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Existence results of infinitely many solutions for a class of $p(x)$-biharmonic problems

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Abstract The existence of infinitely many weak solutions for a Navier doubly eigenvalue boundary value problem involving the $p(x)$-biharmonic operator is established. In our main result, under an appropriate oscillating behaviour of the nonlinearity and suitable assumptions on the variable exponent, a sequence of pairwise distinct solutions is obtained. Furthermore, some applications are pointed out.

Keywords. Ricceri's variational principle, Infinitely many solutions, Navier condition, $p(x)$-biharmonic type operators.

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1. Introduction

The aim of this article is to investigate the following Navier doubly eigenvalue boundary value problem

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, λ is a positive parameter, μ is a non-negative parameter, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $a(\cdot) \in L^\infty(\Omega)$ with $\inf_{x\in\overline{\Omega}} a(x) > 0$, $p(\cdot) \in C^0(\overline{\Omega})$ with

$$\max\{2, \frac{N}{2}\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x),$$

and $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the operator of fourth order called the $p(x)$-biharmonic operator, which is a natural generalization of the p-biharmonic operator (where $p > 1$ is a constant).

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In [14], the authors studied the following $p(x)$-biharmonic elliptic problem with Navier boundary conditions:

$$\begin{cases} \Delta^2_{p(x)}u + e(x)|u|^{p(x)-2}u = \lambda a(x)f(x,u) + \mu g(x,u), \ x \in \Omega, \\ u = \Delta u = 0, \qquad x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, λ, μ are non-negative parameters, $p(\cdot) \in C^0(\overline{\Omega})$ with $p^- > \max\{2, \frac{N}{2}\}$.
By the three critical points theorem obtained by Ricceri [11], they established the existence of three weak solutions to problem (1.2).

For a discussion about the existence of infinitely many solutions for boundary value problems, using Ricceris variational principle [10, Theorem 2.5] and its variant [3, Theorem 2.1], we refer [1, 2, 3, 4, 5].
Our goal in this paper is to obtain some sufficient conditions to guarantee that problem (1.1) has infinitely many weak solutions. To this end, we require that the primitive F of f satisfies a suitable oscillatory behavior either at infinity (for obtaining unbounded solutions) or at zero (for finding arbitrarily small solutions), while G, the primitive of g, has an appropriate growth (see Theorems 3.1 and 4.4). Our approach is fully variational and the main tool is a general critical point theorem (see Lemma 2.1 below) contained in [3]; see also [10].
The plan of the paper is as follows. In Section 2, some known definitions and results on variable exponent Lebesgue and Sobolev spaces, which will be used in sequel, are collected. Moreover, the abstract critical points theorem (Lemma 2.1) is recalled. Section 3 is devoted to main theorem and finally, in Section 4, some examples and applications are presented.

2. Preliminaries

The goal of this work is to establish some new results for problem (1.1) to have infinitely many weak solutions. Our analysis is mainly based on a recent critical point theorem of Bonanno and Molica Bisci [3] (see Lemma (2.1) below) which is a more precise version of Ricceri's variational principle [10, Theorem 2.5].

Lemma 2.1. Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional

$$I_\lambda := \Phi - \lambda\Psi,$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either
  (b_1) I_λ possesses a global minimum, or
  (b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either
  (c_1) there is a global minimum of Φ which is a local minimum of I_λ, or
  (c_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.

For the reader's convenience, we recall some background facts concerning Lebesgue-Sobolev spaces variable exponent and introduce some notation. For more details, we refer the reader to [8, 9, 12, 13].
Set

$$C_+(\Omega) := \{h \in C(\overline{\Omega}) : h(x) > 1,\ x \in \overline{\Omega}\}.$$

For $p(\cdot) \in C_+(\Omega)$, define

$$L^{p(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

We can introduce a norm on $L^{p(\cdot)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf\{\alpha > 0 : \int_\Omega |\frac{u(x)}{\alpha}|^{p(x)} dx \le 1\}.$$

The space $(L^{p(\cdot)}(\Omega), |u|_{p(\cdot)})$ is a Banach space called a variable exponent Lebesgue space. The Sobolev space with variable exponent $W^{m,p(\cdot)}(\Omega)$ is defined as

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \le m\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u$ with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{m,p(\cdot)}(\Omega)$, equipped with the norm

$$\|u\|_{m,p(\cdot)} := \sum_{|\alpha| \le m} |D^\alpha u|_{p(x)},$$

is a separable, reflexive and uniformly convex Banach space (see [7]). We denote by $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$.
Suppose that $a(\cdot) \in L^\infty(\Omega)$ and $a^- := \inf_{x\in\overline{\Omega}} a(x) > 0$, we define

$$L^{p(\cdot)}_{a(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega a(x)|u(x)|^{p(x)} dx < \infty\},$$

with the norm

$$|u|_{(p(\cdot),a(\cdot))} = \inf\{\alpha > 0 : \int_\Omega a(x)|\frac{u(x)}{\alpha}|^{p(x)} dx \le 1\}.$$

Then $L^{p(\cdot)}_{a(\cdot)}(\Omega)$ is a Banach space. Now we denote

$$X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega).$$

For any $u \in X$, define

$$\|u\|_a = \inf \Big\{ \alpha > 0 : \int_\Omega \Big(|\frac{\Delta u(x)}{\alpha}|^{p(x)} + a(x)|\frac{u(x)}{\alpha}|^{p(x)} \Big) dx \leq 1 \Big\}.$$

It is easy to see that X endowed with the above norm is a separable, reflexive Banach space [6]. We denote by X^* its dual.

Remark 2.2. According to [15], the norm $\|u\|_{2,p(\cdot)}$ is equivalent to the norm $|\Delta u|_{p(\cdot)}$ in the space X. Consequently, the norms $\|u\|_{2,p(\cdot)}, \|u\|_a$ and $|\Delta u|_{p(\cdot)}$ are equivalent.

Proposition 2.3. [8, 12] The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{q(\cdot)}(\Omega)$ where $q(\cdot)$ is the conjugate function of $p(\cdot)$; i.e.,

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1.$$

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have

$$\Big| \int_\Omega u(x)v(x)dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-} \Big) |u|_{p(\cdot)} |v|_{q(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{q(\cdot)},$$

where $q^- := \inf_{x\in\overline{\Omega}} q(x)$.

Proposition 2.4. [8, 12] Set $\rho(u) = \int_\Omega |u|^{p(x)}dx$. For $u, u_n \in L^{p(\cdot)}(\Omega)$, we have

(1) $|u|_{p(\cdot)} < (=; >) 1 \Leftrightarrow \rho(u) < (=; >) 1$,
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|^{p^-}_{p(\cdot)} \leq \rho(u) \leq |u|^{p^+}_{p(\cdot)}$,
(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|^{p^+}_{p(\cdot)} \leq \rho(u) \leq |u|^{p^-}_{p(\cdot)}$,
(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$,
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

From Proposition 2.4, for $u \in L^{p(\cdot)}(\Omega)$ the following inequalities hold:

$$\|u\|_a^{p^-} \leq \int_\Omega \Big(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \Big) dx \leq \|u\|_a^{p^+} \text{ if } \|u\|_a > 1; \tag{2.1}$$

$$\|u\|_a^{p^+} \leq \int_\Omega \Big(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \Big) dx \leq \|u\|_a^{p^-} \text{ if } \|u\|_a < 1. \tag{2.2}$$

Proposition 2.5. [14] If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $N/2 < p^-$.

From Proposition 2.5, there exist a positive constant c depending on $p(\cdot), N$ and Ω such that

$$\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)| \leq c\|u\|_a, \quad \forall u \in X. \tag{2.3}$$

Corresponding to f and g we introduce the functions $F, G : \Omega \times \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi, \qquad G(x,t) := \int_0^t g(x,\xi)\,d\xi,$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.
We say that a function $u \in X$ is a *weak solution* of problem (1.1) if

$$\int_\Omega (|\Delta u|^{p(x)-2}\Delta u\,\Delta v + a(x)|u|^{p(x)-2}u\,v)dx$$
$$-\lambda \int_\Omega f(x,u(x))v(x)\,dx - \mu \int_\Omega g(x,u(x))v(x)\,dx = 0,$$

holds for all $v \in X$.

3. Main results

In this section we present our main result. Let

$$A := \liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^{p^-}},$$

$$B := \limsup_{\xi \to +\infty} \frac{\displaystyle\int_\Omega F(x,\xi)\,dx}{\xi^{p^+}},$$

and

$$\lambda_1 := \frac{\|a\|_1}{p^- B}, \qquad \lambda_2 := \frac{1}{p^+ c^{p^-} A},$$

where c is given by (2.3).

Theorem 3.1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Assume that

(A_1) $F(x,t) \ge 0$ for every $(x,t) \in \Omega \times [0,+\infty)$,

(A_2) $A < \frac{p^-}{p^+ c^{p^-}\|a\|_1} B$.

Then, for each $\lambda \in (\lambda_1, \lambda_2)$ and for every Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$G(x,t) \ge 0, \quad \forall (x,t) \in \Omega \times [0,+\infty),$$

$$g_\infty := \lim_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \sup_{|t|\le\xi} G(x,t)\,dx}{\xi^{p^-}} < +\infty, \tag{3.1}$$

and for each $\mu \in [0, \mu_{g,\lambda})$ where

$$\mu_{g,\lambda} := \frac{1}{p^+ c^{p^-} g_\infty}\left(1 - \lambda p^+ c^{p^-} A\right),$$

(where $\mu_{g,\lambda} = +\infty$ when $g_\infty = 0$) problem (1.1) has an unbounded sequence of weak solutions in X.

Proof. Our aim is to apply Lemma 2.1(b) to problem (1.1). To this end, fix $\overline{\lambda} \in (\lambda_1, \lambda_2)$ and g satisfying our assumptions. Since $\overline{\lambda} < \lambda_2$, we have

$$\mu_{g,\overline{\lambda}} = \frac{1}{p^+ c^{p^-} g_\infty} \left(1 - \overline{\lambda} p^+ c^{p^-} A\right) > 0.$$

Now fix $\overline{\mu} \in (0, \mu_{g,\overline{\lambda}})$ and set

$$J(x,\xi) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi)$$

for all $(x,\xi) \in \Omega \times \mathbb{R}$. For each $u \in X$, we let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \int_\Omega \left(\frac{1}{p(x)} |\Delta u(x)|^{p(x)} + \frac{a(x)}{p(x)} |u(x)|^{p(x)} \right) dx,$$

$$\Psi(u) := \int_\Omega J(x, u(x))\, dx,$$

and put

$$I_{\overline{\lambda}}(u) := \Phi(u) - \overline{\lambda}\Psi(u), \qquad u \in X.$$

Note that the weak solutions of (1.1) are exactly the critical points of $I_{\overline{\lambda}}$. The functionals Φ, Ψ satisfy the regularity assumptions of Lemma 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega (|\Delta u|^{p(x)-2} \Delta u \Delta v + a(x)|u|^{p(x)-2} uv) dx,$$

for any $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ admits a continuous inverse (see [14, Lemma 3.1]). On the other hand, the fact that X is compactly embedded into $C^0(\Omega)$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable with compact derivative and whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_\Omega f(x, u(x)) v(x)\, dx + \frac{\overline{\mu}}{\overline{\lambda}} \int_\Omega g(x, u(x)) v(x)\, dx.$$

We claim that the functional Ψ is a sequentially weakly upper semicontinuous functional on X. Indeed, if $u_n \rightharpoonup u$ in X then compactness of embedding $X \hookrightarrow C^0(\Omega)$ implies $u_n \to u$ in $C^0(\Omega)$ i.e., $u_n \to u$ uniformly on Ω. Since $F(x,u)$ and $G(x,u)$ are differentiable with respect to u for a.e. $x \in \Omega$, then, $F(x, u_n(x)) \to F(x, u(x))$ and $G(x, u_n(x)) \to G(x, u(x))$ a.e. $x \in \Omega$. Hence,

$$\begin{aligned}
\limsup_{n\to+\infty} \Psi(u_n) &\le \int_\Omega \limsup_{n\to+\infty} F(x, u_n(x)) dx \\
&\quad + \frac{\overline{\mu}}{\overline{\lambda}} \int_\Omega \limsup_{n\to+\infty} G(x, u_n(x)) dx \\
&= \int_\Omega F(x, u(x)) dx + \frac{\overline{\mu}}{\overline{\lambda}} \int_\Omega G(x, u(x)) dx = \Psi(u),
\end{aligned}$$

which implies Ψ is a sequentially weakly upper semicontinuous functional on X. Furthermore, we have from (2.1) that

$$\Phi(u) \geq \frac{1}{p^+}\|u\|_a^{p^-}, \tag{3.2}$$

for all $u \in X$ such that $\|u\|_a > 1$, and so Φ is coercive. First of all, we will show that $\overline{\lambda} < 1/\gamma$. Hence, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\displaystyle\int_\Omega \sup_{|t|\leq\xi_n} F(x,t)\,dx}{\xi_n^{p^-}} = A.$$

Put

$$r_n := \frac{1}{p^+}\left(\frac{\xi_n}{c}\right)^{p^-},$$

for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (2.1) and (2.2) into account, one has

$$\|v\|_a \leq \max\left\{(p^+ r_n)^{\frac{1}{p^+}}, (p^+ r_n)^{\frac{1}{p^-}}\right\}.$$

So, thanks to the embedding $X \hookrightarrow C^0(\overline{\Omega})$ (see (2.3)), one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)}{r_n} \\
&\leq \frac{\displaystyle\int_\Omega \sup_{|t|\leq\xi_n} J(x,t)\,dx}{\frac{1}{p^+}\left(\frac{\xi_n}{c}\right)^{p^-}} \\
&\leq p^+ c^{p^-}\left[\frac{\displaystyle\int_\Omega \sup_{|t|\leq\xi_n} F(x,t)\,dx}{\xi_n^{p^-}} + \frac{\overline{\mu}}{\overline{\lambda}}\frac{\displaystyle\int_\Omega \sup_{|t|\leq\xi_n} G(x,t)\,dx}{\xi_n^{p^-}}\right].
\end{aligned}$$

Moreover, from the assumption (A_1) and the condition (3.1), we have $A < +\infty$ and

$$\lim_{n\to+\infty} \frac{\displaystyle\int_\Omega \sup_{|t|\leq\xi_n} G(x,t)\,dx}{\xi_n^{p^-}} = g_\infty.$$

Therefore,

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq p^+ c^{p^-}\left(A + \frac{\overline{\mu}}{\overline{\lambda}} g_\infty\right) < +\infty. \tag{3.3}$$

The assumption $\overline{\mu} \in (0, \mu_{G,\overline{\lambda}})$ immediately yields

$$\gamma \leq p^+ c^{p^-} \left(A + \frac{\overline{\mu}}{\overline{\lambda}} g_\infty \right) < p^+ c^{p^-} A + \frac{1 - \overline{\lambda} p^+ c^{p^-} A}{\overline{\lambda}}.$$

Hence,

$$\overline{\lambda} = \frac{1}{p^+ c^{p^-} A + (1 - \overline{\lambda} p^+ c^{p^-} A)/\overline{\lambda}} < \frac{1}{\gamma}.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}} < \frac{p^- B}{\|a\|_1},$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n \to +\infty} \eta_n = +\infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{p^-}{\|a\|_1} \frac{\int_\Omega F(x, \eta_n)\, dx}{\eta_n^{p^+}}, \tag{3.4}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define $w_n \in X$ by

$$w_n(x) = \eta_n, \qquad x \in \overline{\Omega}.$$

For any fixed $n \in \mathbb{N}$, one has

$$\Phi(w_n) = \int_\Omega \frac{1}{p(x)} a(x) \eta_n^{p(x)} dx \leq \frac{\eta_n^{p^+}}{p^-} \|a\|_1. \tag{3.5}$$

On the other hand, bearing (A_1) in mind and since G is non-negative, from the definition of Ψ, we infer

$$\begin{aligned} \Psi(w_n) &= \int_\Omega \left[F(x, w_n(x)) + \frac{\overline{\mu}}{\overline{\lambda}} G(x, w_n(x)) \right] dx \\ &\geq \int_\Omega F(x, \eta_n)\, dx. \end{aligned} \tag{3.6}$$

By (3.4), (3.5) and (3.6), we observe that

$$I_{\overline{\lambda}}(w_n) \leq \frac{\eta_n^{p^+}}{p^-} \|a\|_1 - \overline{\lambda} \int_\Omega F(x, \eta_n)\, dx < \frac{\eta_n^{p^+}}{p^-} \|a\|_1 (1 - \overline{\lambda} \tau), \tag{3.7}$$

for every $n \in \mathbb{N}$ large enough. Since $\overline{\lambda}\tau > 1$ and $\lim_{n \to +\infty} \eta_n = +\infty$, we have

$$\lim_{n \to +\infty} I_{\overline{\lambda}}(w_n) = -\infty.$$

Then, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, by Lemma 2.1(b), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ such that

$$\lim_{n \to +\infty} \|u_n\|_a = +\infty,$$

and the conclusion is achieved. □

Remark 3.2. Under the conditions $A = 0$ and $B = +\infty$, from Theorem 3.1 we see that for every $\lambda > 0$ and for each $\mu \in \big[0, \frac{1}{p^+ c^{p^-} g_\infty}\big)$, problem (1.1) admits a sequence of weak solutions which is unbounded in X. Moreover, if $g_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

4. Applications and examples

In this section, we point out some consequences and applications of the result previously obtained.

First, we present the following consequence of Theorem 3.1 with $\mu = 0$.

Theorem 4.1. Assume that all the assumptions in the Theorem 3.1 hold. Then, for each

$$\lambda \in \left(\frac{\|a\|_1}{p^- B}, \frac{1}{p^+ c^{p^-} A} \right),$$

the problem

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega \end{cases} \tag{4.1}$$

has an unbounded sequence of weak solutions in X.

Here we point out the following consequence of Theorem 3.1.

Corollary 4.2. Assume that the assumption (A_1) in the Theorem 3.1 holds. Suppose that

$$A < \frac{1}{p^+ c^{p^-}}, \qquad B > \frac{\|a\|_1}{p^-}.$$

Then, the problem

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega \end{cases} \tag{4.2}$$

has an unbounded sequence of weak solutions in X.

Corollary 4.3. Let $g_1 : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put

$$G_1(\xi) := \int_0^\xi g_1(t)\, dt, \quad \forall \xi \in \mathbb{R},$$

and assume that

(A_3) $\liminf\limits_{\xi \to +\infty} \frac{G_1(\xi)}{\xi^{p^-}} < +\infty$,

(A_4) $\limsup\limits_{\xi \to +\infty} \frac{G_1(\xi)}{\xi^{p^+}} = +\infty$.

Then, for every $\alpha_i \in L^1(\Omega)$, $1 \leq i \leq n$, with $\min_{x\in\Omega}\{\alpha_i(x) : 1 \leq i \leq n\} \geq 0$ and with $\alpha_1 \neq 0$, and for every non-negative continuous $g_i : \mathbb{R} \to \mathbb{R}$, $2 \leq i \leq n$, satisfying

$$\max\left\{ \sup_{\xi \in \mathbb{R}} G_i(\xi) : 2 \leq i \leq n \right\} \leq 0,$$

and

$$\min\left\{\liminf_{\xi\to+\infty}\frac{G_i(\xi)}{\xi^{p^-}}:2\le i\le n\right\}>-\infty,$$

where

$$G_i(\xi):=\int_0^\xi g_i(t)\,dt,\quad \forall\xi\in\mathbb{R},\ 2\le i\le n,$$

for each

$$\lambda\in\left(0,\frac{1}{p^+c^{p^-}\liminf_{\xi\to+\infty}\frac{G_1(\xi)}{\xi^{p^-}}\int_\Omega\alpha_1(x)\,dx}\right),$$

the problem

$$\begin{cases}\Delta^2_{p(x)}u+a(x)|u|^{p(x)-2}u=\lambda\sum_{i=1}^n\alpha_i(x)g_i(u), & x\in\Omega,\\ u=\Delta u=0, & x\in\partial\Omega\end{cases}$$

has an unbounded sequence of weak solutions in X.

Proof. Set $f(x,t)=\sum_{i=1}^n\alpha_i(x)g_i(t)$ for all $(x,t)\in\Omega\times\mathbb{R}$. From the assumption (A_4) and the condition

$$\min\left\{\liminf_{\xi\to+\infty}\frac{G_i(\xi)}{\xi^{p^-}}:2\le i\le n\right\}>-\infty,$$

we have

$$\limsup_{\xi\to+\infty}\frac{\int_\Omega F(x,\xi)\,dx}{\xi^{p^+}}=\limsup_{\xi\to+\infty}\frac{\sum_{i=1}^n\left(G_i(\xi)\int_\Omega\alpha_i(x)\,dx\right)}{\xi^{p^+}}=+\infty.$$

Moreover, from the assumption (A_3) and the condition

$$\max\left\{\sup_{\xi\in\mathbb{R}}G_i(\xi):2\le i\le n\right\}\le 0,$$

we have

$$\liminf_{\xi\to+\infty}\frac{\int_\Omega\sup_{|t|\le\xi}F(x,t)\,dx}{\xi^{p^-}}\le\left(\int_\Omega\alpha_1(x)\,dx\right)\liminf_{\xi\to+\infty}\frac{G_1(\xi)}{\xi^2}<+\infty.$$

Hence, applying Theorem 3.1 the desired conclusion follows. □

Now, put

$$A' := \liminf_{\xi\to0^+}\frac{\int_\Omega\sup_{|t|\le\xi}F(x,t)\,dx}{\xi^{p^+}},$$

$$B' := \limsup_{\xi\to0^+}\frac{\int_\Omega F(x,\xi)\,dx}{\xi^{p^-}},$$

and

$$\lambda_3 := \frac{\|a\|_1}{p^- B'}, \qquad \lambda_4 := \frac{1}{p^+ c^{p^+} A'}.$$

Using Lemma 2.1(c) and arguing as in the proof of Theorem 3.1, we can obtain the following result.

Theorem 4.4. Assume that the assumption (A_1) in the Theorem 3.1 holds and

(A_5) $A' < \frac{p^-}{p^+ c^{p^+} \|a\|_1} B'$.

Then, for every $\lambda \in (\lambda_3, \lambda_4)$ and for every arbitrary Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$, whose potential $G(x,t) := \int_0^t g(x,\xi)\, d\xi$ for all $(x,t) \in \Omega \times \mathbb{R}$, is a non-negative function satisfying the condition

$$g_0 := \limsup_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} G(x,t)\, dx}{\xi^{p^+}} < +\infty, \tag{4.3}$$

if we put

$$\mu'_{g,\lambda} := \frac{1}{p^+ c^{p^+} g_0} \left(1 - \lambda p^+ c^{p^+} A'\right),$$

where $\mu'_{g,\lambda} = +\infty$ when $g_0 = 0$, for every $\mu \in [0, \mu'_{g,\lambda})$ problem (1.1) has a sequence of weak solutions, which strongly converges to zero in X.

Proof. Fix $\overline{\lambda} \in (\lambda_3, \lambda_4)$ and let g be a function that satisfies the condition (4.3). Since $\overline{\lambda} < \lambda_4$, we obtain

$$\mu'_{g,\overline{\lambda}} := \frac{1}{p^+ c^{p^+} g_0} \left(1 - \overline{\lambda} p^+ c^{p^+} A'\right) > 0.$$

Now fix $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$ and set

$$J(x,t) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi),$$

for all $(x,t) \in \Omega \times \mathbb{R}$. We take Φ, Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Lemma 2.1. As first step, we will prove that $\overline{\lambda} < 1/\delta$. Then, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = 0$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\le\xi_n} F(x,t)\, dx}{\xi_n^{p^+}} = A'.$$

By the fact that $\inf_X \Phi = 0$ and the definition of δ, we have

$$\delta = \liminf_{r\to 0^+} \varphi(r).$$

Then, as in showing (3.3) in the proof of Theorem 3.1, we can prove that $\delta < +\infty$. From $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$, the following inequalities hold

$$\delta \leq p^+ c^{p^+} \left(A' + \frac{\overline{\mu}}{\overline{\lambda}} g_0 \right) < p^+ c^{p^+} A' + \frac{1 - \overline{\lambda} p^+ c^{p^+} A'}{\overline{\lambda}}.$$

Therefore,

$$\overline{\lambda} = \frac{1}{p^+ c^{p^+} A' + \left(1 - \overline{\lambda} p^+ c^{p^+} A'\right)/\overline{\lambda}} < \frac{1}{\delta}.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. Since

$$\frac{1}{\overline{\lambda}} < \frac{p^- B'}{\|a\|_1},$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n \to +\infty} \eta_n = 0^+$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{p^-}{\|a\|_1} \frac{\displaystyle\int_\Omega F(x, \eta_n)\, dx}{\eta_n^{p^-}},$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, define $w_n \in X$ by

$$w_n(x) = \eta_n, \qquad x \in \overline{\Omega}.$$

Note that $\overline{\lambda}\tau > 1$. Then, as in showing (3.7), we can obtain that

$$I_{\overline{\lambda}}(w_n) \leq \frac{\eta_n^{p^-}}{p^-} \|a\|_1 - \overline{\lambda} \int_\Omega F(x, \eta_n)\, dx < \frac{\eta_n^{p^-}}{p^-} \|a\|_1 (1 - \overline{\lambda}\tau) < 0,$$

for every $n \in \mathbb{N}$ large enough. Then, since

$$\lim_{n \to +\infty} I_{\overline{\lambda}}(w_n) = I_{\overline{\lambda}}(0) = 0,$$

we see that zero is not a local minimum of $I_{\overline{\lambda}}$. This, together with the fact that zero is the only global minimum of Φ, we deduce that the energy functional $I_{\overline{\lambda}}$ has not a local minimum at the unique global minimum of Φ. Therefore, by Lemma 2.1(c), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ which converges weakly to zero. In view of the fact that the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Remark 4.5. Under the conditions $A' = 0$ and $B' = +\infty$, Theorem 4.4 ensures that for every $\lambda > 0$ and for each $\mu \in \left[0, \frac{1}{p^+ c^{p^+} g_0}\right)$, problem (1.1) admits a sequence of weak solutions which strongly converges to 0 in X. Moreover, if $g_0 = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Remark 4.6. We can use Theorem 4.4 to obtain similar results to Theorem 4.1 and Corollaries 4.2, 4.3. We omit the discussions here.

We conclude this paper with the following example to illustrate our results.

Example 4.7. Let $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 3\}$. Consider the problem

$$\begin{cases} \Delta^2_{p(x,y)} u + e^{x^2+y^2}|u|^{p(x,y)-2}u = \lambda f(x, y, u) + \mu g(x, y, u), & (x, y) \in \Omega, \\ u = \Delta u = 0, \qquad (x, y) \in \partial\Omega, \end{cases} \tag{4.4}$$

where $p(x, y) = x^2 + y^2 + 3$ for all $(x, y) \in \Omega$,

$$f(x, y, t) = \begin{cases} f^*(x, y)t^6 \Big(7 + \sin(\ln(|t|)) - 7\cos(\ln(|t|))\Big) \\ \qquad \text{if } (x, y, t) \in \Omega \times (\mathbb{R} - \{0\}), \\ 0 \qquad \text{if} (x, y, t) \in \Omega \times \{0\}, \end{cases}$$

where $f^* : \Omega \to \mathbb{R}$ is a non-negative continuous function, and

$$g(x, y, t) = e^{x+y-t^+}(t^+)^{\varsigma-1}(\varsigma - t^+),$$

for all $(x, y) \in \Omega$ and $t \in \mathbb{R}$, where $t^+ = \max\{t, 0\}$ and ς is a positive real number. It is obvious that $p^- = 3$ and $p^+ = 6$. A direct calculation shows

$$F(x, y, t) = \begin{cases} f^*(x, y)t^7 \Big(1 - \cos(\ln(|t|))\Big), & \text{if} \quad (x, y, t) \in \Omega \times (\mathbb{R} - \{0\}), \\ 0, & \text{if} \quad (x, y, t) \in \Omega \times \{0\}. \end{cases}$$

So,

$$\liminf_{\xi \to +\infty} \frac{\int_\Omega \max_{|t| \le \xi} F(x, y, t)\, d\sigma}{\xi^3} = 0,$$

and

$$\limsup_{\xi \to +\infty} \frac{\int_\Omega F(x, y, \xi)\, dx}{\xi^6} = +\infty.$$

Hence, using Theorem 3.1, since $g_\infty = 0$, the problem (4.4) for every $(\lambda, \mu) \in (0, +\infty) \times [0, +\infty)$ admits infinitely many weak solutions in X.

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