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On asymptotic stability of Weber fractional differential systems

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Abstract In this article, we introduce the fractional differential systems in the sense of the Weber fractional derivatives and study the asymptotic stability of these systems. We present the stability regions and then compare the stability regions of fractional differential systems with the Riemann-Liouville and Weber fractional derivatives.

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1. Introduction

Fractional differential equations (the differential equations with the derivative of arbitrary order) are generalizations of the classical differential equations of integer orders and are applicable tools for the modeling of many physical phenomena in physics, biology, fractional dynamics, engineering and control theory. With the developments of theory of fractional calculus the stability analysis of fractional differential systems have been the main point of view in many contributions [12, 16, 17, 20, 1, 3, 19].

In general, the stability analysis of fractional differential equations is related to the stability analysis of the following fractional differential system

$$\mathcal{D}_t^{\alpha} x(t) = A\, x(t), \quad x(0) = x_0, \quad 0 < \alpha \leq 1, \tag{1.1}$$

where $\mathcal{D}_t^{\alpha}$ is a fractional differential operator and $A \in \mathbb{R}^{n\times n}$ is a matrix. The stability analysis of the above fractional differential systems was studied for the first time by Matignon [12] in the sense of the Caputo derivative. Later, other researchers obtained the stability criteria of fractional systems containing other fractional derivatives such as the Riemann-Liouville, Hilfer and Prabhakar fractional derivatives [16, 17, 20, 1,

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3, 19, 4, 5, 6, 18]. In all contributions, the behaviors of eigenvalues of matrix A in the complex plane determine the fundamental criteria for stability analysis of these systems.

In this paper, we intend to introduce the fractional differential systems with respect to a new fractional derivative (*the Weber fractional derivative*) and investigate the asymptotic stability for these types of systems. Among other generalization of fractional derivatives [9, 2, 7, 8, 10, 15], the Weber fractional derivative is defined as a generalization of the Riemann-Liouville and Caputo fractional derivatives with respect to the Weber parabolic cylinder function [13]. We employ the Laplace transforms of the Weber fractional derivatives and use the asymptotic behavior of Weber parabolic cylinder function to study the asymptotic behavior of the solution with respect to the Jordan canonical forms of matrix A.

The rest of paper is organized as follows. In Section 2, some definitions and lemmas in fractional calculus are stated. In Section 3, we introduce the linear differential systems containing the Weber fractional derivative and treat the asymptotic stability analysis of these systems. In Section 4, we compare the stability aspects of Weber fractional differential systems with the Riemann-Liouville fractional differential systems.

2. Preliminaries

In this section, we recall some definitions and lemmas which are used in the next sections. First, we consider the following function

$$\mathbf{e}_\alpha(t) = \frac{1}{2^{\frac{1}{2}-\alpha}\sqrt{\pi}} t^{\alpha-1} e^{-\frac{\nu}{8t}} D_{1-2\alpha}\Big(\sqrt{\frac{\nu}{2t}}\Big), \quad \Re(\alpha) > 0, \Re(\nu) > 0, t > 0, \tag{2.1}$$

where D is the Weber parabolic cylinder function [13, p.448(46:6)]

$$D_{1-2\alpha}(x) = \sqrt{2^{1-2\alpha}\pi}\, e^{\frac{x^2}{4}} \sum_{j=0}^{\infty} \frac{(-x\sqrt{2})^j}{j!\Gamma(\frac{2\alpha-j}{2})}, \tag{2.2}$$

and the Laplace transform of this function is given by

$$\mathcal{L}\Big\{\frac{1}{2^{\frac{1}{2}-\alpha}\sqrt{\pi}} t^{\alpha-1} e^{-\frac{\nu}{8t}} D_{1-2\alpha}\Big(\sqrt{\frac{\nu}{2t}}\Big); s\Big\} = \frac{1}{s^\alpha} e^{-\sqrt{\nu s}}, \ \Re(\nu) > 0, \ s \in \mathbb{C}. \tag{2.3}$$

Definition 2.1. [11, 14]. For $0 < \alpha \le 1$ and $f \in L^1[0, b]$, $0 < t < b \le \infty$, the Riemann-Liouville fractional integral and derivative of the order α are defined as

$${}_{0^+}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \ t > 0, \quad 0 < \alpha \le 1, \tag{2.4}$$

$${}_{0^+}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(\tau)(t-\tau)^{-\alpha} d\tau, \ t > 0, \quad 0 < \alpha \le 1. \tag{2.5}$$

Also, for the absolutely continuous function f, the Caputo fractional derivatives of order α is defined as follows

$${}^{C}_{0^+}D_t^\alpha f(t) = {}_{0^+}I_t^{1-\alpha} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau. \tag{2.6}$$

Definition 2.2. For $m-1<\Re(\alpha)\leq m$ and function $f\in L^1[0,b]$, $0<t<b\leq\infty$, the Weber fractional integral is defined as follows

$$ {}_{0^+}\mathbf{E}_t^{\alpha}f(t)=\frac{1}{2^{\frac{1}{2}-\alpha}\sqrt{\pi}}\int_0^t(t-\tau)^{\alpha-1}e^{-\frac{\nu}{8(t-\tau)}}D_{1-2\alpha}(\sqrt{\frac{\nu}{2(t-\tau)}})f(\tau)d\tau. \tag{2.7}$$

Definition 2.3. For the function $f\in L^1[0,b]$, the Weber fractional derivative is defined as

$$ {}_{0^+}\mathbf{D}_t^{\alpha}f(t)=\frac{d^m}{dt^m}{}_{0^+}\mathbf{E}_t^{m-\alpha}f(t),\quad t>0, \tag{2.8}$$

where $m-1<\Re(\alpha)\leq m$.

Lemma 2.4. *The Laplace transform of the Weber fractional derivative for $m-1<\Re(\alpha)\leq m$ is given by*

$$\mathcal{L}\{{}_{0^+}\boldsymbol{D}_t^{\alpha}f(t);s\}=s^{\alpha}e^{-\sqrt{\nu s}}F(s)-\sum_{k=0}^{m-1}s^{m-k-1}\boldsymbol{D}_t^{\alpha-m+k}f(t)\mid_{t=0}, \tag{2.9}$$

or

$$\mathcal{L}\{{}_{0^+}\boldsymbol{D}_t^{\alpha}f(t);s\}=s^{\alpha}e^{-\sqrt{\nu s}}F(s)-\sum_{k=0}^{m-1}s^{m-k-1}\frac{d^k}{dt^k}\boldsymbol{E}_t^{m-\alpha}f(t)\mid_{t=0}, \tag{2.10}$$

where $D=\frac{d}{dx}$ and $F(s)$ is the Laplace transform of $f(t)$

$$F(s)=\int_0^{\infty}e^{-st}f(t)dt. \tag{2.11}$$

Proof. Applying the Laplace transform on the right hand side of relation (2.8), we get

$$\begin{aligned}\mathcal{L}\{{}_{0^+}\mathbf{D}_t^{\alpha}f(t);s\}&=\mathcal{L}\{\frac{d^m}{dt^m}{}_{0^+}\mathbf{E}_t^{m-\alpha}f(t);s\}\\&=s^m\mathcal{L}\{{}_{0^+}\mathbf{E}_t^{m-\alpha}f(t);s\}-s^{m-1}{}_{0^+}\mathbf{E}_t^{m-\alpha}f(t)\mid_{t=0}\\&-s^{m-2}\frac{d}{dt}\mathbf{E}_t^{m-\alpha}f(t)\mid_{t=0}\cdots-\frac{d^{m-1}}{dt^{m-1}}\mathbf{E}_t^{m-\alpha}f(t)\mid_{t=0}.\end{aligned} \tag{2.12}$$

We now use the following fact for the Laplace transform of fractional Weber integral

$$\mathcal{L}\{{}_{0^+}\mathbf{E}_t^{m-\alpha}f(t);s\}=\frac{1}{s^{m-\alpha}}e^{-\sqrt{\nu s}}F(s), \tag{2.13}$$

and substitute the relation (2.13) into (2.12), to obtain the claimed result. □

Definition 2.5. For $m-1<\Re(\alpha)\leq m$ and function $f\in L^1[0,b]$, $0<t<b\leq\infty$, the Caputo-Weber fractional derivative of order α which is a generalization of the Caputo fractional derivative is defined as

$$ {}_{0^+}^{C}\mathbf{D}_t^{\alpha}f(t)=\mathbf{E}_{0^+}^{m-\alpha}\frac{d^m}{dt^m}f(t), \tag{2.14}$$

or equivalently

$$ {}_{0^+}^{C}\mathbf{D}_t^{\alpha} f(t) = {}_{0^+}\mathbf{D}_t^{\alpha} f(t) - \sum_{k=0}^{m-1} \frac{1}{2^{\frac{1}{2}-(k-\alpha+1)}\sqrt{\pi}} t^{k-\alpha} e^{-\frac{\nu}{8t}} D_{1-2(k-\alpha+1)}(\sqrt{\frac{\nu}{2t}}). \tag{2.15} $$

Lemma 2.6. *The asymptotic expansion of the parabolic cylinder function is given by* [13, p. 449(46:6:6)]

$$ \begin{aligned} D_\nu(x) &\simeq x^\nu e^{-\frac{x^2}{4}} \Big[1 - \frac{(-\nu)(1-\nu)}{2x^2} + \frac{(-\nu)(1-\nu)(2-\nu)(3-\nu)}{2!(2x^2)^2} \\ &\quad - \cdots + \frac{(-\nu)_{2j}}{j!(-2x^2)^j} + \cdots \Big] \\ &= x^\nu e^{-\frac{x^2}{4}} \sum_{j=0}^{\infty} \frac{(-\nu)_{2j}}{j!(-2x^2)^j}, \ x \to \infty, \end{aligned} \tag{2.16} $$

where $(-\nu)_{2j} = -\nu(1-\nu)\cdots(-\nu+2j-1)$. *Also, for small* x *we have* [13, p. 450(46:9:1)]

$$ D_\nu(x) \simeq \sqrt{2^\nu \pi}\Big[\frac{1-\left(\frac{1}{4}+\frac{\nu}{2}\right)x^2}{\Gamma(\frac{1-\nu}{2})} - \frac{x\sqrt{2}}{\Gamma(-\frac{\nu}{2})}\Big]. \tag{2.17} $$

3. Asymptotic Stability Analysis of Linear Autonomous Weber Fractional Differential Systems

3.1. **Main theorem.** In this section, we introduce the linear autonomous Weber fractional differential systems and treat the asymptotic stability of these systems. We consider the following fractional system

$$ \begin{aligned} &{}_{0^+}\mathbf{D}_t^{\alpha} x(t) = A\, x(t), \ t > 0, \ 0 < \alpha \le 1, \\ &{}_{0^+}\mathbf{E}_t^{1-\alpha} x(0^+) = x_0, \end{aligned} \tag{3.1} $$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is matrix, $x_0 = (x_{10}, x_{20}, \cdots, x_{n0})$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ such that $0 < \alpha_i < 1$ for $i = 1, 2, \cdots, n$.

Definition 3.1. The system (3.1) is called commensurate order system if

$$ \alpha_1 = \alpha_2 = \cdots = \alpha_n. $$

Definition 3.2. The Weber fractional derivatives system (3.1)

- **i):** is stable if for any initial value x_0 and $t > 0$, i.e., there exists an $\epsilon > 0$ such that $\|x(t)\| < \epsilon$.
- **ii):** is asymptotically stable if it is stable and $\lim_{t\to\infty} \|x(t)\| = 0$.

We discuss the asymptotic stability of system (3.1) when A has non-zero eigenvalues.

Theorem 3.3. *The solution of the linear commensurate order system (3.1) is given by*

$$x(t) = -\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} 2^{-n\alpha} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} D_{1+2n\alpha}\Big(\sqrt{\frac{n^2\nu}{2t}}\Big)(A^{-1})^{n+1} x_0. \tag{3.2}$$

Proof: Since A is an invertible matrix, then we have $A^{-1}\,{}_{0^+}\mathbf{D}_t^{\alpha} x(t) = x(t)$. We apply the Laplace transform on both sides of (3.1) and use the relation (2.10) to get

$$A^{-1}\Big\{s^{\alpha} e^{-\sqrt{\nu s}} X(s) - x_0\Big\} = X(s). \tag{3.3}$$

We write the Taylor expansion of $X(s)$ with the condition $||A^{-1}\, s^{\alpha} e^{-\sqrt{\nu s}}|| < 1$, to obtain

$$\begin{aligned} X(s) &= -\frac{A^{-1}x_0}{I - A^{-1}\, s^{\alpha} e^{-\sqrt{\nu s}}} = -A^{-1}x_0\big(I - A^{-1}\, s^{\alpha} e^{-\sqrt{\nu s}}\big)^{-1} \\ &= -A^{-1}x_0 \sum_{n=0}^{\infty} (A^{-1} s^{\alpha} e^{-\sqrt{\nu s}})^n = \sum_{n=0}^{\infty} (A^{-1})^{n+1} s^{n\alpha} e^{-n\sqrt{\nu s}} x_0. \end{aligned} \tag{3.4}$$

We now consider the relation (2.3) for the inverse Laplace transform of the above expression in terms of the parabolic cylinder functions

$$x(t) = -\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} 2^{-n\alpha} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} D_{1+2n\alpha}\Big(\sqrt{\frac{n^2\nu}{2t}}\Big)(A^{-1})^{n+1} x_0. \tag{3.5}$$

Hence, the proof is completed.

Theorem 3.4. *If all the eigenvalues of A ($\lambda(A)$) satisfy the following condition for any $r > 0$ and $\nu > 0$*

$$|\arg(\lambda(A))| > \frac{\pi}{2}\alpha - \frac{1}{2}\sqrt{2r\nu}, \tag{3.6}$$

then the solution of system (3.1) is asymptotically stable.

Proof: It is well known that the linear differential system

$$x'(t) = Ax(t), \quad x(0) = x_0,$$

is asymptotically stable if and only if all roots of $\det(sI - A) = 0$ have negative real parts. So the system (3.1) is asymptotically stable if and only if all roots of $\det(s^{\alpha} e^{-\sqrt{\nu s}} I - A) = 0$ have negative real parts. In this sense, we apply the transformation $W(s) = s^{\alpha} e^{-\sqrt{\nu s}}$ that maps the region $\arg(\lambda(s)) > \frac{\pi}{2}$ onto desired stability region. For this purpose, we choose the boundaries $s = re^{\pm i\frac{\pi}{2}}$ of region

$$R := \{s \in \mathbb{C} |\, \arg(\lambda(s)) > \frac{\pi}{2}\},$$

and use the mapping function $W(s)$ to find the necessary condition of asymptotic stability. Therefore, we construct the mapped region as

$$|\arg(\lambda(A))| > \arg\Big[s^{\alpha} e^{-\sqrt{\nu s}}\Big]_{s=re^{i\frac{\pi}{2}}}, \tag{3.7}$$

which implies that

$$|\arg(\lambda(A))| > \frac{\pi}{2}\alpha - \frac{1}{2}\sqrt{2r\nu}. \tag{3.8}$$

We now discuss the asymptotic stability of system (3.1) in two cases as follows.

Case 1: Suppose that the matrix A^{-1} is diagonalizable and J has the Jordan canonical form of the matrix A^{-1} such that $J = P^{-1}A^{-1}P = \mathrm{diag}(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n})$ where P is an invertible matrix and λ_i, $i = 1, 2, \cdots, n$, are the eigenvalues of A. In this case, we have

$$\left(A^{-1}\right)^{n+1} = (PJP^{-1})^{n+1} = PJ^{n+1}P^{-1} = P(\mathrm{diag}(\frac{1}{\lambda_1^{n+1}}, \frac{1}{\lambda_2^{n+1}}, \cdots, \frac{1}{\lambda_n^{n+1}}))P^{-1}. \tag{3.9}$$

So, by applying relation (2.17) for the solution (3.2), we have

$$\begin{aligned} x(t) &= -\sum_{n=0}^{\infty} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} \Big[\frac{1 - \left(\frac{3}{4} + n\alpha\right)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)} - \frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big] \frac{1}{\lambda_i^{n+1}} x_0 \to 0, \\ &\quad t \to \infty,\ 1 \le i \le n. \end{aligned} \tag{3.10}$$

Hence for the value

$$S = \sum_{n=0}^{\infty} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} \Big[\frac{1 - \left(\frac{3}{4} + n\alpha\right)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)} - \frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big] \mathrm{diag}(\frac{1}{\lambda_1^{n+1}}, \frac{1}{\lambda_2^{n+1}}, \cdots, \frac{1}{\lambda_n^{n+1}}) x_0$$

we get

$$\lim_{t\to\infty} \|S\| = 0, \tag{3.11}$$

and consequently

$$\begin{aligned} \lim_{t\to\infty} \|x(t)\| &= \lim_{t\to\infty} \|\sum_{n=0}^{\infty} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} \Big[\frac{1 - \left(\frac{3}{4} + n\alpha\right)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)} - \frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big] (A^{-1})^{n+1} x_0\| \\ &= \lim_{t\to\infty} \|P(\sum_{n=0}^{\infty} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} \Big[\frac{1 - \left(\frac{3}{4} + n\alpha\right)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)} - \frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big] J^{n+1} x_0) P^{-1}\| = 0. \end{aligned} \tag{3.12}$$

Case 2: Suppose that the matrix A^{-1} has a Jordan canonical form $J = (J_1, J_2, \cdots, J_s)$, where $J_i, 1 \leq i \leq s$, is given by

$$J_i = \begin{pmatrix} \frac{1}{\lambda_i} & 1 & & & \\ & \frac{1}{\lambda_i} & 1 & & \\ & & \ddots & \ddots & \\ & & & \frac{1}{\lambda_i} & 1 \\ & & & & \frac{1}{\lambda_i} \end{pmatrix}_{n_i \times n_i}, \lambda_i \in \mathbb{C}, \tag{3.13}$$

and $\sum_{i=1}^{s} n_i = n$. So, we have

$$(A^{-1})^{n+1} = (PJP^{-1})^{n+1} = PJ^{n+1}P^{-1} = P(\mathrm{diag}(J_1^{n+1}, J_2^{n+1}, \cdots, J_h^{n+1}))P^{-1}, \tag{3.14}$$

and for the solution (3.2), we obtain

$$\begin{aligned} x(t) &= -\sum_{n=0}^{\infty} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} \Big[\frac{1-\left(\frac{3}{4}+n\alpha\right)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)} - \frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big] J_i^{n+1} x_0 \\ &= -\sum_{n=0}^{\infty} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} \Big[\frac{1-\left(\frac{3}{4}+n\alpha\right)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)} - \frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big] \\ &\times \begin{pmatrix} \frac{1}{\lambda_i^{n+1}} & D_{n+1}^{1}\frac{1}{\lambda_i^{n}} & \cdots & D_{n+1}^{n_i-1}\frac{1}{\lambda_i^{n-n_i+2}} \\ 0 & \frac{1}{\lambda_i^{n+1}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & D_{n+1}^{1}\frac{1}{\lambda_i^{n}} \\ 0 & \cdots & 0 & \frac{1}{\lambda_i^{n+1}} \end{pmatrix} x_0, \end{aligned} \tag{3.15}$$

where $D_k^j, 1 \leq j \leq n_i$, are the binomial coefficients such that

$$D_k^j = \begin{pmatrix} k \\ j \end{pmatrix} = \begin{cases} \frac{k!}{j!(k-j)!} & 0 \leq j \leq k, \\ 0 & \text{etc.} \end{cases}$$

If we consider the non-zero elements of the above matrix and apply the relation (2.17) once again, then for solution (3.2) we obtain

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}} \Big[\frac{1-\left(\frac{3}{4}+n\alpha\right)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)} - \frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big] \\ &\times \frac{(-\lambda)^j}{(j-1)!}\Big\{\Big(\frac{\partial}{\partial\lambda}\Big)^{j-1}\Big(\frac{1}{\lambda}\Big)^{n-j+3}\Big\} |_{\lambda=\lambda_i}\; x_0, \\ & j = 1, 2, \cdots, n_i. \end{aligned} \tag{3.16}$$

which implies that for $t \to \infty$

$$|\sum_{n=0}^{\infty} t^{-n\alpha-1} e^{-\frac{n^2\nu}{8t}}$$

$$\times\Big[\frac{1-\big(\frac{3}{4}+n\alpha\big)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)}-\frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big]\frac{(-\lambda_i)^j}{(j-1)!}\Big\{\Big(\frac{\partial}{\partial\lambda_i}\Big)^{j-1}\Big(\frac{1}{\lambda_i}\Big)^{n-j+3}\Big\}x_0|\to 0. \tag{3.17}$$

Consequently

$$\lim_{t\to\infty}\|x(t)\|=\lim_{t\to\infty}\|\sum_{n=0}^{\infty}t^{-n\alpha-1}e^{-\frac{n^2\nu}{8t}}\Big[\frac{1-\big(\frac{3}{4}+n\alpha\big)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)}-\frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big](A^{-1})^{n+1}x_0\|$$

$$=\lim_{t\to\infty}\|P(\sum_{n=0}^{\infty}t^{-n\alpha-1}e^{-\frac{n^2\nu}{8t}}\Big[\frac{1-\big(\frac{3}{4}+n\alpha\big)\frac{n^2\nu}{2t}}{\Gamma(-n\alpha)}-\frac{\sqrt{\frac{n^2\nu}{t}}}{\Gamma(\frac{-1-2n\alpha}{2})}\Big]J^{n+1}x_0)P^{-1}\|=0. \tag{3.18}$$

4. Comparison with Riemann-Liouville fractional differential systems

It is obvious that for $\nu = 0$, the Weber fractional integral (2.1) coincides with the Riemann-Liouville fractional integral of order α. In this case, the stability region of fractional systems with the Riemann-Liouville fractional derivative can be plotted by the following condition

$$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}. \tag{4.1}$$

The stability regions of the Weber and Riemann-Liouville fractional derivatives differ by the term $\frac{\pi}{2}\alpha - \frac{1}{2}\sqrt{2r\nu}$ which is plotted in Figure 2.

Remark 4.1. The shaded region in Figure 2 shows that the fractional differential systems with the Riemann-Liouville and Weber derivatives have not the same stability for order $0 < \alpha \leq 1$. It means that, for a determined parameter α, the fractional differential system with the Riemann-Liouville derivative is asymptotically stable, but the associated fractional differential system with the Weber derivative is unstable.

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FIGURE 1. The asymptotic stability regions of system (3.1) for parameters $\alpha = 0.25, 0.5, 0.75, 0.95, 1$ and $\nu = 1$.

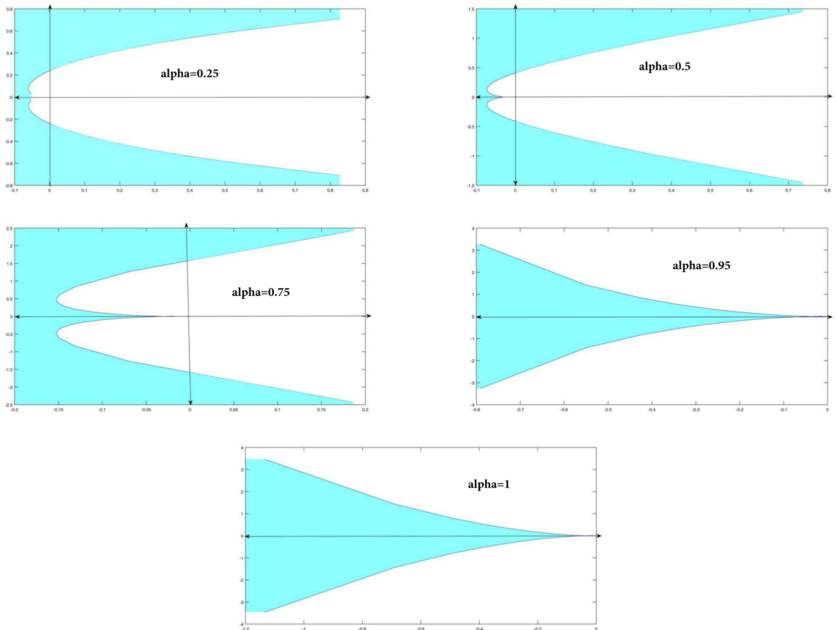
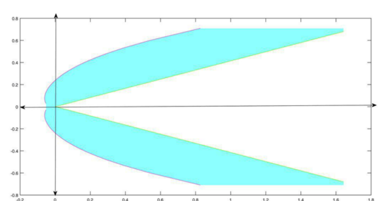


FIGURE 2. The difference region between the fractional differential systems with the Riemann-Liouville and Weber derivatives of order $\alpha = 0.25$, $\nu = 1$.



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