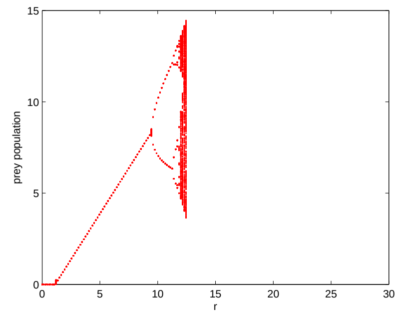
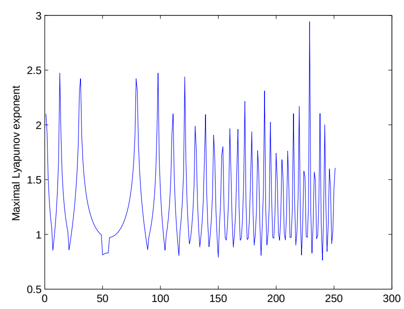


FIGURE 5. Maximal Lyapunov exponent $a = 1.3, \eta = 0.8, d = 0.4, b = 1, \theta = 0.9, s = 0.01$.



Then the bifurcation diagram of model (1.3) in (r, x) plane is shown in Figure 6 for $\theta = 0.5$ and $s = 0.09$. Thus, it shows that the discretized model (1.3) is stabilized only for relatively small step-sizes(s approaches to zero) and for large fractional order θ (θ approaches to one).

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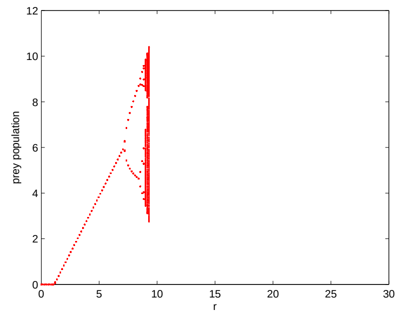
FIGURE 6. Bifurcation diagram $a = 1.3, \eta = 0.8, d = 0.4, b = 1, \theta = 0.5, s = 0.09$.



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