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Convergence of Legendre wavelet collocation method for solving nonlinear Stratonovich Volterra integral equations

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Abstract In this paper, we apply Legendre wavelet collocation method to obtain the approximate solution of nonlinear Stratonovich Volterra integral equations. The main advantage of this method is that Legendre wavelet has orthogonality property and therefore coefficients of expansion are easily calculated. By using this method, the solution of nonlinear Stratonovich Volterra integral equation reduces to the nonlinear system of algebraic equations which can be solved by using a suitable numerical method such as Newton's method. Convergence analysis with error estimate are given with full discussion. Also, we provide an upper error bound under weak assumptions. Finally, accuracy of this scheme is checked with two numerical examples. The obtained results reveal efficiency and capability of the proposed method.

Keywords. Stochastic integrals; Operational matrix of integration ; Wavelet; Legendre polynomials; Error analysis.

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1. Introduction

In stochastic processes, stochastic integrals are divided into two parts; Itô integral and Stratonovich integral. The Itô integral named after Kiyoshi Itô and usually is used in applied mathematics whereas the Stratonovich integral developed simultaneously by R. L. Stratonovich [19] and D. L. Fisk [5] and is frequently applied in physics. Stratonovich calculus has some advantages to Itô calculus, for example, in some positions, integrals in the Stratonovich definition are easier to manipulate. Unlike the Itô calculus, Stratonovich integrals are defined such that the chain rule of ordinary calculus hold. Because of this feature has satisfied, stochastic differential equations in the Stratonovich sense are more straightforward to define on differentiable manifolds. Stochastic integral equations can rarely be solved in analytic form and therefore provide a numerical method for solving them is an essential requirement in applied

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mathematics. In recent years, various numerical methods such as finite difference method [4], radial basis function method [2], Galerkin method [9], operational matrix method [6, 11, 12, 16], Legendre wavelets Galerkin method [7], linear analytic approximation method [8], collocation method [10], orthonormal Bernstein polynomial method [13], wavelet-based computational method [17], Euler method [20], Chebyshev wavelet method [1] are used to solve ordinary and stochastic integral equations. But the number of papers on the numerical solution of stochastic integral equations are still very few. Also, few researchers have worked on numerical method for solving Stratonovich integral equations. Recently Mirzaee and Samadyar have applied operational matrix method for solving nonlinear Stratonovich Volterra integral equation [15] and system of linear Stratonovich Volterra integral equations [14].
In this paper, we solve the specific case of nonlinear Stratonovich Volterra integral equation as follows

$$X(t) = X_0 + \lambda_1 \int_0^t a\big(s, X(s)\big)ds + \lambda_2 \int_0^t b\big(s, X(s)\big) \circ dB(s), \tag{1.1}$$

where $t \in [0, 1)$, λ_1 and λ_2 are constant parameters, $a\big(s, X(s)\big)$ and $b\big(s, X(s)\big)$ are known functions, $X(t)$ is unknown function which should be determined and $B(t)$ is Brownian motion process defined on probability space $(\Omega, \mathcal{F}, P)$ consisting of the sample space Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω called events, and a real-valued set function P defined on $\mathcal{F}$ called a probability. Note that the symbol $\circ$ between integrand and the stochastic differential is used to show Stratonovich integral.
Eq. (1.1) arise in many applications such as mathematical finance, engineering, biology, medical, and social sciences. Solving this equation analytically is very difficult or even sometimes impossible, so we develop an efficient numerical method to solve it. In this paper, operational matrix of integration and stochastic operational matrix of integration based on Legendre wavelet are used to solve Eq. (1.1) numerically. By using these matrices and collocation points this equation converts to the nonlinear system of algebraic equations which can be solved by using a suitable numerical method such as Newton's method. Also, we know that the Stratonovich integral equations can be transformed to Itô integral equations. The existence and uniqueness of the solution of Itô integral equations is discussed in [3].
The reminder of this work is organized as follows. In Section 2, we obtain some elementary definitions and properties of wavelet and Legendre wavelet. Operational matrix of integration and stochastic operational matrix of integration are obtained in this section. In Section 3, the proposed method are used to estimate the solution of linear Stratonovich Volterra integral equation. In Section 4, convergence analysis of the proposed method is proved. Numerical examples are included in Section 5. Finally, we give the conclusion of this paper in Section 6.

2. Wavelet and Legendre wavelet

Wavelets are a family of functions which are constructed from dilation and translation of a single function ψ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [18]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\Big(\frac{t-b}{a}\Big), \qquad a,b \in \mathbb{R}, \quad a \neq 0. \tag{2.1}$$

If we restrict the dilation parameter a and translation parameter b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, where n and k are positive integer numbers, the family of discrete wavelets are constructed as

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0), \qquad n,k \in \mathbb{Z}^+, \tag{2.2}$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(\mathbb{R})$. When $a_0 = 2$ and $b_0 = 1$, the functions $\psi_{k,n}(t)$ form an orthonormal basis.
Legendre wavelets $\psi_{n,m}(t) = \psi(k,n,m,t)$ have four arguments; $n = 1,2,\dots,2^{k-1}$, k is assumed to be any positive integer numbers, m is the degree of the Legendre polynomials and t is on the interval $[0,1)$. They are defined on the interval $[0,1)$ as follows

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} P_m(2^k t - 2n + 1), & \frac{2n-2}{2^k} \leq t < \frac{2n}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

where $m = 0,1,\dots,M-1$ and $n = 1,2,\dots,2^{k-1}$ and $P_m(t)$ are the Legendre polynomials of degree m. These polynomials are orthogonal with respect to the weight function $w(t) = 1$, on the interval $[-1,1]$ and satisfy the following recursive relation [7]

$$\begin{aligned} &P_0(t) = 1, \\ &P_1(t) = t, \\ &\vdots \\ &P_{m+1}(t) = \Big(\frac{2m+1}{m+1}\Big)tP_m(t) - \Big(\frac{m}{m+1}\Big)P_{m-1}(t), \qquad m = 1,2,3,\dots. \end{aligned}$$

2.1. **Function approximation.** We consider the set of Legendre wavelet as

$$\begin{aligned} \Psi(t) = \big[&\psi_{1,0}(t), \psi_{2,0}(t), \dots, \psi_{2^{k-1},0}(t), \psi_{1,1}(t), \dots, \psi_{2^{k-1},1}(t) \\ &, \dots, \psi_{1,M-1}(t), \dots, \psi_{2^{k-1},M-1}(t)\big]^T \subset L^2[0,1], \end{aligned} \tag{2.4}$$

and suppose that

$$\begin{aligned} Y = \operatorname{span}\{&\psi_{1,0}(t), \psi_{2,0}(t), \dots, \psi_{2^{k-1},0}(t), \psi_{1,1}(t), \dots, \psi_{2^{k-1},1}(t) \\ &, \dots, \psi_{1,M-1}(t), \dots, \psi_{2^{k-1},M-1}(t)\}, \end{aligned}$$

also, suppose that f be an arbitrary function in $L^2[0,1]$. Because Y is a finite dimensional vector space, so f has the best approximation out of Y such as $f^* \in Y$, that is

$$\forall g \in Y, \ \ \|f - f^*\| \leq \|f - g\|.$$

Since $f^* \in Y$, there exist unique coefficients $c_{1,0}, c_{2,0}, \dots, c_{2^{k-1},M-1}$ such that

$$f(t) \simeq f^*(t) = \sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}} c_{n,m}\psi_{n,m}(t) = C^T\Psi(t), \tag{2.5}$$

where $\Psi(t)$ is a vector of order $2^{k-1}M \times 1$ defined in Eq. (2.4) and C is an $2^{k-1}M \times 1$ vector given by

$$C = [c_{1,0}, c_{2,0}, \dots, c_{2^{k-1},0}, \dots, c_{1,M-1}, \dots, c_{2^{k-1},M-1}]^T. \tag{2.6}$$

Also, the coefficients $c_{n,m}$ in Eq. (2.5) can be computed from the following relation

$$c_{n,m} = \langle\, f, \psi_{n,m}\rangle = \int_0^1 f(t)\psi_{n,m}(t)dt. \tag{2.7}$$

2.2. **Operational matrix of integration.** We can approximate the integration of the $\Psi(t)$ defined in Eq. (2.4) as follows

$$\int_0^t \Psi(s)ds \simeq P\Psi(t), \tag{2.8}$$

where P is a matrix of order $2^{k-1}M \times 2^{k-1}M$ and is named operational matrix of integration. For example, for $k = 2$ and $M = 3$, we have

$$\Psi(t) = [\psi_{1,0}(t), \psi_{2,0}(t), \psi_{1,1}(t), \psi_{2,1}(t), \psi_{1,2}(t), \psi_{2,2}(t)]^T, \tag{2.9}$$

$$\begin{aligned}\int_0^t \psi_{1,0}(s)ds &= \begin{cases} \sqrt{2}t & 0 \le t < \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq \frac{1}{4}\psi_{1,0}(t) + \frac{1}{2}\psi_{2,0}(t) + \frac{1}{4\sqrt{3}}\psi_{1,1}(t),\end{aligned} \tag{2.10}$$

$$\begin{aligned}\int_0^t \psi_{2,0}(s)ds &= \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{2}(t - \frac{1}{2}) & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq \frac{1}{4}\psi_{2,0}(t) + \frac{1}{4\sqrt{3}}\psi_{2,1}(t),\end{aligned} \tag{2.11}$$

$$\begin{aligned}\int_0^t \psi_{1,1}(s)ds &= \begin{cases} \sqrt{6}\big(2t^2 - t\big) & 0 \le t < \frac{1}{2} \\ 0 & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq \frac{-1}{4\sqrt{3}}\psi_{1,0}(t) + \frac{\sqrt{6}}{12\sqrt{10}}\psi_{1,2}(t),\end{aligned} \tag{2.12}$$

$$\begin{aligned}\int_0^t \psi_{2,1}(s)ds &= \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{6}\big(2t^2 - 3t + 1\big) & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq \frac{-1}{4\sqrt{3}}\psi_{2,0}(t) + \frac{\sqrt{6}}{12\sqrt{10}}\psi_{2,2}(t),\end{aligned} \tag{2.13}$$

$$\begin{aligned}\int_0^t \psi_{1,2}(s)ds &= \begin{cases} \sqrt{10}\big(8t^3 - 6t^2 + t\big) & 0 \le t < \frac{1}{2} \\ 0 & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq \frac{-\sqrt{6}}{12\sqrt{10}}\psi_{1,1}(t) + \frac{\sqrt{10}}{20\sqrt{14}}\psi_{1,3}(t),\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
\int_0^t \psi_{2,2}(s)ds &= \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{10}\big(8t^3 - 18t^2 + 13t - 3\big) & \frac{1}{2} \le t < 1 \end{cases} \\
&\simeq \frac{-\sqrt{6}}{12\sqrt{10}}\psi_{2,1}(t) + \frac{\sqrt{10}}{20\sqrt{14}}\psi_{2,3}(t).
\end{aligned} \tag{2.15}
$$

By applying Eqs. (2.9)-(2.15) and omitting the terms $\psi_{1,3}(t)$ and $\psi_{2,3}(t)$, we conclude

$$
\int_0^t \Psi(s)ds \simeq \begin{pmatrix}
\frac{1}{4} & \frac{1}{2} & \frac{1}{4\sqrt{3}} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4\sqrt{3}} & 0 & 0 \\
\frac{-1}{4\sqrt{3}} & 0 & 0 & 0 & \frac{\sqrt{6}}{12\sqrt{10}} & 0 \\
0 & \frac{-1}{4\sqrt{3}} & 0 & 0 & 0 & \frac{\sqrt{6}}{12\sqrt{10}} \\
0 & 0 & \frac{-\sqrt{6}}{12\sqrt{10}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-\sqrt{6}}{12\sqrt{10}} & 0 & 0
\end{pmatrix} \Psi(t).
$$

In general, we can demonstrate operational matrix of integration P is in the following form

$$
P = \frac{1}{2^k}\begin{pmatrix}
P_0 & A_1 & O & O & \cdots & O \\
-A_1 & O & A_2 & O & \cdots & O \\
O & -A_2 & O & A_3 & \cdots & O \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
O & \cdots & O & -A_{M-2} & O & A_{M-1} \\
O & \cdots & O & O & -A_{M-1} & O
\end{pmatrix}, \tag{2.16}
$$

where O is a zero matrix of order $2^{k-1} \times 2^{k-1}$ and P_0 is a matrix of order $2^{k-1} \times 2^{k-1}$ defined as

$$
P_0 = \begin{pmatrix}
1 & 2 & 2 & \cdots & 2 & 2 \\
0 & 1 & 2 & \cdots & 2 & 2 \\
0 & 0 & 1 & \ddots & 2 & 2 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 2 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},
$$

and A_i is an $2^{k-1} \times 2^{k-1}$ matrix defined as follows

$$
A_i = \mathrm{diag}\Big(\frac{1}{\sqrt{(2i-1)(2i+1)}}\Big), \qquad i = 1, 2, \ldots, M-1.
$$

By using Eqs. (2.5) and (2.8), we can approximate integral of every function f as follows

$$
\int_0^t f(s)ds \simeq \int_0^t C^T \Psi(s)ds \simeq C^T P \Psi(t). \tag{2.17}
$$

2.3. **Stochastic operational matrix of integration.** The Stratonovich integral of $\Psi(t)$ can be approximated as follows

$$\int_0^t \Psi(s) \circ dB(s) \simeq P_s \Psi(t), \tag{2.18}$$

where P_s is an $2^{k-1}M \times 2^{k-1}M$ matrix and is called stochastic operational matrix of integration. For example, for $k = 2$ and $M = 3$, we obtain

$$\begin{aligned}\int_0^t \psi_{1,0}(s) \circ dB(s) &= \begin{cases} \sqrt{2}B(t) & 0 \le t < \frac{1}{2} \\ \sqrt{2}B(\frac{1}{2}) & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq B(\frac{1}{4})\psi_{1,0}(t) + B(\frac{1}{2})\psi_{2,0}(t),\end{aligned} \tag{2.19}$$

$$\begin{aligned}\int_0^t \psi_{2,0}(s) \circ dB(s) &= \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{2}\big(B(t) - B(\frac{1}{2})\big) & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq \Big(B(\frac{3}{4}) - B(\frac{1}{2})\Big)\psi_{2,0}(t),\end{aligned} \tag{2.20}$$

$$\begin{aligned}\int_0^t \psi_{1,1}(s) \circ dB(s) &= \begin{cases} \sqrt{6}(4t-1)B(t) - \int_0^t 4\sqrt{6}B(s)ds & 0 \le t < \frac{1}{2} \\ \sqrt{6}B(\frac{1}{2}) - \int_0^{\frac{1}{2}} 4\sqrt{6}B(s)ds & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq (-\frac{1}{\sqrt{2}}\int_0^{\frac{1}{4}} 4\sqrt{6}B(s)ds)\psi_{1,0}(t) \\ &+ (\frac{\sqrt{6}B(\frac{1}{2}) - \int_0^{\frac{1}{2}} 4\sqrt{6}B(s)ds}{\sqrt{2}})\psi_{2,0}(t) + B(\frac{1}{4})\psi_{1,1}(t),\end{aligned} \tag{2.21}$$

$$\begin{aligned}\int_0^t \psi_{2,1}(s) \circ dB(s) &= \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{6}\Big[(4t-3)B(t) + B(\frac{1}{2}) - \int_{\frac{1}{2}}^t 4B(s)ds\Big] & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq (\frac{\sqrt{6}B(\frac{1}{2}) - \int_{\frac{1}{2}}^{\frac{3}{4}} 4\sqrt{6}B(s)ds}{\sqrt{2}})\psi_{2,0}(t) + B(\frac{3}{4})\psi_{2,1}(t),\end{aligned} \tag{2.22}$$

$$\begin{aligned}\int_0^t \psi_{1,2}(s) \circ dB(s) &= \begin{cases} \sqrt{10}\Big[(24t^2 - 12t + 1)B(t) - \int_0^t (48s - 12)B(s)ds\Big] & 0 \le t < \frac{1}{2} \\ \sqrt{10}B(\frac{1}{2}) - \int_0^{\frac{1}{2}} \sqrt{10}(48s-12)B(s)ds & \frac{1}{2} \le t < 1 \end{cases} \\ &\simeq (-\frac{\int_0^{\frac{1}{4}} \sqrt{10}(48s-12)B(s)ds}{\sqrt{2}})\psi_{1,0}(t) \\ &+ (\frac{\sqrt{10}B(\frac{1}{2}) - \int_0^{\frac{1}{2}} \sqrt{10}(48s-12)B(s)ds}{\sqrt{2}})\psi_{2,0}(t) \\ &+ B(\frac{1}{4})\psi_{1,2}(t),\end{aligned} \tag{2.23}$$

$$\int_0^t \psi_{2,2}(s) \circ dB(s) = \begin{cases} 0 & 0 \leq t < \frac{1}{2} \\ \sqrt{10}\Big[(24t^2 - 36t + 13)B(t) - B(\frac{1}{2}) - \int_{\frac{1}{2}}^t (48s - 36)B(s)ds\Big] & \frac{1}{2} \leq t < 1 \end{cases}$$

$$\simeq (-\frac{\sqrt{10}B(\frac{1}{2}) + \int_{\frac{1}{2}}^{\frac{3}{4}} \sqrt{10}(48s - 36)B(s)ds}{\sqrt{2}})\psi_{2,0}(t)$$

$$+ B(\frac{3}{4})\psi_{2,2}(t). \tag{2.24}$$

Now, by using Eqs. (2.19)-(2.24), we have

$$\int_0^t \Psi(s)\circ dB(s) \simeq \underbrace{\begin{pmatrix} B(\frac{1}{4}) & B(\frac{1}{2}) & 0 & 0 & 0 & 0 \\ 0 & B(\frac{3}{4})-B(\frac{1}{2}) & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}\int_0^{\frac{1}{4}} 4\sqrt{6}B(s)ds & \frac{\sqrt{6}B(\frac{1}{2})-\int_0^{\frac{1}{2}} 4\sqrt{6}B(s)ds}{\sqrt{2}} & B(\frac{1}{4}) & 0 & 0 & 0 \\ 0 & \frac{\sqrt{6}B(\frac{1}{2})-\int_{\frac{1}{2}}^{\frac{3}{4}} 4\sqrt{6}B(s)ds}{\sqrt{2}} & 0 & B(\frac{3}{4}) & 0 & 0 \\ -\frac{\int_0^{\frac{1}{4}} \sqrt{10}(48s-12)B(s)ds}{\sqrt{2}} & \frac{\sqrt{10}B(\frac{1}{2})-\int_0^{\frac{1}{2}} \sqrt{10}(48s-12)B(s)ds}{\sqrt{2}} & 0 & 0 & B(\frac{1}{4}) & 0 \\ 0 & -\frac{\sqrt{10}B(\frac{1}{2})+\int_{\frac{1}{2}}^{\frac{3}{4}} \sqrt{10}(48s-36)B(s)ds}{\sqrt{2}} & 0 & 0 & 0 & B(\frac{3}{4}) \end{pmatrix}}_{P_s} \Psi(t).$$

In general, we get P_s as follows

$$P_s = \begin{pmatrix} P_1 & O & O & \cdots & O \\ C_1 & D & O & \cdots & O \\ C_2 & O & D & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{M-1} & O & O & \cdots & D \end{pmatrix}, \tag{2.25}$$

where O is a zero matrix of order $2^{k-1} \times 2^{k-1}$ and

$$D = \begin{pmatrix} B(\frac{1}{2^k}) & 0 & \cdots & 0 \\ 0 & B(\frac{3}{2^k}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B(\frac{2^k-1}{2^k}) \end{pmatrix},$$

and

$$P_1 = \begin{pmatrix} B(\frac{1}{2^k}) & B(\frac{2}{2^k}) & B(\frac{2}{2^k}) & \cdots & B(\frac{2}{2^k}) \\ 0 & B(\frac{3}{2^k}) - B(\frac{2}{2^k}) & B(\frac{4}{2^k}) - B(\frac{2}{2^k}) & \cdots & B(\frac{4}{2^k}) - B(\frac{2}{2^k}) \\ 0 & 0 & B(\frac{7}{2^k}) - B(\frac{6}{2^k}) & \cdots & B(\frac{8}{2^k}) - B(\frac{6}{2^k}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B(\frac{2^k-1}{2^k}) - B(\frac{2^k-2}{2^k}) \end{pmatrix},$$

and for $i = 1, 2, \ldots, M-1$,

$$C_i = \begin{pmatrix} \frac{-\int_0^{\frac{1}{2^k}} (\psi_{1,i}(t))' B(t) dt}{2^{\frac{k-1}{2}}} & \frac{\psi_{1,i}(\frac{2}{2^k}^+) B(\frac{2}{2^k}) - \int_0^{\frac{2}{2^k}} (\psi_{1,i}(t))' B(t) dt}{2^{\frac{k-1}{2}}} & \cdots & \frac{\psi_{1,i}(\frac{2}{2^k}^+) B(\frac{2}{2^k}) - \int_0^{\frac{2}{2^k}} (\psi_{1,i}(t))' B(t) dt}{2^{\frac{k-1}{2}}} \\ 0 & \frac{-\psi_{2,i}(\frac{2}{2^k}^+) B(\frac{2}{2^k}) - \int_{\frac{2}{2^k}}^{\frac{3}{2^k}} (\psi_{2,i}(t))' B(t) dt}{2^{\frac{k-1}{2}}} & \cdots & \frac{\psi_{2,i}(\frac{4}{2^k}^+) B(\frac{4}{2^k}) - \psi_{2,i}(\frac{2}{2^k}^+) B(\frac{2}{2^k}) - \int_{\frac{2}{2^k}}^{\frac{4}{2^k}} (\psi_{2,i}(t))' B(t) dt}{2^{\frac{k-1}{2}}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{-\psi_{n,i}(\frac{2^k-2}{2^k}^+) B(\frac{2^k-2}{2^k}) - \int_{\frac{2^k-2}{2^k}}^{\frac{2^k-1}{2^k}} (\psi_{n,i}(t))' B(t) dt}{2^{\frac{k-1}{2}}} \end{pmatrix},$$

where for $r = 1, 2, \dots, k$

$$\psi_{i,j}(\frac{2^r - 2}{2^r}^+) = \lim_{t \to \frac{2^r-2}{2^r}^+} \psi_{i,j}(t).$$

The Stratonovich integral of every arbitrary function f can be approximated as follows

$$\int_0^t f(s) \circ dB(s) \simeq \int_0^t C^T \Psi(s) \circ dB(s) \simeq C^T P_s \Psi(t). \tag{2.26}$$

3. Description of the proposed computational method

In this section, we consider nonlinear Stratonovich Volterra integral equation (1.1). For solving this equation, we apply operational matrices of integration based on Legendre wavelet. First, we let

$$z_1(t) = a(t, X(t)), \qquad z_2(t) = b(t, X(t)). \tag{3.1}$$

From Eqs. (1.1) and (3.1), we have

$$\begin{cases} z_1(t) = a\big(t, X_0 + \lambda_1 \int_0^t z_1(s)ds + \lambda_2 \int_0^t z_2(s) \circ dB(s)\big), \\ z_2(t) = b\big(t, X_0 + \lambda_1 \int_0^t z_1(s)ds + \lambda_2 \int_0^t z_2(s) \circ dB(s)\big). \end{cases} \tag{3.2}$$

We approximate $z_1(t)$ and $z_2(t)$ by applying Legendre wavelet as follows

$$z_1(t) \simeq A_1^T \Psi(t), \qquad z_2(t) \simeq A_2^T \Psi(t), \tag{3.3}$$

where $\Psi(t)$ defined in Eq. (2.4) and A_1 and A_2 are Legendre wavelet coefficient vectors of $z_1(t)$ and $z_2(t)$, respectively. Now, we substitute Eq. (3.3) into Eq. (3.2). We get

$$\begin{cases} A_1^T \Psi(t) = a\big(t, X_0 + \lambda_1 \int_0^t A_1^T \Psi(s)ds + \lambda_2 \int_0^t A_2^T \Psi(s) \circ dB(s)\big), \\ A_2^T \Psi(t) = b\big(t, X_0 + \lambda_1 \int_0^t A_1^T \Psi(s)ds + \lambda_2 \int_0^t A_2^T \Psi(s) \circ dB(s)\big). \end{cases}$$

From Eqs. (2.8) and (2.18), we get

$$\begin{cases} A_1^T \Psi(t) = a\big(t, X_0 + \lambda_1 A_1^T P \Psi(t) + \lambda_2 A_2^T P_s \Psi(t)\big), \\ A_2^T \Psi(t) = b\big(t, X_0 + \lambda_1 A_1^T P \Psi(t) + \lambda_2 A_2^T P_s \Psi(t)\big). \end{cases} \tag{3.4}$$

Now, we collocate Eq. (3.4) at $2^{k-1}M$ Newton-Cotes nodes defined as

$$t_i = \frac{2i-1}{2^k M}, \qquad i = 1, 2, \dots, 2^{k-1}M. \tag{3.5}$$

Then, the solution of nonlinear Stratonovich Volterra integral equation reduces to the solution of following nonlinear systems of algebraic equations

$$\begin{cases} A_1^T \Psi(t_i) = a\big(t_i, X_0 + \lambda_1 A_1^T P \Psi(t_i) + \lambda_2 A_2^T P_s \Psi(t_i)\big), \\ A_2^T \Psi(t_i) = b\big(t_i, X_0 + \lambda_1 A_1^T P \Psi(t_i) + \lambda_2 A_2^T P_s \Psi(t_i)\big). \end{cases} \tag{3.6}$$

This nonlinear system can be solved by using an appropriate numerical method such as Newton's method. After solving this nonlinear system and finding unknown vectors A_1 and A_2, the approximate solution of Eq. (1.1) can be obtained by

$$X(t) = X_0 + \lambda_1 A_1^T P \Psi(t) + \lambda_2 A_2^T P_s \Psi(t).$$

4. Convergence analysis and error estimate

Lemma 1. Let $X(t)$ be a continuous function defined on $[0,1)$ and $X^*(t)$ be the approximation of $X(t)$ by applying Legendre wavelet. Also, suppose that $X(t)$ is bounded by a positive constant η, i.e. $|X(t)| < \eta$. Then the Legendre wavelet coefficients of $X(t)$ are bounded as

$$|c_{n,m}| \leq \frac{\eta}{2^{\frac{k-2}{2}}}\sqrt{m+\frac{1}{2}}. \tag{4.1}$$

Proof. Any arbitrary function $X(t) \in L^2([0,1))$ can be approximated by using the Legendre wavelet as follows

$$X^*(t) = \sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}} c_{n,m}\psi_{n,m}(t), \tag{4.2}$$

where the coefficients $c_{n,m}$ can be determined as

$$\begin{aligned} c_{n,m} &= \langle X, \psi_{n,m}\rangle = \int_0^1 X(t)\psi_{n,m}(t)dt \\ &= 2^{\frac{k}{2}}\sqrt{m+\frac{1}{2}}\int_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}} X(t)P_m(2^k t - 2n + 1)dt. \end{aligned}$$

Now, by using change the variable $2^k t - 2n + 1 = u$, we have

$$c_{n,m} = \frac{1}{2^{\frac{k}{2}}}\sqrt{m+\frac{1}{2}}\int_{-1}^{1} X\Big(\frac{u+2n-1}{2^k}\Big)P_m(u)du. \tag{4.3}$$

By using assumption $|X(t)| < \eta$, we have

$$\begin{aligned} |c_{n,m}| &= \frac{1}{2^{\frac{k}{2}}}\sqrt{m+\frac{1}{2}}\Big|\int_{-1}^{1} X\Big(\frac{u+2n-1}{2^k}\Big)P_m(u)du\Big| \\ &\leq \frac{1}{2^{\frac{k}{2}}}\sqrt{m+\frac{1}{2}}\int_{-1}^{1}\Big|X\Big(\frac{u+2n-1}{2^k}\Big)\Big||P_m(u)|du \\ &\leq \frac{\eta}{2^{\frac{k}{2}}}\sqrt{m+\frac{1}{2}}\int_{-1}^{1}|P_m(u)|du. \end{aligned} \tag{4.4}$$

On the other hand, $|P_m(u)| \leq 1, \forall u \in [-1,1]$. So

$$\int_{-1}^{1}|P_m(u)|du \leq 2. \tag{4.5}$$

By using Eqs. (4.4) and (4.5), we conclude

$$|c_{n,m}| \leq \frac{\eta}{2^{\frac{k-2}{2}}}\sqrt{m+\frac{1}{2}}.$$

□

Theorem 4.1. Suppose $X(t)$ be a continuous function defined on $[0,1)$ and $X^*(t)$ be the approximation of $X(t)$ by using Legendre wavelet. Then we have the following upper bound of error

$$\|X(t)-X^*(t)\|_2 \leq \Big(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}(\alpha(m))^2\Big)^{\frac{1}{2}}+\Big(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}(\alpha(m))^2\Big)^{\frac{1}{2}}, \tag{4.6}$$

where

$$\alpha(m)=\frac{\eta}{2^{\frac{k-2}{2}}}\sqrt{m+\frac{1}{2}}. \tag{4.7}$$

Proof. An arbitrary function $X(t)\in L^2([0,1))$ can be expanded by using the Legendre wavelet as follows

$$X(t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}c_{n,m}\psi_{n,m}(t).$$

Suppose $X^*(t)$ be the truncated Legendre wavelet expansion defined in Eq. (4.2). Then the truncated error term can be calculated as

$$X(t)-X^*(t)=\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}c_{n,m}\psi_{n,m}(t)+\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}c_{n,m}\psi_{n,m}(t). \tag{4.8}$$

On the other hand, $\psi_{n,m}(t)$ have orthogonality property, i.e.,

$$\int_0^1\psi_{n,m}(t)\psi_{r,s}(t)dt=\begin{cases}0 & n\neq r, m\neq s,\\ 1 & n=r, m=s.\end{cases} \tag{4.9}$$

By applying Eqs. (4.8) and (4.9) and Lemma 1, conclude

$$\begin{aligned}\|X(t)-X^*(t)\|_2 &\leq \Big\|\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}c_{n,m}\psi_{n,m}(t)\Big\|_2+\Big\|\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}c_{n,m}\psi_{n,m}(t)\Big\|_2\\ &=\Big(\int_0^1\Big|\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}c_{n,m}\psi_{n,m}(t)\Big|^2dt\Big)^{\frac{1}{2}}\\ &+\Big(\int_0^1\Big|\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}c_{n,m}\psi_{n,m}(t)\Big|^2dt\Big)^{\frac{1}{2}}\\ &\leq\Big(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}|c_{n,m}|^2\Big)^{\frac{1}{2}}+\Big(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}|c_{n,m}|^2\Big)^{\frac{1}{2}}\\ &\leq\Big(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}(\alpha(m))^2\Big)^{\frac{1}{2}}+\Big(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}(\alpha(m))^2\Big)^{\frac{1}{2}}.\end{aligned}$$

□

Theorem 4.2. Let $X(t)$ and $X^*(t)$ be the exact solution and the approximate solution of Eq. (1.1), respectively. Moreover assume that functions $a(t,X(t))$ and $b(t,X(t))$, satisfy the Lipschitz condition, i.e. there is a constant L where,

$$\|a(t,X(t))-a(t,X^*(t))\|_2+\|b(t,X(t))-b(t,X^*(t))\|_2\leq L\|X(t)-X^*(t)\|_2. \tag{4.10}$$

Also, assume that

$$1-L\big(|\lambda_1|+|\lambda_2|\|B(t)\|_\infty\big)>0. \tag{4.11}$$

Then, the upper error bound would be obtained as follows

$$\|X(t)-X^*(t)\|_2 \leq \frac{|\lambda_1|\beta_1(m)+|\lambda_2|\|B(t)\|_\infty\beta_2(m)}{1-L\big(|\lambda_1|+|\lambda_2|\|B(t)\|_\infty\big)}, \tag{4.12}$$

where

$$\Big\|a\big(t,X^*(t)\big)-a^*\big(t,X^*(t)\big)\Big\|_2 \leq \beta_1(m),$$
$$\Big\|b\big(t,X^*(t)\big)-b^*\big(t,X^*(t)\big)\Big\|_2 \leq \beta_2(m),$$

and $\beta_1(m)$ and $\beta_2(m)$ are obtained from Theorem 4.1.

Proof. Suppose that $z_i(t)$ and $z_i^*(t)$ be the exact solution and the approximate solution of Eq. (3.2), respectively. Then we have

$$z_1^*(t)=a^*\big(t,X^*(t)\big), \qquad z_2^*(t)=b^*\big(t,X^*(t)\big), \tag{4.13}$$

also, we define

$$\hat{z}_1(t)=a\big(t,X^*(t)\big), \qquad \hat{z}_2(t)=b\big(t,X^*(t)\big). \tag{4.14}$$

According to Eq. (4.10) for $i=1,2$, we have

$$\|z_i(t)-z_i^*(t)\|_2 \leq \|z_i(t)-\hat{z}_i(t)\|_2+\|\hat{z}_i(t)-z_i^*(t)\|_2 \leq L\|X(t)-X^*(t)\|_2+\beta_i(m). \tag{4.15}$$

Furthermore,

$$X(t)=X_0+\lambda_1\int_0^t z_1(s)ds+\lambda_2\int_0^t z_2(s)\circ dB(s),$$
$$X^*(t)=X_0+\lambda_1\int_0^t z_1^*(s)ds+\lambda_2\int_0^t z_2^*(s)\circ dB(s).$$

So,

$$\begin{aligned}\|X(t)-X^*(t)\|_2 &\leq |\lambda_1|\|z_1(t)-z_1^*(t)\|_2+|\lambda_2|\|B(t)\|_\infty\|z_2(t)-z_2^*(t)\|_2\\ &\leq |\lambda_1|\big(L\|X(t)-X^*(t)\|_2+\beta_1(m)\big)\\ &\quad+|\lambda_2|\|B(t)\|_\infty\big(L\|X(t)-X^*(t)\|_2+\beta_2(m)\big).\end{aligned} \tag{4.16}$$

By applying Eq. (4.11) , we have

$$\|X(t)-X^*(t)\|_2 \leq \frac{|\lambda_1|\beta_1(m)+|\lambda_2|\|B(t)\|_\infty\beta_2(m)}{1-L\big(|\lambda_1|+|\lambda_2|\|B(t)\|_\infty\big)}.$$

□

Table 1. Numerical results of Example 5.1.

t_i	M=3,k=2			M=4,k=2		
	Exact	Approximate	Error	Exact	Approximate	Error
0.0	0.50000000	0.51681700	0.01681700	0.50000000	0.46406247	0.03593752
0.1	0.62485278	0.54137869	0.08347409	0.55214132	0.48587488	0.06626643
0.2	0.65756740	0.56548209	0.09208530	0.59216016	0.50767980	0.08448036
0.3	0.64050650	0.58912721	0.05137928	0.62277744	0.52939085	0.09338658
0.4	0.64277721	0.61231403	0.03046317	0.63674930	0.55092162	0.08582768
0.5	0.63484394	0.55022209	0.08462184	0.62597896	0.57471701	0.05126194
0.6	0.62069731	0.57115107	0.04954623	0.67281937	0.59534008	0.07747929
0.7	0.59722355	0.59165301	0.00557053	0.67511074	0.61554205	0.05956869
0.8	0.59606938	0.61172791	0.01565853	0.65874043	0.63526485	0.02347557
0.9	0.59013957	0.63137576	0.04123619	0.58853005	0.65445040	0.06592034

5. Illustrative test problems

In this section, we present two examples to show accuracy and efficiency of the proposed method. We compare the values of approximate solution and exact solution at the some selected points via definition of absolute error which defined as

$$e(t) = \Big|X(t) - X^*(t)\Big|, \qquad t \in [0, 1), \tag{5.1}$$

where $X(t)$ and $X^*(t)$ denote exact and approximate solution, respectively. Also, we compare this method with Block-pulse method in paper [15].

Example 5.1. Consider the following nonlinear Stratonovich Volterra integral equation [15]

$$X(t) = 0.5 + \int_0^t X(s)\big(0.96875 - X(s)\big)ds + \int_0^t 0.25X(s) \circ dB(s), \quad t \in [0, 1], \tag{5.2}$$

where $X(t)$ is unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, and $B(t)$ is a Brownian motion process. The exact solution of this equation is as follows

$$X(t) = \frac{0.5\exp\big(0.96875t + 0.25B(t)\big)}{1 + 0.5\int_0^t \exp\big(0.96875s + 0.25B(s)\big)ds}.$$

We report the values of exact and approximate solution at some points in Table 1. Also, in Table 2, maximum absolute error obtained from present method and Block-pulse method [15] is compared.

Example 5.2. Consider the following nonlinear Stratonovich Volterra integral equation [15]

$$X(t) = 1 + \int_0^t X^3(s)ds + \int_0^t 0.25X(s) \circ dB(s), \qquad t \in [0, 1], \tag{5.3}$$

where $X(t)$ is unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, and $B(t)$ is a Brownian motion process. The exact solution of this equation is as

Table 2. Comparison of maximum absolute error for Example 5.1.

Methods	E_m
Block-pulse method	
n=4	0.30146607
n=8	0.13059298
Present method	
M=3,k=2	0.09208530
M=4,k=2	0.09338658

Table 3. Numerical results of Example 5.2.

	M=3,k=2			M=4,k=2		
t_i	Exact	Approximate	Error	Exact	Approximate	Error
0.0	1.00000000	1.04920729	0.04920729	1.00000000	1.00985549	0.00985549
0.1	0.81605830	0.95384236	0.13778406	0.85943213	0.92004934	0.06061720
0.2	0.76247127	0.87360818	0.11113691	0.75674073	0.84927527	0.09253454
0.3	0.71941158	0.80850475	0.08909317	0.69162951	0.79312805	0.10149853
0.4	0.61832343	0.75853207	0.14020864	0.57737828	0.74720241	0.16982413
0.5	0.52399925	0.56544977	0.04145052	0.49015291	0.65113354	0.16098063
0.6	0.51062701	0.55109972	0.04047271	0.44120839	0.62670257	0.18549417
0.7	0.45304151	0.53764140	0.08459988	0.42197050	0.60477576	0.18280526
0.8	0.42205036	0.52507479	0.10302443	0.34060181	0.58501805	0.24441624
0.9	0.39641390	0.51339992	0.11698601	0.31846102	0.56709437	0.24863334

Table 4. Comparison of maximum absolute error for Example 5.2.

Methods	E_m
Block-pulse method	
n=4	0.22808556
n=8	0.30057406
Present method	
M=3,k=2	0.14020864
M=4,k=2	0.24863334

follows

$$X(t) = \frac{\exp\big(0.25B(t)\big)}{\sqrt{1 + 2\int_0^t \exp\big(0.5B(s)\big)ds}}.$$

We report the values of exact and approximate solution at some points in Table 3. Also, in Table 4, maximum absolute error obtained from present method and Block-pulse method [15] is compared.

6. Conclusion

There are many stochastic integral equations which can not be solved analytically. In recent decade, many researcher are trying to develop the numerical methods for solving stochastic integral equations such as Itô Volterra integral equation and Stratonovich Volterra integral equations. In this paper, we calculate operational matrix of integration and stochastic operational matrix of integration based on Legendre wavelet. By applying these matrices, the nonlinear Stratonovich Volterra integral equation reduces to nonlinear system of algebraic equations which can be solved by using Newton's method. Also, the error analysis of the present method were investigated. In Section 5, we solve two examples by using present technique and finally compare this method with Block-pulse method. Numerical results show present method is more accurate than Block-pulse method.

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