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Matrix Mittag-Leffler functions of fractional nabla calculus

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Abstract In this article, we propose the definition of one parameter matrix Mittag-Leffler functions of fractional nabla calculus and present three different algorithms to construct them. Examples are provided to illustrate the applicability of suggested algorithms.

Keywords. Fractional order, Nabla difference, Mittag-Leffler function, Spectral radius, N-transform.

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1. Introduction

Fractional nabla calculus is a new branch of mathematics which deals with arbitrary order sums and differences and their properties in nabla perspective. The concept of fractional nabla difference dates back to the works of Gray & Zhang [10] and Miller & Ross [20]. Since then, several authors gave valuable contributions to the development of the theory of fractional nabla difference equations [9].

In 1988, Gray and Zhang [10] introduced the fractional nabla difference operator and developed Leibniz formula, a limited composition rule and a version of a power rule for differentiation. Following the work of Gray and Zhang, Miller & Ross [20], Atici & Eloe [4], Anastassiou [3] and Abdeljawad [1] defined Riemann-Liouville and Caputo types of fractional nabla differences and established several properties of these operators. Atsushi Nagai [21] and Atici & Eloe [6] proposed the definitions of one and two parameter Mittag-Leffler functions of fractional nabla calculus, respectively. Atici & Eloe [6], Hein et al. [11] and the author [19] developed discrete Laplace transform method to find the solutions of initial value problems involving fractional nabla difference equations. Atici & Eloe [5] obtained Gronwall's inequality and established sufficient conditions for the continuous dependence of solutions of initial value problems on initial conditions. Čermák et al. [8] discussed stability and asymptotic properties of a two term linear fractional nabla difference equation. In [2], Acar & Atici studied exponential functions of discrete fractional calculus with the nabla operator, sequential linear nabla difference equations of fractional order with constant coefficients and defined a generalized Casoratian for a set of discrete functions. Jia et al. [13] established comparison theorems and obtained asymptotic behaviour of solutions of fractional nabla difference equations.

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Atici & Eloe [6] studied linear systems of fractional nabla difference equations with constant coefficients and constructed the fundamental matrix for the homogeneous system and the causal Green's function for the nonhomogeneous system. Wyrwas et al. [23] discussed the stability of discrete nonautonomous systems with the nabla Caputo fractional difference using the Lyapunov's direct method. Čermák et al. [7] derived stability regions for linear fractional nabla difference systems including a precise description of their asymptotics. The author [14, 16, 17, 18] established sufficient conditions on existence, uniqueness, stability and periodic properties of solutions of nonlinear fractional nabla difference systems.

Exponential and one parameter Mittag-Leffler functions of fractional nabla calculus act as eigenfunctions of the corresponding eigenvalue problems involving Riemann-Liouville and Caputo types of fractional nabla difference operators, respectively. Recently, Atici & Eloe [6] and the author [15] introduced the notion of matrix exponential function and devised different algorithms to compute matrix exponential functions. Motivated by the works in [6, 15], in this article, we introduce the idea of one parameter matrix Mittag-Leffler functions and propose three different algorithms to construct them.

2. Preliminaries

Throughout, we use the following notations, definitions and known results of fractional nabla calculus [9]: Denote the set of all real numbers and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. For any $a \in \mathbb{R}$, define $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$.

Definition 2.1. (Gamma Function) For any $t \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, the gamma function is defined by

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} ds.$$

Definition 2.2. (Rising Factorial Function) For any $\alpha \in \mathbb{R}$ and $t \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$ such that $(t+\alpha) \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, the rising factorial function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0.$$

Definition 2.3. Let $u : \mathbb{N}_a \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$.

(1) The first order backward (nabla) difference of u is defined by

$$\big(\nabla u\big)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1}.$$

(2) The α^{th}-order nabla sum of u is given by

$$\big(\nabla_a^{-\alpha} u\big)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a.$$

(3) The Caputo type α^{th}-order nabla difference of u is given by

$$\big(\nabla_{a*}^{\alpha} u\big)(t) = \big(\nabla_a^{-(1-\alpha)}\big(\nabla u\big)\big)(t), \quad t \in \mathbb{N}_{a+1}.$$

Theorem 2.4. *(Power Rule) Let $\alpha \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$. Assume that the following factorial functions are well defined. Then,*

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}}, \quad t \in \mathbb{N}_a.$$

Definition 2.5. [21, 6] The one parameter Mittag-Leffler function of fractional nabla calculus is defined by

$$F_\alpha(\lambda, (t-a)^{\overline{\alpha}}) = \sum_{n=0}^{\infty} \lambda^n \frac{(t-a)^{\overline{\alpha n}}}{\Gamma(\alpha n+1)},$$

where $\alpha > 0$, $|\lambda| < 1$ and $t \in \mathbb{N}_a$.

Definition 2.6. [2] The exponential function of fractional nabla calculus is defined by

$$\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) = (1-\lambda)\sum_{n=0}^{\infty} \lambda^n \frac{(t-a+1)^{\overline{\alpha(n+1)-1}}}{\Gamma(\alpha(n+1))},$$

where $0 < \alpha < 1$, $|\lambda| < 1$ and $t \in \mathbb{N}_a$.

Now, we recall the definition and properties of discrete Laplace transform (N-transform) [6, 19].

Definition 2.7. [6] Let $u : \mathbb{N}_a \to \mathbb{R}$. The N-transform of u is defined by

$$N_a\big[u(t)\big] = \sum_{j=a}^{\infty} u(j)(1-z)^{j-1} = U(z),$$

for each $z \in \mathbb{C}$ for which the series converges.

Definition 2.8. [6] Let u, $v : \mathbb{N}_a \to \mathbb{R}$. The convolution of u and v is defined by

$$\big(u *_a v\big)(t) = \sum_{s=a}^{t} u(t+a-\rho(s))v(s).$$

Theorem 2.9. *We observe the following properties of N-transform.*

(1) $N_a\Big[\big(u *_a v\big)(t)\Big] = N_1\big[u(t+a)\big]N_a\big[v(t)\big]$.

(2) $N_a\Big[(t-a+1)^{\overline{\alpha}}\Big] = (1-z)^{a-1}\frac{\Gamma(\alpha+1)}{z^{\alpha+1}}, \quad \alpha \in \mathbb{R} \setminus \{\cdots, -3, -2, -1\}$.

(3) $N_{a+1}\Big[\big(\nabla_a^{-\alpha}u\big)(t)\Big] = z^{-\alpha}N_{a+1}\big[u(t)\big], \quad \alpha > 0$.

(4) $N_{a+1}\Big[\big(\nabla u\big)(t)\Big] = zN_{a+1}\big[u(t)\big] - (1-z)^a u(a)$.

(5) $N_{a+1}\Big[\big(\nabla_{a*}^{\alpha}u\big)(t)\Big] = z^\alpha N_{a+1}\big[u(t)\big] - (1-z)^a z^{\alpha-1}u(a), \quad 0 < \alpha < 1$.

(6) $N_a\Big[F_\alpha(\lambda, (t-a+1)^{\overline{\alpha}})\Big] = (1-z)^{a-1}\frac{z^{\alpha-1}}{(z^\alpha-\lambda)}, \quad |\lambda| < z^\alpha$.

(7) $N_a\Big[\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}})\Big] = (1-z)^{a-1}\frac{(1-\lambda)}{(z^\alpha-\lambda)}, \quad |\lambda| < z^\alpha$.

Proof. For proofs of (1), (2), (3) and (7), we refer [6]. For proof of (4), we refer [17]. To prove (5), consider

$$\begin{aligned} N_{a+1}\Big[\big(\nabla^{\alpha}_{a*}u\big)(t)\Big] &= N_{a+1}\Big[\big(\nabla_a^{-(1-\alpha)}\big(\nabla u\big)\big)(t)\Big] \\ &= z^{-(1-\alpha)}N_{a+1}\big[\big(\nabla u\big)(t)\big] \\ &= z^{-(1-\alpha)}\Big[zN_{a+1}\big[u(t)\big]-(1-z)^{a}u(a)\Big] \\ &= z^{\alpha}N_{a+1}\big[u(t)\big]-(1-z)^{a}z^{\alpha-1}u(a). \end{aligned}$$

To prove (6), consider

$$\begin{aligned} N_a\Big[F_\alpha(\lambda,(t-a+1)^{\overline{\alpha}})\Big] &= \sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma(\alpha n+1)}N_a\Big[(t-a+1)^{\overline{\alpha n}}\Big] \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma(\alpha n+1)}(1-z)^{a-1}\frac{\Gamma(\alpha n+1)}{z^{\alpha n+1}} \\ &= \frac{(1-z)^{a-1}}{z}\sum_{n=0}^{\infty}\big(\lambda z^{-\alpha}\big)^n \\ &= (1-z)^{a-1}\frac{z^{\alpha-1}}{(z^\alpha-\lambda)},\quad |\lambda|<z^\alpha. \end{aligned}$$

□

Consider the nabla fractional initial value problem (IVP)

$$\big(\nabla^{\alpha}_{0*}u\big)(t)=\lambda u(t),\quad 0<\alpha<1,\quad |\lambda|<1,\quad t\in\mathbb{N}_1, \tag{2.1}$$

$$u(0)=u_0, \tag{2.2}$$

where $u:\mathbb{N}_0\to\mathbb{R}$. The condition $\lambda\neq 1$ guarantees the existence of unique solution of (2.1) - (2.2). Atsushi Nagai [21] established a formal proof of Theorem 2.10. Now, we prove Theorems 2.10 and 2.11 using N-transform.

Theorem 2.10. *The unique solution of (2.1) - (2.2) is given by*

$$u(t)=u_0F_\alpha(\lambda,t^{\overline{\alpha}}),\quad t\in\mathbb{N}_0. \tag{2.3}$$

Proof. Applying N_1-transform on both sides of (2.1), we get

$$N_1\big[u(t)\big]=\frac{z^{\alpha-1}}{(z^\alpha-\lambda)}u_0. \tag{2.4}$$

Applying inverse N_1-transform on both sides of (2.4) and using Theorem 2.9, we obtain (2.3). □

Theorem 2.11. *Let u, $g:\mathbb{N}_0\to\mathbb{R}$. The unique solution of*

$$\big(\nabla^{\alpha}_{0*}u\big)(t)=\lambda u(t)+g(t),\quad 0<\alpha<1,\quad |\lambda|<1,\quad t\in\mathbb{N}_1, \tag{2.5}$$

$$u(0)=u_0, \tag{2.6}$$

is given by

$$u(t) = u_0 F_\alpha(\lambda, t^{\overline{\alpha}}) + \frac{1}{(1-\lambda)} \sum_{s=1}^{t} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) g(s), \quad t \in \mathbb{N}_0. \tag{2.7}$$

Proof. Applying N_1-transform on both sides of (2.5), we get

$$N_1\big[u(t)\big] = \frac{z^{\alpha-1}}{(z^\alpha - \lambda)} u_0 + \frac{1}{(z^\alpha - \lambda)} N_1\big[g(t)\big]. \tag{2.8}$$

Applying inverse N_1-transform on both sides of (2.8) and using Theorem 2.9, we obtain (2.7). □

Definition 2.12. Consider the vector spaces $\mathbb{R}^k$ of all ordered k-tuples of real numbers and M_k of all $k \times k$ matrices over $\mathbb{R}$. Corresponding to each vector norm on $\mathbb{R}^k$, we define an operator norm on M_k by

$$\|A\| = \max_{\|u\|=1} \|Au\|,$$

for any $u \in \mathbb{R}^k$ and $A \in M_k$. We observe that $\|I\| = 1$, where I is the $k \times k$ identity matrix.

Theorem 2.13. *Let R be the radius of convergence of a scalar power series*

$$\sum_{n=0}^{\infty} a_n x^n,$$

and let $A \in M_k$ be given with $\|A\| < R$. Then, the matrix power series

$$\sum_{n=0}^{\infty} a_n A^n$$

converges if $\rho(A) < R$.

3. One Parameter Matrix Mittag-Leffler Function

Fix $t \in \mathbb{N}_a$. We know that the radius of convergence of the scalar power series

$$\sum_{n=0}^{\infty} \frac{(t-a)^{\overline{\alpha n}}}{\Gamma(\alpha n + 1)} \lambda^n$$

is 1. Let $A \in M_k$ such that $\|A\| < 1$. Then, by Theorem 2.13, the matrix power series

$$\sum_{n=0}^{\infty} \frac{(t-a)^{\overline{\alpha n}}}{\Gamma(\alpha n + 1)} A^n$$

converges if $\rho(A) < 1$. Analogous to Definition 2.5, we propose the definition of one parameter matrix Mittag-Leffler function as follows:

$$F_\alpha(A, (t-a)^{\overline{\alpha}}) = \sum_{n=0}^{\infty} \frac{(t-a)^{\overline{\alpha n}}}{\Gamma(\alpha n + 1)} A^n, \tag{3.1}$$

where $\alpha > 0$, $t \in \mathbb{N}_a$ and $A \in M_k$ such that $\rho(A) < 1$. Atici & Eloe [6] and the author [15] introduced the matrix exponential function of fractional nabla calculus as follows:

$$\hat{e}_{\alpha,\alpha}(A,(t-a)^{\overline{\alpha}}) = \sum_{n=0}^{\infty} \frac{(t-a+1)^{\overline{\alpha n+\alpha-1}}}{\Gamma(\alpha n+\alpha)} A^n(I-A), \quad t \in \mathbb{N}_a. \tag{3.2}$$

Consequently, one can prove the following two theorems using N-transform.

Theorem 3.1. *Consider the nabla fractional IVP*

$$\left(\nabla_{0*}^{\alpha} u\right)(t) = Au(t), \quad 0 < \alpha < 1, \quad \rho(A) < 1, \quad t \in \mathbb{N}_1, \tag{3.3}$$

$$u(0) = u_0, \tag{3.4}$$

where $u : \mathbb{N}_0 \to \mathbb{R}^k$. *The condition* $\det(I-A) \neq 1$ *guarantees the existence of unique solution of (3.3) - (3.4). The unique solution of (3.3) - (3.4) is given by*

$$u(t) = F_\alpha(A, t^{\overline{\alpha}})u_0, \quad t \in \mathbb{N}_0. \tag{3.5}$$

Theorem 3.2. *Let* u, $g : \mathbb{N}_0 \to \mathbb{R}^k$. *The unique solution of*

$$\left(\nabla_{0*}^{\alpha} u\right)(t) = \lambda u(t) + g(t), \quad 0 < \alpha < 1, \quad \rho(A) < 1, \quad t \in \mathbb{N}_1, \tag{3.6}$$

$$u(0) = u_0, \tag{3.7}$$

is given by

$$u(t) = F_\alpha(A, t^{\overline{\alpha}})u_0 + \sum_{s=1}^{t} \hat{e}_{\alpha,\alpha}(A,(t-s)^{\overline{\alpha}})(I-A)^{-1}g(s), \quad t \in \mathbb{N}_0. \tag{3.8}$$

4. Construction of Matrix Mittag-Leffler Function

In [15], the author has presented three different algorithms to construct the matrix exponential function. In a similar way, we construct the one parameter matrix Mittag-Leffler function as follows:

(I) Assume that A is diagonalizable. Let λ_i, $1 \le i \le k$ be the eigenvalues of A and let ξ_i, $1 \le i \le k$ be the corresponding linearly independent eigenvectors of A. Then, for $t \in \mathbb{N}_0$,

$$F_\alpha(A, t^{\overline{\alpha}}) = \Big[F_\alpha(\lambda_1, t^{\overline{\alpha}})\xi_1, F_\alpha(\lambda_2, t^{\overline{\alpha}})\xi_2, \cdots, F_\alpha(\lambda_k, t^{\overline{\alpha}})\xi_k\Big] \times \Big[\xi_1, \xi_2, \cdots, \xi_k\Big]^{-1}.$$

Example 1. Consider a linear homogeneous fractional nabla difference system

$$\left(\nabla_{0*}^{\alpha} u\right)(t) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} u(t), \quad t \in \mathbb{N}_1.$$

Solution: The eigenvalues of $A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$ are $\lambda_1 = \lambda_2 = \frac{1}{3}$ and $\lambda_3 = \frac{1}{2}$. Obviously, $\rho(A) = \frac{1}{2} < 1$. The corresponding linearly independent eigenvectors are

$$\xi_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \xi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively. Clearly, A is diagonalizable. Then,

$$\begin{aligned} & F_\alpha(A, t^{\overline{\alpha}}) \\ = & \begin{pmatrix} -2F_\alpha(\frac{1}{3}, t^{\overline{\alpha}}) & 0 & 0 \\ 0 & F_\alpha(\frac{1}{3}, t^{\overline{\alpha}}) & 0 \\ F_\alpha(\frac{1}{3}, t^{\overline{\alpha}}) & 0 & F_\alpha(\frac{1}{2}, t^{\overline{\alpha}}) \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \\ = & \begin{pmatrix} F_\alpha(\frac{1}{3}, t^{\overline{\alpha}}) & 0 & 0 \\ 0 & F_\alpha(\frac{1}{3}, t^{\overline{\alpha}}) & 0 \\ -\frac{1}{2}F_\alpha(\frac{1}{3}, t^{\overline{\alpha}}) + \frac{1}{2}F_\alpha(\frac{1}{2}, t^{\overline{\alpha}}) & 0 & F_\alpha(\frac{1}{2}, t^{\overline{\alpha}}) \end{pmatrix}. \end{aligned}$$

(II) If A is not diagonalizable, then it is similar to the Jordan form, i.e., there exists a non-singular matrix P such that $P^{-1}AP = J$, where

$$J = \begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \ddots & & \\ & & & J_{r-1} & \\ & & & & J_r \end{pmatrix}_{k \times k}, \quad 1 \le r \le k,$$

and

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \lambda_i \end{pmatrix}_{s_i \times s_i} \quad \text{such that } \sum_{i=1}^{r} s_i = k.$$

Then, for $1 \le r \le k$ and $t \in \mathbb{N}_0$, we have

$$\begin{aligned} F_\alpha(A, t^{\overline{\alpha}}) &= P F_\alpha(J, t^{\overline{\alpha}}) P^{-1} \\ &= P \begin{pmatrix} F_\alpha(J_1, t^{\overline{\alpha}}) & & & \\ & F_\alpha(J_2, t^{\overline{\alpha}}) & & \\ & & \ddots & \\ & & & F_\alpha(J_r, t^{\overline{\alpha}}) \end{pmatrix}_{k \times k} P^{-1}, \end{aligned}$$

where

$$P = [\xi_1, \xi_2, \cdots, \xi_k]$$

and

$$F_\alpha(J_i, t^{\overline{\alpha}})$$
$$= \begin{pmatrix} F_\alpha(\lambda_i, t^{\overline{\alpha}}) & DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\lambda_i} & D^2F_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\lambda_i} & \cdots & D^{s_i-1}F_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\lambda_i} \\ 0 & F_\alpha(\lambda_i, t^{\overline{\alpha}}) & DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\lambda_i} & \cdots & D^{s_i-2}F_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\lambda_i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\lambda_i} \\ 0 & 0 & 0 & \cdots & F_\alpha(\lambda_i, t^{\overline{\alpha}}) \end{pmatrix}.$$

Here

$$D^m = \frac{1}{m!}\frac{d^m}{d\lambda^m}, \quad m \in \mathbb{N}_0.$$

Example 2. Consider a linear homogeneous fractional nabla difference system

$$\left(\nabla^\alpha_{0*}u\right)(t) = \begin{pmatrix} \frac{1}{4} & -\frac{1}{16} & -\frac{1}{8} \\ 0 & \frac{3}{8} & \frac{1}{4} \\ 0 & -\frac{1}{16} & \frac{1}{8} \end{pmatrix} u(t), \quad t \in \mathbb{N}_1.$$

Solution: The eigenvalues of $A = \begin{pmatrix} \frac{1}{4} & -\frac{1}{16} & -\frac{1}{8} \\ 0 & \frac{3}{8} & \frac{1}{4} \\ 0 & -\frac{1}{16} & \frac{1}{8} \end{pmatrix}$ are $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{4}$. Obviously, $\rho(A) = \frac{1}{4} < 1$. The eigenvectors of A corresponding to the eigenvalue $\frac{1}{4}$ is given by

$$\xi_1 = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix}, \xi_2 = \begin{pmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{8} \end{pmatrix}.$$

The generalized eigenvector of A is given by

$$\xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We observe that A is not diagonalizable. Thus,

$$P = \begin{pmatrix} 0 & -\frac{1}{8} & 0 \\ 1 & \frac{1}{4} & 0 \\ -\frac{1}{2} & -\frac{1}{8} & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{4} \end{pmatrix},$$

and

$$
\begin{aligned}
&F_\alpha(A, t^{\overline{\alpha}}) \\
= \; & P \begin{pmatrix} F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) & 0 & 0 \\ 0 & F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) & DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} \\ 0 & 0 & F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) \end{pmatrix} P^{-1} \\
= \; & \begin{pmatrix} 0 & -\frac{1}{8}F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) & -\frac{1}{8}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} \\ F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) & \frac{1}{4}F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) & \frac{1}{4}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} \\ -\frac{1}{2}F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) & -\frac{1}{8}F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) & -\frac{1}{8}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} + F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) \end{pmatrix} P^{-1} \\
= \; & \begin{pmatrix} F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) & -\frac{1}{16}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} & -\frac{1}{8}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} \\ 0 & F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) + \frac{1}{8}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} & \frac{1}{4}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} \\ 0 & -\frac{1}{16}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} & -\frac{1}{8}DF_\alpha(\lambda, t^{\overline{\alpha}})\Big|_{\lambda=\frac{1}{4}} + F_\alpha(\frac{1}{4}, t^{\overline{\alpha}}) \end{pmatrix}.
\end{aligned}
$$

(III) Putzer's Algorithm:

Theorem 4.1. *Let $A \in M_k$ such that $\rho(A) < 1$. If λ_i, $1 \leq i \leq k$ are eigenvalues (not necessarily distinct) of A, then*

$$
F_\alpha(A, t^{\overline{\alpha}}) = \sum_{i=1}^{k} p_i(t) M_{i-1}, \tag{4.1}
$$

where

$$
\begin{aligned}
&M_0 = I, \\
&M_i = (A - \lambda_i I) M_{i-1}, \quad 1 \leq i \leq (k-1), \\
&M_k = 0,
\end{aligned}
$$

and the vector valued function $p(t) = (p_1(t), p_2(t), \cdots, p_k(t))^T$ is the solution of the initial value problem

$$
\left(\nabla_{0*}^{\alpha} p\right)(t) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \lambda_k \end{pmatrix} p(t), \quad t \in \mathbb{N}_1, \tag{4.2}
$$

$$
p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.3}
$$

Example 3. Consider a linear homogeneous fractional nabla difference system

$$\left(\nabla^{\alpha}_{0*}u\right)(t) = \begin{pmatrix} 0.25 & 0 & 0 \\ 1 & 0.50 & 1 \\ 0 & 0 & 0.75 \end{pmatrix} u(t), \quad t \in \mathbb{N}_1.$$

Solution: The eigenvalues of $A = \begin{pmatrix} 0.25 & 0 & 0 \\ 1 & 0.50 & 1 \\ 0 & 0 & 0.75 \end{pmatrix}$ are $\lambda_1 = 0.25$, $\lambda_2 = 0.50$ and $\lambda_3 = 0.75$. Clearly, $\rho(A) = 0.75 < 1$. We have

$$M_0 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_1 = (A - \lambda_1 I)M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0.25 & 1 \\ 0 & 0 & 0.50 \end{pmatrix},$$

$$M_2 = (A - \lambda_2 I)M_1 = \begin{pmatrix} -0.25 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0.25 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0.25 & 1 \\ 0 & 0 & 0.50 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0.125 \end{pmatrix},$$

$$M_3 = 0,$$

and $p(t) = (p_1(t), p_2(t), p_3(t))^T$ is the solution of the initial value problem

$$\left(\nabla^{\alpha}_{0*}p\right)(t) = \begin{pmatrix} 0.25 & 0 & 0 \\ 1 & 0.50 & 0 \\ 0 & 1 & 0.75 \end{pmatrix} p(t), \quad t \in \mathbb{N}_1, \tag{4.4}$$

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.5}$$

The equivalent form of (4.4) - (4.5) is

$$\left(\nabla^{\alpha}_{0*}p_1\right)(t) = (0.25)p_1(t), \quad t \in \mathbb{N}_1, \tag{4.6}$$

$$p_1(0) = 1, \tag{4.7}$$

$$\left(\nabla^{\alpha}_{0*}p_2\right)(t) = (0.50)p_2(t) + p_1(t), \quad t \in \mathbb{N}_1, \tag{4.8}$$

$$p_2(0) = 0, \tag{4.9}$$

and

$$\left(\nabla^{\alpha}_{0*}p_3\right)(t) = (0.75)p_3(t) + p_2(t), \quad t \in \mathbb{N}_1, \tag{4.10}$$

$$p_3(0) = 0. \tag{4.11}$$

Using Theorem 2.10, the unique solution of the initial value problem (4.6) - (4.7) is

$$p_1(t) = F_{\alpha}(0.25, t^{\overline{\alpha}}), \quad t \in \mathbb{N}_0. \tag{4.12}$$

Using Theorem 2.11, the unique solution of the initial value problem (4.8) - (4.9) is

$$p_2(t) = \frac{1}{(0.50)} \sum_{s=1}^{t} \hat{e}_{\alpha,\alpha}(0.50, (t-s)^{\overline{\alpha}}) p_1(s), \quad t \in \mathbb{N}_0. \tag{4.13}$$

Using Theorem 2.11, the unique solution of the initial value problem (4.10) - (4.11) is

$$p_3(t) = \frac{1}{(0.25)} \sum_{s=1}^{t} \hat{e}_{\alpha,\alpha}(0.75, (t-s)^{\overline{\alpha}}) p_2(s), \quad t \in \mathbb{N}_0. \tag{4.14}$$

Thus,

$$\begin{aligned} F_\alpha(A, t^{\overline{\alpha}}) &= p_1(t)M_0 + p_2(t)M_1 + p_3(t)M_2 \\ &= \begin{pmatrix} p_1(t) & 0 & 0 \\ p_2(t) & p_1(t) + (0.25)p_2(t) & p_2(t) + (0.5)p_3(t) \\ 0 & 0 & p_1(t) + (0.50)p_2(t) + (0.125)p_3(t) \end{pmatrix}. \end{aligned}$$

Corollary 4.2. *Let $A \in M_k$ such that $\rho(A) < 1$. If λ_i, $1 \le i \le k$ are eigenvalues (not necessarily distinct) of A, then*

$$F_\alpha(A, t^{\overline{\alpha}}) = \sum_{i=1}^{k} p_i(t) M_{i-1},$$

where

$$\begin{aligned} &M_0 = I, \\ &M_i = (A - \lambda_i I)M_{i-1}, \quad 1 \le i \le (k-1), \\ &M_k = 0, \end{aligned}$$

and

$$p_1(t) = F_\alpha(\lambda_1, t^{\overline{\alpha}}), \tag{4.15}$$

$$p_i(t) = \frac{1}{(1-\lambda_i)} \sum_{s=1}^{t} \hat{e}_{\alpha,\alpha}(\lambda_i, (t-s)^{\overline{\alpha}}) p_{i-1}(s), \; 2 \le i \le k, \tag{4.16}$$

for $t \in \mathbb{N}_0$.

5. Conclusion

In this article, we defined the one parameter matrix Mittag-Leffler functions and presented various methods to compute them. We also observed that the one parameter matrix Mittag - Leffler function acts as an eigenfunction of the Caputo type fractional nabla difference eigenvalue problem.

In continuation to this work, we can also define the two parameter matrix Mittag - Leffler function as

$$F_{\alpha,\beta}(A, (t-a)^{\overline{\alpha}}) = \sum_{n=0}^{\infty} \frac{(t-a)^{\overline{\alpha n}}}{\Gamma(\alpha n + \beta)} A^n,$$

where α, $\beta > 0$, $t \in \mathbb{N}_a$ and $A \in M_k$ such that $\rho(A) < 1$. Obviously, we have

$$\hat{e}_{\alpha,\alpha}(A, (t-a)^{\overline{\alpha}}) = (t+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(A, (t-a+\alpha)^{\overline{\alpha}})(I-A).$$

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