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Inverse nodal problem for p-Laplacian with two potential functions

Abdol Hadi Dabbaghian
Department of Mathematics, Neka Branch,
Islamic Azad University, Neka, Iran.
E-mail: a.dabbaghian@iauneka.ac.ir

Abstract In this study, inverse nodal problem is solved for the p-Laplacian operator with two potential functions. We present some asymptotic formulas which have been proved in [17,18] for the eigenvalues, nodal points and nodal lengths, provided that a potential function is unknown. Then, using the nodal points we reconstruct the potential function and its derivatives. We also introduce a solution of inverse nodal problem when the two potential functions are unknown.

Keywords. Prüfer substitution; Inverse nodal problem; P-Laplacian operator.

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1. Introduction

In a typical inverse nodal problem, the operators are to be determined from the given nodes(zeros) of their eigenfunctions. Mclaughlin seems to be the first to consider this sort of inverse problem. This kind of problem is called inverse nodal problem. By using nodal points which are zeros of eigenfunctions, she [23] showed that potential function can be determined uniquely for the Sturm-Liouville problem. Especially, inverse nodal problems were highly studied by many authors [4,6,7,19,25,26,28].
On the other hand, there are many studies of nonlinear operators such as the p-Laplacian $\Delta_p y = div(|\Delta y|^{p-2}\Delta y)$ with $p > 1$ on a bounded domain $\Omega \subseteq R^n$. The equation with p-Laplacian operator arises in some modeling of different physical events; such as non-Newtonian mechanics [10,15], nonlinear elasticity and glaciology [14], population biology [24], nonlinear flow laws [14], petroleum extraction [9].
Consider the following p-Laplacian equation

$$-\Delta_p u + q|u|^{p-2}u = \lambda|u|^{p-2}u,$$

with the condition

$$u|_{\partial\Omega} = 0,$$

where $p > 1$, $q \in L^2(\Omega)$ and $\Omega \subseteq R^N$. It shows a quasi-linear equation, when $p \neq 2$. In linear and nonlinear case, two problems have the similiar properties when $p = 2$. For examples, the Sturm-Liouville theory for p-Laplacian [2], the comparison theorem for p-Laplacian [26] in one dimension, and a type of the Courant nodal domain theorem

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($q \equiv 0$) [11] in high dimensional case also hold. But, two-Laplacian case is not the same completely. For example, in one dimension the Fredholm alternative may not hold [8]. Also, for a periodic boundary condition, there exist some non-variational eigenvalues [1] and the multiplicity of the periodic eigenvalue can be arbitrary [3].
For the one-dimensional case and $\Omega = (0,1)$, after some changing the problem turns into the following problem

$$-[(u')^{(p-1)}]' = (p-1)(\lambda - q(x))u^{(p-1)}, \qquad u(0) = u(1) = 0, \tag{1.1}$$

where $u^{(p-1)} := |u|^{p-1}Sgn(u)$.
Inverse problems of the problem (1.1) have been investigated by several authors [1,5,8,20,25,27].
For $q = 0$, let us consider

$$\begin{aligned} -[(u')^{(p-1)}]' &= (p-1)\lambda u^{(p-1)}, \\ u(0) &= u(1) = 0. \end{aligned}$$

In case of $q = 0$, the eigenvalues of the problem were [20]

$$\lambda_n = (n\pi_p)^p, n = 1, 2, 3, ...,$$

where

$$\pi_p = 2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}} = \frac{2\pi}{p\sin(\frac{\pi}{p})}.$$

Its associated eigenfunction is denoted by $S_p(x)$. This function and its derivative $S_p'(x)$ are periodic functions satisfying

$$[S_p(x)]^p + [S_p'(x)]^p = 1,$$

for arbitrary $x \in R$. Most of properties of the functions S_p and S_p' are p-similiar to *sine* and *cosine* functions for $p = 2$ [22].
The following Lemma, is about some properties of S_p.

Lemma 1.1. *([20])*

$$(a) For S_p' \neq 0, \qquad (S_p')' = -|\frac{S_p}{S_p'}|^{p-2}S_p;$$

$$(b)(S_pS_p'^{(p-1)})' = |S_p'|^p - (p-1)|S_p|^p = 1 - p|S_p|^p = (1-p) + p|S_p'|^p.$$

Consider the p-Laplacian eigenvalue problem

$$-[(u')^{(p-1)}]' = (p-1)(\lambda^2 - q(x) - 2\lambda r(x))u^{(p-1)}, \qquad 0 < x < 1, \tag{1.2}$$

with the Dirichlet conditions

$$u(0) = u(1) = 0, \tag{1.3}$$

where $q \in L^2(0,1)$ and $r \in W_2^1(0,1)$ are real functions.
For $p = 2$, equation (1.2) becomes

$$-u'' + [q + 2\lambda r]u = \lambda^2 u, \tag{1.4}$$

which is known as diffusion equation in the spectral theory. Equation (1.4) is so crucial in quantum theory. For instance, such problems arise in solving the Klein-Gordon equations defining the motion of massless particles such as photons. Inverse diffusion problems have been considered and obtained some new results such as inverse nodal problem and trace formulas in [12,16,19,29,30].
In [17], the author considered the problem (1.2)-(1.3). It is assumed that r is known as priori and he has reconstructed the unknown function q by zeros of eigenfunctions i.e. [17, theorem 3.1]. In this paper, we plan to get derivatives of q. In addition, we consider the inverse nodal problem for (1.2) with two potential functions q and r unknown.
This paper is structured as follows: In section 2, we present some preliminary results and notations that have been studied in [17] which will be useful in the sequel. Furthermore, in this section Theorem 2.6 will be proved. In section 3, we reconstruct the derivatives of q and also determine the smoothness of the potential function. Finally in section 4, we prove the uniqueness of recovering the functions $q(x)$, $r(x)$ from a dense set of nodal points and obtain a constructive procedure for solving the inverse nodal problem.

2. Preliminaries

In this section, we present some preliminary results and notations that have been studied in [17,18]. Furthermore, we give a formula for $q \in L^1(0,1)$. considering λ_n be the nth eigenvalue and $0 < x_1^{(n)} < ... < x_{n-1}^{(n)} < 1$ as the nodes of the nth eigenfunction u_n. The set $X := \{x_j^{(n)}\}_{n\geq 1, j=\overline{1,n-1}}$ is called the set of nodal points of (1.2)-(1.3) and $l_j^n = x_{j+1}^{(n)} - x_j^{(n)}$ is defined as the nodal length of u_n. Also, let $x_0^{(n)} = 0$ and $x_n^{(n)} = 1$. Furthermore, we take the function $j_n(x)$ by $j_n(x) = max\{i : x_i^{(n)} \leq x\}$.
Define a modified Prüfer substitution

$$u(x) = c(x) S_p(\lambda^{\frac{2}{p}} \theta(x)),$$

$$u'(x) = \lambda^{\frac{2}{p}} c(x) S'_p(\lambda^{\frac{2}{p}} \theta(x)),$$

or

$$\frac{u'(x)}{u(x)} = \lambda^{\frac{2}{p}} \frac{S'_p(\lambda^{\frac{2}{p}} \theta(x))}{S_p(\lambda^{\frac{2}{p}} \theta(x))}. \tag{2.1}$$

Differentiating both side of (2.1) for x and by using Lemma 1.1, we can obtain easily

$$\theta' = 1 - \frac{q}{\lambda^2} S_p^p - \frac{2}{\lambda} r S_p^p. \tag{2.2}$$

Then we may obtain a detailed asymptotic of the eigenvalue $\lambda^{\frac{2}{p}}$ [17,18].

Theorem 2.1. *([18, Theorem2.1]) The eigenvalues λ_n of the Dirichlet problem given in (1.2), (1.3) have the from*

$$\lambda^{\frac{2}{p}} = n\pi_p + \frac{1}{p(n\pi_p)^{p-1}} \int_0^1 q(t)dt + \frac{2}{p(n\pi_p)^{\frac{p-2}{p}}} \int_0^1 r(t)dt + O(\frac{1}{n^{\frac{p}{2}}}). \tag{2.3}$$

as $n \to \infty$.

Based on Theorem 2.1 the estimate about the nodal points is given in the following theorem.

Theorem 2.2. *([18,Theorem2.2]) For problem (1.2), (1.3) the nodal points expansion satisfies*

$$x_j^n = \frac{j}{n} + \frac{j}{pn^{p+1}(\pi_p)^p}\int_0^1 q(t)dt + \frac{2j}{pn^{\frac{p}{2}+1}(\pi_p)^{\frac{p}{2}}}\int_0^1 r(t)dt + \frac{2}{(n\pi_p)^{\frac{p}{2}}}\int_0^{x_j^n} r(x)S_p^p dx$$
$$+ \frac{1}{(n\pi_p)^p}\int_0^{x_j^n} q(x)S_p^p dx + O(\frac{1}{n^{\frac{p}{2}+2}}) \tag{2.4}$$

as $n \to \infty$.

Moreover, we have the following asymptotic formula for the nodal length.

Theorem 2.3. *([17,Theorem2.3]) For problem (1.2), (1.3) the nodal length expansion satisfies*

$$l_j^n = \frac{\pi_p}{\lambda_n^{\frac{2}{p}}} + \frac{2}{p\lambda_n}\int_{x_j^n}^{x_{j+1}^n} r(t)dt + \frac{1}{p\lambda_n^2}\int_{x_j^n}^{x_{j+1}^n} q(t)dt + O(\frac{1}{\lambda_n^{1+\frac{4}{p}}}). \tag{2.5}$$

Theorem 2.4. *([17,Theorem3.1]) Let* $q \in L^2(0,1)$, $r \in W_2^1(0,1)$ *and assume* r *that on the interval [0,1] is given a priori. Then*

$$q(x) = \lim_{n\to\infty} p\lambda_n^2(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - \frac{2r(x)}{p\lambda_n} - 1). \tag{2.6}$$

Example 2.5. *We consider the initial value problem (1.2)-(1.3) for the special case of* $q(x) = x$ *and* $r(x) = \frac{1}{\sqrt{x}}$ *i.e.*

$$-[(u')^{(p-1)}]' = (p-1)(\lambda^2 - x - 2\lambda\frac{1}{\sqrt{x}})u^{(p-1)}, \qquad 0 < x < 1,$$

We can obtain that

$$\theta' = 1 - \frac{x}{\lambda^2}S_p^p - \frac{2}{\lambda}\frac{1}{\sqrt{x}}S_p^p.$$

Hence, asymptotic estimate of l_j^n is as following:

$$l_j^n = \frac{\pi_p}{\lambda_n^{\frac{2}{p}}} + \frac{2}{p\lambda_n}\int_{x_j^n}^{x_{j+1}^n} \frac{1}{\sqrt{t}}dt + \frac{1}{p\lambda_n^2}\int_{x_j^n}^{x_{j+1}^n} tdt + O(\frac{1}{n^{1+\frac{4}{p}}}).$$

Conversely, if we consider $r(x) = \frac{1}{\sqrt{x}}$ and use asymptotic formula of l_j^n as in Example 2.5, we get

$$\lim_{n\to\infty} p\lambda_n^2(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - \frac{2r(x)}{p\lambda_n} - 1)$$

$$= \lim_{n\to\infty} p\lambda_n^2 \Big(\frac{\lambda_n^{\frac{2}{p}}}{\pi_p}\Big[\frac{\pi_p}{\lambda_n^{\frac{2}{p}}} + \frac{2}{p\lambda_n}\int_{x_j^n}^{x_{j+1}^n} \frac{1}{\sqrt{t}}dt + \frac{1}{p\lambda_n^2}\int_{x_j^n}^{x_{j+1}^n} tdt\Big] - \frac{2}{p\lambda_n\sqrt{x}} - 1\Big)$$

$$= x = q(x).$$

Thus, we have reconstructed the potential function $q(x)$ using nodal data.$\diamond$
Define the function $F_n(x)$ by

$$F_n(x) := p(n\pi_p)^p(nl_j^n - 1) + \int_0^1 q(t)dt. \tag{2.7}$$

We show that $F_n(x)$ is in fact a direct approximation of $q \in L^1(0,1)$.

Theorem 2.6. *Let $q \in L^1(0,1)$ and $r \in W_2^1(0,1)$, For the Dirichlet problem (1.2),(1.3). Then, F_n converges to q pointwisely and in $L^1(0,1)$.*

Proof. By the eigenvalue estimates (2.3), we have

$$p\lambda_n^2\Big(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - \frac{2r(x)}{p\lambda_n} - 1\Big) = \lambda_n\Big(\frac{p\lambda_n\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - 2r(x) - p\lambda_n\Big) \tag{2.8}$$

$$= (n\pi)^{\frac{p}{2}}\big(p(n\pi)^{\frac{p}{2}} nl_j^n - 2r(x) - p(n\pi)^{\frac{p}{2}}\big) + O(1).$$

Hence, applying Theorem 2.4, $F_n(x)$ converges to q pointwisely and in $L^1(0,1)$.$\diamond$

3. Reconstructing the derivatives of the potential function

The interval [0,1] is divided by the nodal points x_j^n, $j = 1, 2, ..., n-1$, into n subintervals. Denote $I_j^n = [x_j^n, x_{j+1}^n]$ be the jth nodal domain of the nth eigenvalue. Suppose $q \in [0,1]$, so both $q_{m,j} = min\{q(x) : x \in I_j^n\}$ and $q_{M,j} = max\{q(x) : x \in I_j^n\}$ exist. Let Δ denotes the difference operator $\Delta a_i = a_{i+1} - a_i$. For $k > 1$, $\Delta^k a_i = \Delta^k a_{i+1} - \Delta^k a_i$, and define δ:

$$\delta a_i = \frac{a_{i+1} - a_i}{x_{i+1} - x_i} = \frac{\Delta a_i}{l_i}, \qquad \delta^k a_i = \frac{\delta^{k-1} a_{i+1} - \delta^{k-1} a_i}{l_i}.$$

Consider two equations

$$-[(u')^{(p-1)}]' = (p-1)(\lambda_n^2 - q_{m,j}(x) - 2\lambda_n r(x))u^{(p-1)},$$

$$-[(u')^{(p-1)}]' = (p-1)(\lambda_n^2 - q_{M,j}(x) - 2\lambda_n r(x))u^{(p-1)}.$$

So fix x in [0,1], for any n, exists a subinterval I_j^n such that $x \in I_j^n$,

$$\lambda_n^2 - q_{M,j} \le \lambda_n^2 - q(x) \le \lambda_n^2 - q_{m,j}.$$

By sturm comparison theorem, the nodal length of the equation

$$-[(u')^{(p-1)}]' = (p-1)(\lambda^2 - q(x) - 2\lambda_n r(x))u^{(p-1)},$$

lies between that for the other two equations i.e.

$$\frac{\pi_p}{\sqrt[p]{\lambda_n^2 - q_{m,j}}} \le l_j^n \le \frac{\pi_p}{\sqrt[p]{\lambda_n^2 - q_{M,j}}}.$$

Hence, we have

$$\frac{\pi_p}{\sqrt[p]{1-\frac{q_{m,j}}{\lambda_n^2}}} \leq l_j^n \lambda_n^{\frac{2}{p}} \leq \frac{\pi_p}{\sqrt[p]{1-\frac{q_{M,j}}{\lambda_n^2}}}. \tag{3.1}$$

Proposition 3.1. *If q is continuous function, then*
(a) $\lim_{n\to\infty} l_j^n \lambda_n^{\frac{2}{p}} = \pi_p$, $l_j^n = \frac{1}{n} + O(\frac{1}{n^{\frac{p}{2}}})$
(b) $\frac{l_{j+k}^n}{l_{j+m}^n} = 1 + O(\frac{1}{n^{\frac{p}{2}-1}})$ *for any fixed* $k, m \in N$;
(c) $q_{m,j} \leq \lambda_n^2 - \frac{\pi_p^p}{(l_j^n)^p} \leq q_{M,j}$.

Lemma 3.2. *([19]) If $q \in C^N[0,1]$, then for $k = 1, ..., N$, $\Delta^k l_j = O(n^{-(k+3)})$ as $n \to \infty$ and the order estimate is independent of j.*

Lemma 3.3. *([19]) Let $\Phi_j = \Sigma_{i=1}^m \phi_{j,i}$ with each $\phi_{j,i} = \prod_{p=1}^{k_i} \varphi_{j,i,p}$, where each $\varphi_{j,i,p} \in U_j^{(n)}$. Suppose $\Phi_j = O(n^{-\nu})$ and q is sufficiently smooth. Then $\delta^k \Phi_j = O(n^{-\nu})$ for all $k \in N$.*

Lemma 3.4. *([19]) Suppose $f \in C^N[0,1]$ and $\Phi_j = \int_{x_j^n}^{x_{j+1}^n} f(x)dx$. Then $\delta^k \Phi_j = O(n^{-1})$ for any $k = 0, 1, ..., N$.*

Theorem 3.5. *([19])Let $\Phi_m(x_j^n) = \psi_1(x_j^n)\psi_2(x_j^n)...\psi_m(x_j^n)$, where $\psi_j(x_j^n) = x_{j+k_i}^n$ and $k_i \in N \cup 0$. If q is C^k on $[0,1]$, then*

$$\delta^k \Phi(x_j^n) == \begin{cases} O(1), & 0 \leq k \leq m-1, \\ m! + O(n^{-1}), & k = m, \\ O(n^{-2}), & k \geq m+1. \end{cases}$$

Theorem 3.6. *([19]) If $q \in C^{N+1}[0,1]$, then*

$$q^{(k)}(x) = \delta^k q(x_j^n) - 2\lambda_n r(x) + O(\frac{1}{n})$$

for $k = 0, 1, ..., N$, where $j = j_n(x)$. The order estimate is uniformly valid for compact subsets of $[0,1]$.

Theorem 3.7. *Suppose that q in (1.2) is C^{N+1} on [0,1] $(N \geq 1)$, and let $j = j_n(x)$ for each $x \in [0,1]$. Then, as $n \to \infty$,*

$$q(x) = p\lambda_n^2\left(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - \frac{2r(x)}{p\lambda_n} - 1\right) + O(\frac{1}{n^{\frac{2}{p}}}), \tag{3.2}$$

and, for all $k = 1, 2, ..., N$,

$$q^{(k)}(x) = \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \delta^k l_j^n - 2\lambda_n \delta^k r(x_j^n) - 2\lambda_n r(x) + O(1). \tag{3.3}$$

Proof. It is clear that q is uniform. Assume that q is differentiable on [0,1]. By intermediate value theorem on (3.1). There is some $\xi_j^{(n)} \in (x_j^n, x_{j+1}^n)$ such that

$$\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} = \left(1 - \frac{q(\xi_j^{(n)})}{\lambda_n^2}\right)^{-\frac{1}{p}} = 1 + \frac{q(\xi_j^{(n)})}{p\lambda_n^2} + O(\frac{1}{\lambda_n^4}).$$

Hence

$$p\lambda_n^2(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - 1) - q(\xi_j^{(n)}) = O(\frac{1}{n^p}),$$

$$p\lambda_n^2(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - \frac{2r(x)}{p\lambda_n} - 1) + 2\lambda_n r(x) - q(\xi_j^{(n)}) = O(\frac{1}{n^p}).$$

By the mean value theorem, when n is sufficiently large,

$$q(x) = q(\xi_j^{(n)}) - 2\lambda_n r(x) + O(\frac{1}{n^{\frac{p}{2}}}).$$

By using the estimate of nodal lengths, we obtain

$$l_j^n = \frac{\pi_p}{\lambda_n^{\frac{2}{p}}} + \frac{2}{p\lambda_n} r(x_j^n) l_j^n + \frac{1}{p\lambda_n^2} q(x_j^n) l_j^n + O(\frac{1}{n^{1+\frac{4}{p}}}).$$

Hence,

$$q(x_j^n) = p\lambda_n^2(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - \frac{2r(x_j^n)}{p\lambda_n} - 1) + O(1).$$

Therefore

$$\delta q(x_j^n) = \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \delta l_j^n - 2\lambda_n \delta r(x_j^n) + O(1),$$

and so, for $k = 1, 2, ..., N$,

$$\delta^k q(x_j^n) = \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \delta^k l_j^n - 2\lambda_n \delta^k r(x_j^n) + O(1).$$

If we use the result of Theorem 3.6, we get

$$q^{(k)}(x) = \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \delta^k l_j^n - 2\lambda_n \delta^k r(x_j^n) - 2\lambda_n r(x) + O(1). \diamond$$

Now, we can prove the smoothness of the q with nodes. Let

$$F_n^{(k)} = \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \delta^k l_j^n - 2\lambda_n \delta^k r(x_j^n).$$

Our theorem on the smoothness of q is as follows.

Theorem 3.8. *Assume that q is continuous on [0,1]. If $F_n^{(k)}$ is uniformly convergent on [0,1], for each $k = 1, ..., N$, then q is C^N, and $F_n^{(k)}$ converge to $q^{(k)}$ uniformly on [0,1].*

Proof. The proof is similar to Theorem 1.2 in [21]. Take $x \in (0, 1)$. Also, let $G^{(1)}(x) = \lim_{n\to\infty} F_n^{(1)}(x)$. Hence

$$\int_0^x G^{(1)}(t)dt = \lim_{n\to\infty} \int_0^{x_j^n} G^{(1)}(t)dt$$

$$= \lim_{n\to\infty} \int_0^{x_j^n} F_n^{(1)}(t)dt$$

$$= \lim_{n\to\infty} \int_0^{x_j^n} [\frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \delta l_j^n - 2\lambda_n \delta r(x_j^n)]dt$$

$$= \lim_{n\to\infty} \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \sum_{i=0}^{j-1} \frac{(l_{i+1}^n - l_i^n)}{l_i^n} l_i^n - \lim_{n\to\infty} 2\lambda_n \sum_{i=0}^{j-1} \frac{(r(x_{i+1}^n) - r(x_i^n))}{l_i^n} l_i^n$$

$$= \lim_{n\to\infty} \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} (l_j^n - l_0^n) - \lim_{n\to\infty} 2\lambda_n (r(x_j^n) - r(0))$$

$$= [\lim_{n\to\infty} \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} l_j^n - 2\lambda_n r(x_j^n) - p\lambda_n^2] - [\lim_{n\to\infty} \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} l_0^n - 2\lambda_n r(0) - p\lambda_n^2]$$

$$= q(x) - q(0).$$

Since $G^{(1)}$ is continuous, we finalized $q'(x) = G^{(1)}(x)$ then $q \in C^1$.
If for any nonnegative integer N,

$$q^{(N)}(x) = \lim_{n\to\infty} F_n^{(N)}(x).$$

for any $x \in (0,1)$ and $F_n^{(N+1)}$ is uniformly convergent on [0,1], let $G^{(N+1)}(x) = \lim_{n\to\infty} F_n^{(N+1)}$. Hence,

$$\int_0^x G^{(N+1)}(t)dt = \lim_{n\to\infty} \int_0^{x_j^n} G^{(N+1)}(t)dt$$

$$= \lim_{n\to\infty} \int_0^{x_j^n} F_n^{(N+1)}(t)dt$$

$$= \lim_{n\to\infty} \int_0^{x_j^n} [\frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \delta^{N+1} l_j^n - 2\lambda_n \delta^{N+1} r(x_j^n)]dt$$

$$= \lim_{n\to\infty} \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} \sum_{i=0}^{j-1} \frac{(\delta^N l_{i+1}^n - \delta^N l_i^n)}{l_i^n} l_i^n - \lim_{n\to\infty} 2\lambda_n \sum_{i=0}^{j-1} \frac{(\delta^N r(x_{i+1}^n) - \delta^N r(x_i^n))}{l_i^n} l_i^n$$

$$= \lim_{n\to\infty} \frac{p\lambda_n^2 \lambda_n^{\frac{2}{p}}}{\pi_p} (\delta^N l_j^n - \delta^N l_0^n) - \lim_{n\to\infty} 2\lambda_n (\delta^N r(x_j^n) - \delta^N r(0))$$

$$= \lim_{n\to\infty} (F_n^N(x) - F_n^N(0))$$

$$= q^{(N)}(x) - q^{(N)}(0).$$

Therefore, $q^{(N+1)}(x) = G^{(N+1)}(x)$ and q is C^{N+1}.$\diamond$

4. Reconstructing of the potential functions

In this section we investigate the inverse nodal problem for (1.2) provided $q(x)$ and $r(x)$ are unknown from a dense subset of nodal points. The uniqueness theorem is given and a constructive algorithm for the solution is provided.

Corollary 4.1. *From (2.4) it follows that the set X of nodal points is dense in [0,1].*

Problem 4.2. *Given the set X of nodal points, find the functions $q(x)$ and $r(x)$.*

The following assertion is an immediate corollary of Theorem 2.2.

Lemma 4.3. *Fix $x \in [0,1]$. Chosen j_n such that $\lim_{n\to\infty} x^n_{j_n} = x$. Then the following finite limits exist and the corresponding equalities hold:*

$$R(x) = \lim_{n\to\infty} \frac{p(\pi_p)^{\frac{p}{2}} n^{\frac{p}{2}-1}}{2}(nx^n_{j_n} - j_n) \tag{4.1}$$

$$Q(x) = \lim_{n\to\infty} (n\pi_p)^{\frac{p}{2}} [\frac{p(n\pi_p)^{\frac{p}{2}}}{2n}(nx_{j_n} - j_n) - R(x^n_{j_n})] \tag{4.2}$$

and

$$R(x) = \int_0^x r(t)dt \tag{4.3}$$

$$Q(x) = \int_0^x q(t)dt. \tag{4.4}$$

Now, we are ready for a uniqueness theorem and solution of inverse nodal problem.

Theorem 4.4. *Let $X_0 \subset X$ be a dense subset. Then, the specification of X_0 uniquely determines the functions $q(x)$ and $r(x)$. The functions $q(x)$ and $r(x)$ can be constructed via the following algorithm:*

(i) For each $x \in [0,1]$ choose a sequence $\{x^n_{j_n}\} \subset X_0$ such that $x^n_{j_n} \to x$ as $n \to \infty$.
(ii) Find the function $R(x)$ via (4.3) and calculate $r(x) = R'(x)$.
(iii) Find the function $Q(x)$ via (4.4) and calculate $q(x) = Q'(x)$.

We denote the boundary value problem (1.2) and (1.3) by $L = L(q(x), r(x))$. Together with L we consider a boundary value problem $\widetilde{L} = L(\widetilde{q}(x), \widetilde{r}(x))$. We agree that if a certain symbol α denotes an object related to L, then the same symbol with the tilde $\widetilde{\alpha}$ denotes the analogous object related to $\widetilde{L}$.
From Theorem 9 we notice that if $X = \widetilde{X}$, then $q(x) = \widetilde{q}(x)$ and $r(x) = \widetilde{r}(x)$.

Lemma 4.5. *([20]) Suppose that $f \in L^1[0,1]$. Then for almost every $x \in [0,1]$ with $j = j_n(x)$,*

$$\lim_{n\to\infty} \frac{\lambda_n^{\frac{2}{p}}}{\pi_p} \int_{x^n_j}^{x^n_{j+1}} f(t)dt = f(x). \tag{4.5}$$

Alternatively, we can obtain the formulae using the nodal length, which allows one to approximate $r(x)$. Asymptotic formulae for $q(x)$ has already obtained in [16].

Theorem 4.6. *The function $r(x)$ is given by*

$$r(x) = \lim_{n\to\infty} \frac{1}{2} p\lambda_n \Big(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - 1\Big) \tag{4.6}$$

for a.e. $x \in (0,1)$ with $j = j_n(x)$.

Proof. According to (2.5) we have

$$l_j^n = \frac{\pi_p}{\lambda_n^{\frac{2}{p}}} + \frac{2}{p\lambda_n} \int_{x_j^n}^{x_{j+1}^n} r(t)dt + O\Big(\frac{1}{\lambda_n^2}\Big).$$

Hence,

$$p\lambda_n\Big(\frac{\lambda_n^{\frac{2}{p}} l_j^n}{\pi_p} - 1\Big) = \frac{2\lambda_n^{\frac{2}{p}}}{\pi_p} \int_{x_j^n}^{x_{j+1}^n} r(t)dt + O\Big(\frac{1}{\lambda_n^{1-\frac{2}{p}}}\Big).$$

By Lemma 4.5, (4.6) is proved.$\diamond$

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