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Stability analysis of two classes of improved backward Euler methods for stochastic delay differential equations of neutral type

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Abstract This paper examines stability analysis of two classes of improved backward Euler methods, namely split-step (θ, λ)-backward Euler (SSBE) and semi-implicit (θ, λ)-Euler (SIE) methods, for nonlinear neutral stochastic delay differential equations (NSDDEs). It is proved that the SSBE method with $\theta, \lambda \in (0, 1]$ can recover the exponential mean-square stability with some restrictive conditions on stepsize Δ, drift and diffusion coefficients, but the SIE method can reproduce the exponential mean-square stability unconditionally. Moreover, for sufficiently small stepsize, we show that the decay rate as measured by the Lyapunov exponent can be reproduced arbitrarily accurately. Finally, numerical experiments are included to confirm the theorems.

Keywords. Neutral stochastic delay differential equations, Exponential mean-square stability, Split-step (θ, λ)-backward Euler method, Semi-implicit (θ, λ)-Euler method, Lyapunov exponent.

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1. Introduction

Stochastic functional differential equations (SFDEs), as an important mathematical model, appear in science and engineering applications, especially for systems whose evolution in time is influenced by random forces as well as its history information. Both the theory and numerical methods for SFDEs have been well developed in the recent decades (see [21], [1], [5] and [11]). If the time delay in SFDEs reduces to a constant, it is usually called stochastic delay differential equations (SDDEs) (see [15], [16] and [20]). For the theory of NSFDEs we refer to [10], [12], [13] and [7]. The scalar neutral stochastic differential equations with fixed time delay (NSDDE) has the following general form

$$\begin{cases} d[x(t) - N(x(t-\tau))] = f\big(x(t), x(t-\tau)\big)dt + g\big(x(t), x(t-\tau)\big)dW(t), \quad t > 0, \\ x(t) = \psi(t) \in C([-\tau, 0]; \mathbb{R}^n), \end{cases}$$

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where $\tau > 0$ is a fixed constant.
In practice, many system models are described by NSDDEs. The models involve not only time delays in the state but also has time delay included in the state derivatives (see [2] and [4]). Since most of these equations cannot be solved explicitly, numerical approximations became to be an important tool in studying stochastic systems of neutral type (see [22], [18] and [19]).
Mean-square stability analysis of numerical solution for system of stochastic differential equations (SDEs) is one of the key problems in stochastic analysis (see [8], [17] and [14]). However, the study on stability of numerical method for neutral stochastic differential systems is relatively scarce due to their technical difficulties, which is the main topic of the present paper. Chen and Wu [3] showed that almost sure exponential stability of the backward Euler-Maruyama scheme for stochastic delay differential equations with monotone-type condition. Li and Cao [9] showed that asymtotic mean-square stability of two-step Maruyama methods for nonlinear neutral stochastic differential equations with constant time delay (NSDDEs). In [24], [25], [6] and [23], authors examined the theta's effects on the exponential mean-square stability and revealed that the linear growth condition on the drift coefficient is necessary for the two classes of theta approximations when $\theta \in [0, \frac{1}{2}]$ to be mean-square stable, but for $\theta \in (\frac{1}{2}, 1]$, both of the approximations can reproduce the exponential mean-square stability without the linear growth condition.

The rest of the paper is organized as follows. Section 2 begins with notations and preliminaries, then it introduces the SSBE and SIE methods for NSDDEs. Section 3 examines the conditions under which the SSBE method can reproduce the exponential mean-square stability of the exact solution with some restriction on stepsize Δ, but the SIE method can reproduce the exponential mean-square stability unconditionally. Section 4 describes the numerical experiments to confirm the theoretical results.

2. Preliminaries and notations

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ denotes both the Euclidean norm in $\mathbb{R}^n$ and the trace (or Frobenius) norm in $\mathbb{R}^{n\times d}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^T A)}$. $a \vee b$ represents $max\{a, b\}$ and $a \wedge b$ denotes $min\{a, b\}$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is right continuous and satisfies that each $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets, and $W(t)$ be a d-dimensional standard Wiener process defined on this probability space.
Let $N : \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$ be Borel measurable functions. Consider the n-dimensional NSDDE of the form

$$d[x(t) - N(x(t-\tau))] = f\big(x(t), x(t-\tau)\big)dt + g\big(x(t), x(t-\tau)\big)dW(t), \quad t > 0, \tag{2.1}$$

with initial data $x(t) = \psi(t) \in C([-\tau, 0]; \mathbb{R}^n)$, $\mathbb{E}\|\psi\|^2 < \infty$, where $\tau > 0$ is delay time.

Assumption 2.1. (Contractive Mapping) Assume that for all $x, y \in \mathbb{R}^n$, there exists a positive constant $\kappa \in (0, 1)$ such that

$$|N(x) - N(y)| \leq \kappa |x - y|. \tag{2.2}$$

Assumption 2.2. (Local Lipschitz Condition) Let f and g satisfy the local Lipschitz condition, that is, for each $j > 0$ there exists a positive constant K_j such that for any $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\overline{x}| \vee |\overline{y}| \leq j$,

$$|f(x, y) - f(\overline{x}, \overline{y})| \vee |g(x, y) - g(\overline{x}, \overline{y})| \leq K_j(|x - \overline{x}| + |y - \overline{y}|). \tag{2.3}$$

Theorem 2.1. *(See [13]) Let Assumption 2.1 and 2.2 hold. Assume that there exist two positive constants μ, σ such that for any $x, y \in \mathbb{R}^n$,*

$$2[x - N(y)]^T f(x, y) + |g(x, y)|^2 \leq -\mu |x|^2 + \sigma |y|^2. \tag{2.4}$$

If $\mu > \sigma$, then the trivial solution of equation (2.1) is exponential mean-square stability and the solution $x(t)$ satisfying in following relations

$$\mathbb{E}|x(t) - N(x(t - \tau))|^2 \leq C(\psi) e^{-\gamma t}, \tag{2.5}$$

and

$$\mathbb{E}|x(t)|^2 \leq C(\psi) e^{-\overline{\gamma} t}, \tag{2.6}$$

where $C(\psi)$ represents a generic positive constant, depending on the initial data ψ, whose value may changes with each appearance. Let $\overline{\gamma} := \gamma \wedge r$ with γ and r defined by

$$\gamma = \max\left\{ q > 0;\ q(1 + \varepsilon) - \mu + \left[q(1 + \varepsilon)\frac{\kappa^2}{\varepsilon} + \sigma\right] e^{q\tau} = 0,\ \varepsilon > 0 \right\},$$

and $r := \dfrac{2}{\tau} \ln \dfrac{1}{\kappa} - \ell$ for sufficiently small $\ell > 0$.

Remark 2.2. It is easy to see that the coupled monotone condition (2.4) implies that

$$2[x - N(y)]^T f(x, y) \leq -\mu |x|^2 + \sigma |y|^2.$$

Now we introduce the split-step (θ, λ)-backward Euler (SSBE) approximation $\{x_k\}_{k \geq 0}$ as follows:

$$\begin{cases} y_k = x_k - N(x_{k-N_\tau}) + N(y_{k-N_\tau}) + \theta f(y_k, y_{k-N_\tau})\Delta, \quad k \geq 0, \\ x_{k+1} = x_k + N(x_{k+1-N_\tau}) - N(x_{k-N_\tau}) + f(y_k, y_{k-N_\tau})\Delta + \lambda g(y_k, y_{k-N_\tau})\Delta W_k, \end{cases} \tag{2.7}$$

where stepsize $\Delta = \frac{\tau}{N_\tau}$ for a integer N_τ, $x_k = y_k = \psi(k\Delta)$ for $k = -N_\tau, -N_\tau + 1, \cdots, -1$, $y_0 = \psi(0)$, θ and λ are fixed parameters in interval $(0, 1]$. The Wiener increments is defined as $\Delta W_k := W((k+1)\Delta) - W(k\Delta)$, where $W(k\Delta)$ denotes the Wiener process at time $k\Delta$. It is interesting to deduce that the approximation $\{y_k\}_{k \geq 0}$ in (2.7) has the form

$$y_{k+1} = y_k + N(y_{k+1-N_\tau}) - N(y_{k-N_\tau}) + \theta f(y_{k+1}, y_{k+1-N_\tau})\Delta + \lambda g(y_k, y_{k-N_\tau})\Delta W_k. \tag{2.8}$$

We refer to (2.8) as semi-implicit (θ,λ)-Euler (SIE) method which includes the backward Euler (BE) method when $\theta=\lambda=1$.

3. Exponential mean-square stability analysis

In this section, we first prove the exponential mean-square stability of SSBE approximation $\{x_k\}_{k\geq 0}$ for $\theta,\lambda\in(0,1]$. For the purpose of stability, assume that $N(0)=f(0,0)=0$, $g(0,0)=0$. This shows that (2.1) admits a trivial solution.

Theorem 3.1. *Let all the conditions in Theorem 2.1 hold and $\theta,\lambda\in(0,1]$. If there exists two positive constants K_1 and K_2 such that the functions f and g satisfy the linear growth conditions*

$$|f(x,y)|^2\leq K_1(|x|^2+|y|^2),\qquad |g(x,y)|^2\leq K_2(|x|^2+|y|^2), \tag{3.1}$$

for all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, then there is a stepsize bound $\Delta^*=\dfrac{(\mu-\sigma)\theta+2K_2\lambda^2}{2(1-2\theta)K_1}$, such that for any $\Delta<\Delta^*$, the SSBE approximation $\{x_k\}_{k\geq 0}$ has the properties

$$\mathbb{E}|x_k-N(x_{k-N_\tau})|^2\leq C(\psi)e^{-\gamma_\Delta(\theta)k\Delta}, \tag{3.2}$$

and

$$\mathbb{E}|x_k|^2\leq C(\psi)e^{-\overline{\gamma}_\Delta(\theta)k\Delta}, \tag{3.3}$$

where $\overline{\gamma}_\Delta(\theta)=\gamma_\Delta(\theta)\wedge r$ and $\gamma_\Delta(\theta)\in(0,\frac{1}{\tau}\ln(\frac{\mu}{\sigma}))$ is the unique root of the equation

$$\begin{aligned}&-\mu+\frac{1-e^{-\gamma_\Delta(\theta)\Delta}}{\Delta}(1+\theta\Delta)(1+\varepsilon_0+K_1\theta\Delta)+(1-2\theta)K_1\Delta+\lambda^2K_2\\&+\left[\frac{1-e^{-\gamma_\Delta(\theta)\Delta}}{\Delta}(1+\theta\Delta)\Big(\frac{1+\varepsilon_0}{\varepsilon_0}\kappa^2+K_1\theta\Delta\Big)+\sigma+(1-2\theta)K_1\Delta+\lambda^2K_2\right]e^{\gamma_\Delta(\theta)\tau}=0,\end{aligned} \tag{3.4}$$

and

$$\lim_{\Delta\to 0}\gamma_\Delta(\theta)=\gamma. \tag{3.5}$$

Proof. Let $z_k=x_k-N(x_{k-N_\tau})$. It is easy to deduce from (2.7) that

$$\begin{aligned}|z_{k+1}|^2=&|z_k|^2+|f(y_k,y_{k-N_\tau})|^2\Delta^2+\lambda^2|g(y_k,y_{k-N_\tau})|^2|\Delta W_k|^2\\&+2z_k^Tf(y_k,y_{k-N_\tau})\Delta+2\langle z_k+f(y_k,y_{k-N_\tau})\Delta,\lambda g(y_k,y_{k-N_\tau})\Delta W_k\rangle.\end{aligned} \tag{3.6}$$

Note that $z_k=y_k-N(y_{k-N_\tau})-\theta f(y_k,y_{k-N_\tau})\Delta$. Substituting this equality into (3.6), we have

$$\begin{aligned}|z_{k+1}|^2=&|z_k|^2+(1-2\theta)|f(y_k,y_{k-N_\tau})|^2\Delta^2+\lambda^2|g(y_k,y_{k-N_\tau})|^2|\Delta W_k|^2\\&+2[y_k-N(y_{k-N_\tau})]^Tf(y_k,y_{k-N_\tau})\Delta+m_k^\Delta,\end{aligned} \tag{3.7}$$

where

$$m_k^\Delta=2\langle y_k-N(y_{k-N_\tau})+(1-\theta)f(y_k,y_{k-N_\tau})\Delta,\lambda g(y_k,y_{k-N_\tau})\Delta W_k\rangle.$$

Note that

$$\mathbb{E}\big(\Delta W_k\big) = 0, \tag{3.8}$$

therefore we conclude

$$\mathbb{E}(m_k^{\Delta}) = 0. \tag{3.9}$$

Now by using (2.4) and (3.1), and then taking the expectation on the both sides of (3.7), we obtain

$$\begin{aligned}\mathbb{E}|z_{k+1}|^2 \leq \mathbb{E}|z_k|^2 &+ \left[-\mu + (1-2\theta)K_1\Delta + \lambda^2 K_2\right] \Delta\mathbb{E}|y_k|^2 \\ &+ \left[\sigma + (1-2\theta)K_1\Delta + \lambda^2 K_2\right] \Delta\mathbb{E}|y_{k-N_\tau}|^2.\end{aligned} \tag{3.10}$$

Subsequently for any positive number $P \geq 1$, we have

$$\begin{aligned}P^{(k+1)\Delta}\mathbb{E}|z_{k+1}|^2 - P^{k\Delta}\mathbb{E}|z_k|^2 \leq{}& (1-P^{-\Delta})P^{(k+1)\Delta}\mathbb{E}|z_k|^2 \\ &+ [-\mu + (1-2\theta)K_1\Delta + \lambda^2 K_2]\Delta P^{(k+1)\Delta}\mathbb{E}|y_k|^2 \\ &+ [\sigma + (1-2\theta)K_1\Delta + \lambda^2 K_2]\Delta P^{(k+1)\Delta}\mathbb{E}|y_{k-N_\tau}|^2.\end{aligned} \tag{3.11}$$

Also by using the linear growth condition (3.1) and the elementary inequality

$$|a+b|^2 \leq (1+\varepsilon)(a^2 + \frac{1}{\varepsilon}b^2),$$

for any positive constants $a, b \in \mathbb{R}^n$ and ε, we obtain

$$\begin{aligned}\mathbb{E}|z_k|^2 &= \mathbb{E}\left(|y_k - N(y_{k-N_\tau}) - \theta f(y_k, y_{k-N_\tau})\Delta|^2\right) \\ &\leq (1+\theta\Delta)(1+\varepsilon+K_1\theta\Delta)\mathbb{E}|y_k|^2 + (1+\theta\Delta)\big(\frac{1+\varepsilon}{\varepsilon}\kappa^2 + K_1\theta\Delta\big)\mathbb{E}|y_{k-N_\tau}|^2.\end{aligned} \tag{3.12}$$

Substituting inequality (3.12) into (3.11) and letting $\varepsilon = \varepsilon_0$ yields

$$P^{(k+1)\Delta}\mathbb{E}|z_{k+1}|^2 - P^{k\Delta}\mathbb{E}|z_k|^2 \leq -\mu_\Delta(P)\Delta P^{(k+1)\Delta}\mathbb{E}|y_k|^2 + \sigma_\Delta(P)\Delta P^{(k+1)\Delta}\mathbb{E}|y_{k-N_\tau}|^2, \tag{3.13}$$

where

$$\mu_\Delta(P) = \mu - \frac{1-P^{-\Delta}}{\Delta}(1+\theta\Delta)(1+\varepsilon_0+K_1\theta\Delta) - (1-2\theta)K_1\Delta - \lambda^2 K_2, \tag{3.14}$$

and

$$\sigma_\Delta(P) = \sigma + \frac{1-P^{-\Delta}}{\Delta}(1+\theta\Delta)\big(\frac{1+\varepsilon_0}{\varepsilon_0}\kappa^2 + K_1\theta\Delta\big) + (1-2\theta)K_1\Delta + \lambda^2 K_2. \tag{3.15}$$

Summing relation (3.13) from $j=0$ to $j=k$, which implies

$$\begin{aligned} P^{(k+1)\Delta}\mathbb{E}|z_{k+1}|^2 &\le \mathbb{E}|z_0|^2 - \mu_\Delta(P)\Delta\sum_{j=0}^{k}P^{(j+1)\Delta}\mathbb{E}|y_j|^2 + \sigma_\Delta(P)\Delta\sum_{j=0}^{k}P^{(j+1)\Delta}\mathbb{E}|y_{j-N_\tau}|^2 \\ &\le \mathbb{E}|z_0|^2 + \sigma_\Delta(P)\Delta\sum_{j=-N_\tau}^{-1}P^{(j+1)\Delta}\mathbb{E}|y_{j-N_\tau}|^2 + h(P)\Delta\sum_{j=0}^{k}P^{(j+1)\Delta}\mathbb{E}|y_j|^2, \end{aligned} \tag{3.16}$$

where $h(P)=\mu_\Delta(P)-P^\tau\sigma_\Delta(P)$.

Let

$$\Delta^* = \begin{cases} +\infty, & \theta=\frac{1}{2},\ \lambda\in(0,1], \\ \dfrac{(\mu-\sigma)+2K_2\lambda^2}{2(1-2\theta)K_1}, & \theta\in[0,\frac{1}{2}),\ \lambda\in(0,1]. \end{cases} \tag{3.17}$$

For $\Delta<\Delta^*$, $h(1)<0$ and for $\underline{P}=(\frac{\mu}{\sigma})^{\frac{1}{\tau}}$, $h(\underline{P})>0$. Moreover, $h^{'}(P)>0$ for any $P>1$. Hence, for any $\theta,\lambda\in(0,1]$ and $\Delta<\Delta^*$, there is a unique positive constant $\gamma_\Delta(\theta)$ such that $e^{\gamma_\Delta(\theta)}\in(1,(\frac{\mu}{\sigma})^{\frac{1}{\tau}})$ and $h(e^{\gamma_\Delta(\theta)})=0$, which implies (3.4). The limitation (3.5) follows from (3.4), directly. Taking $P=e^{\gamma_\Delta(\theta)}$ in (3.16) it yields

$$e^{\gamma_\Delta(\theta)(k+1)\Delta}\mathbb{E}|z_{k+1}|^2 \le \mathbb{E}|z_0|^2 + \sigma_\Delta(e^{\gamma_\Delta(\theta)})\Delta\sum_{j=-N_\tau}^{-1}\mathbb{E}|y_{j-N_\tau}|^2 := C(\psi), \tag{3.18}$$

then

$$\mathbb{E}|z_k|^2 \le C(\psi)e^{-\gamma_\Delta(\theta)k\Delta},$$

which gives (3.2). Then from the definition of z_k and the contractive condition (2.2) we obtain that for any $\epsilon>0$ and $0\le i\le k$,

$$\begin{aligned} e^{\overline{\gamma}_\Delta(\theta)i\Delta}\mathbb{E}|x_i|^2 &\le (1+\epsilon)\big[e^{\overline{\gamma}_\Delta(\theta)i\Delta}\mathbb{E}|z_i|^2 + \frac{1}{\epsilon}\kappa^2 e^{\overline{\gamma}_\Delta(\theta)i\Delta}\mathbb{E}|x_{i-N_\tau}|^2\big] \\ &\le (1+\epsilon)\big[e^{\overline{\gamma}_\Delta(\theta)i\Delta}C(\psi)e^{-\gamma_\Delta(\theta)i\Delta} + \frac{1}{\epsilon}\kappa^2 e^{\overline{\gamma}_\Delta(\theta)i\Delta}\mathbb{E}|x_{i-N_\tau}|^2\big], \end{aligned} \tag{3.19}$$

and then we know that $e^{\overline{\gamma}_\Delta(\theta)i\Delta}e^{-\gamma_\Delta(\theta)i\Delta}=1$, we conclude

$$e^{\overline{\gamma}_\Delta(\theta)i\Delta}\mathbb{E}|x_i|^2 \le (1+\epsilon)C(\psi) + \frac{1+\epsilon}{\epsilon}\kappa^2 e^{\overline{\gamma}_\Delta(\theta)\tau}\sup_{-m\le j\le k}e^{\overline{\gamma}_\Delta(\theta)j\Delta}\mathbb{E}|x_j|^2. \tag{3.20}$$

Note that this inequality also holds for all $-m\le i\le 0$. In view of $\overline{\gamma}_\Delta(\theta)\le r<\frac{2}{\tau}\ln(\frac{1}{\kappa})$, there exists a positive constant ϵ_0 such that $R(\epsilon_0):=\dfrac{1+\epsilon_0}{\epsilon_0}\kappa^2 e^{\overline{\gamma}_\Delta(\theta)\tau}<1$, therefore we have

$$\sup_{-m\le j\le k}e^{\overline{\gamma}_\Delta(\theta)j\Delta}\mathbb{E}|x_j|^2 \le (1+\epsilon_0)C(\psi) + R(\epsilon_0)\sup_{-m\le j\le k}e^{\overline{\gamma}_\Delta(\theta)j\Delta}\mathbb{E}|x_j|^2, \tag{3.21}$$

which gives (3.3). This completes the proof of the Theorem 3.1. ☐

Theorem 3.1 shows that for sufficiently small stepsize, the bound of the Lyapunov exponent of the exact solution can also be preserved. Now we examine the exponential mean-square stability of SIE approximation $\{y_k\}_{k\geq 0}$ for $\theta, \lambda \in (0, 1]$.

Remark 3.2. It is easy to see that the coupled monotone condition (2.4) for some nonnegative constants $\eta_1, \eta_2, \eta_3, \eta_4$ implies that

$$2[x - N(y)]^T f(x, y) \leq -\eta_1 |x|^2 + \eta_2 |y|^2, \tag{3.22}$$

$$|g(x, y)|^2 \leq \eta_3 |x|^2 + \eta_4 |y|^2, \tag{3.23}$$

for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\eta_1 - \eta_3 > \eta_2 + \eta_4$.

Theorem 3.3. *Under Assumption 2.1 and Assumption 2.2, assume that conditions (3.22) and (3.23) hold and $\theta, \lambda \in (0, 1]$. Then the SIE method is exponential mean-square stable for any stepsize Δ with the properties*

$$\mathbb{E}|y_k - N(y_{k-N_\tau})|^2 \leq C(\psi) e^{-\gamma_\Delta(\theta) k\Delta}, \tag{3.24}$$

and

$$\mathbb{E}|y_k|^2 \leq e^{-\overline{\gamma}_\Delta(\theta) k\Delta}, \tag{3.25}$$

where $\overline{\gamma}_\Delta(\theta) = \gamma_\Delta(\theta) \wedge r$ and $\gamma_\Delta(\theta) \in (0, \frac{1}{\tau} \ln(\frac{\eta_1}{\eta_2}))$ is the unique root of the equation

$$\begin{aligned}
&- \eta_1 \theta + \eta_3 \lambda e^{\gamma_\Delta(\theta)\Delta} + (1 + \varepsilon_0)(\frac{e^{\gamma_\Delta(\theta)\Delta} - 1}{\Delta}) \\
&\quad + \left[\eta_2 \theta + \eta_4 \lambda e^{\gamma_\Delta(\theta)\Delta} + \kappa^2 (1 + \frac{1}{\varepsilon_0})(e^{\gamma_\Delta(\theta)\Delta} - 1)/\Delta\right] e^{\gamma_\Delta(\theta)\tau} = 0,
\end{aligned} \tag{3.26}$$

and

$$\lim_{\Delta \to 0} \gamma_\Delta(\theta) = \gamma. \tag{3.27}$$

Proof. Let $Y_k = y_k - N(y_{k-N_\tau})$. Then from conditions (3.22) and (3.23), we have

$$\begin{aligned}
|Y_{k+1}|^2 &= \langle Y_{k+1}, Y_k + \theta f(y_{k+1}, y_{k+1-N_\tau})\Delta + \lambda g(y_k, y_{k-N_\tau})\Delta W_k \rangle \\
&= \langle Y_{k+1}, \theta f(y_{k+1}, y_{k+1-N_\tau})\Delta \rangle + \langle Y_{k+1}, Y_k + \lambda g(y_k, y_{k-N_\tau})\Delta W_k \rangle \\
&\leq \frac{\theta}{2}(-\eta_1 |y_{k+1}|^2 + \eta_2 |y_{k+1-N_\tau}|^2)\Delta + \frac{1}{2}\left[|Y_{k+1}|^2 + |Y_k + \lambda g(y_k, y_{k-N_\tau})\Delta W_k|^2\right] \\
&\leq \frac{\theta}{2}(-\eta_1 |y_{k+1}|^2 + \eta_2 |y_{k+1-N_\tau}|^2)\Delta + \frac{1}{2}\left[|Y_{k+1}|^2 + |Y_k|^2 + \lambda(\eta_3 |y_k|^2 + \eta_4 |y_{k-N_\tau}|^2)\Delta\right] \\
&\quad + \frac{1}{2} m_k^\Delta,
\end{aligned} \tag{3.28}$$

where

$$m_k^\Delta = |g(y_k, y_{k-N_\tau})|^2 (|\Delta W_k|^2 - \Delta) + 2\lambda Y_k^T g(y_k, y_{k-N_\tau})\Delta W_k. \tag{3.29}$$

Then we have

$$|Y_{k+1}|^2 \leq |Y_k|^2 + \theta\big[-\eta_1 |y_{k+1}|^2 + \eta_2 |y_{k+1-N_\tau}|^2\big]\Delta + \lambda\big[\eta_3 |y_k|^2 + \eta_4 |y_{k-N_\tau}|^2\big]\Delta + m_k^\Delta.$$

$$(3.30)$$

Note that $\mathbb{E}\big(|\Delta W_k|^2\big) = \Delta$ and $\mathbb{E}\big(\Delta W_k\big) = 0$, which implies $\mathbb{E}(m_k^\Delta) = 0$. Taking expectation on the both sides of inequality (3.30), we have

$$\mathbb{E}|Y_{k+1}|^2 \leq \mathbb{E}|Y_k|^2 + \theta\left[-\eta_1\mathbb{E}|y_{k+1}|^2 + \eta_2\mathbb{E}|y_{k+1-N_\tau}|^2\right]\Delta + \lambda\left[\eta_3\mathbb{E}|y_k|^2 + \eta_4\mathbb{E}|y_{k-N_\tau}|^2\right]\Delta. \tag{3.31}$$

Subsequently for any positive number $P > 1$, we derive

$$\begin{aligned}\left[P^{(k+1)\Delta}\mathbb{E}|Y_{k+1}|^2 - P^{k\Delta}\mathbb{E}|Y_k|^2\right] \leq{}& \theta\left[-\eta_1\mathbb{E}|y_{k+1}|^2 + \eta_2\mathbb{E}|y_{k+1-N_\tau}|^2\right]\Delta P^{(k+1)\Delta} \\ &+ \left(P^{(k+1)\Delta} - P^{k\Delta}\right)\mathbb{E}|Y_k|^2 \\ &+ \lambda\left[\eta_3\mathbb{E}|y_k|^2 + \eta_4\mathbb{E}|y_{k-N_\tau}|^2\right]\Delta P^{(k+1)\Delta}.\end{aligned} \tag{3.32}$$

Summing relation (3.32) from $i = 0$ to $i = k-1$, which implies

$$\begin{aligned}\sum_{i=0}^{k-1}\left[P^{(i+1)\Delta}\mathbb{E}|Y_{i+1}|^2 - P^{i\Delta}\mathbb{E}|Y_i|^2\right] \leq{}& \sum_{i=0}^{k-1}\theta\left[-\eta_1\mathbb{E}|y_{i+1}|^2 + \eta_2\mathbb{E}|y_{i+1-N_\tau}|^2\right]\Delta P^{(i+1)\Delta} \\ &+ \sum_{i=0}^{k-1}\lambda\left[\eta_3\mathbb{E}|y_i|^2 + \eta_4\mathbb{E}|y_{i-N_\tau}|^2\right]\Delta P^{(i+1)\Delta} \\ &+ \sum_{i=0}^{k-1}\left(P^{(i+1)\Delta} - P^{i\Delta}\right)\mathbb{E}|Y_i|^2.\end{aligned}$$

By using the contractive condition (2.2) and the elementary inequality $|a+b|^2 \leq (1+\varepsilon)(a^2 + \frac{1}{\varepsilon}b^2)$ for any positive constants $a, b \in \mathbb{R}^n$ and ε, we obtain

$$\begin{aligned}P^{k\Delta}\mathbb{E}|Y_k|^2 \leq{}& \mathbb{E}|Y_0|^2 - \eta_1\theta\Delta\sum_{i=0}^{k-1}P^{(i+1)\Delta}\mathbb{E}|y_{i+1}|^2 + \eta_2\theta\Delta\sum_{i=0}^{k-1}P^{(i+1)\Delta}\mathbb{E}|y_{i+1-N_\tau}|^2 \\ &+ \left[\eta_3\lambda\Delta P^\Delta + (1+\varepsilon_0)(P^\Delta - 1)\right]\sum_{i=0}^{k-1}P^{i\Delta}\mathbb{E}|y_i|^2 \\ &+ \left[\eta_4\lambda\Delta P^\Delta + \kappa^2(1+\frac{1}{\varepsilon_0})(P^\Delta - 1)\right]\sum_{i=0}^{k-1}P^{i\Delta}\mathbb{E}|y_{i-N_\tau}|^2.\end{aligned} \tag{3.33}$$

Note that

$$-\sum_{i=0}^{k-1}P^{(i+1)\Delta}\mathbb{E}|y_{i+1}|^2 = -\sum_{i=0}^{k-1}P^{i\Delta}\mathbb{E}|y_i|^2 + \mathbb{E}|y_0|^2 - P^{k\Delta}\mathbb{E}|y_k|^2, \tag{3.34}$$

and

$$\begin{aligned}\sum_{i=0}^{k-1} P^{(i+1)\Delta}\mathbb{E}|y_{i+1-N_\tau}|^2 &= \sum_{i=-N_\tau+1}^{k-N_\tau} P^{(i+N_\tau)\Delta}\mathbb{E}|y_i|^2 \\ &= P^{N_\tau\Delta}\sum_{i=-N_\tau+1}^{-1} P^{i\Delta}\mathbb{E}|y_i|^2 + P^{N_\tau\Delta}\sum_{i=0}^{k-1} P^{i\Delta}\mathbb{E}|y_i|^2 \\ &\quad - P^{N_\tau\Delta}\sum_{i=k-N_\tau+1}^{k-1} P^{i\Delta}\mathbb{E}|y_i|^2. \end{aligned} \tag{3.35}$$

Substituting (3.34) and (3.35) into (3.33) it yields

$$\begin{aligned} P^{k\Delta}\mathbb{E}|Y_k|^2 \le& \left[\eta_2\theta + \eta_4\lambda P^{\Delta} + \kappa^2(1+\frac{1}{\varepsilon_0})(P^{\Delta}-1)/\Delta\right] P^{\tau}\Delta \sum_{i=-N_\tau+1}^{-1}\mathbb{E}|y_i|^2 \\ &+ \mathbb{E}|Y_0|^2 + \eta_1\theta\Delta\mathbb{E}|y_0|^2 + h(P)\Delta\sum_{i=0}^{k-1}P^{i\Delta}\mathbb{E}|y_i|^2, \end{aligned} \tag{3.36}$$

where

$$h(P) = -\eta_1\theta + \eta_3\lambda P^{\Delta} + (1+\varepsilon_0)\frac{P^{\Delta}-1}{\Delta} + \left[\eta_2\theta + \eta_4\lambda P^{\Delta} + \kappa^2(1+\frac{1}{\varepsilon_0})(P^{\Delta}-1)/\Delta\right] P^{\tau}.$$

For $\underline{P} = (\frac{\eta_1}{\eta_2})^{\frac{1}{\tau}}$, it is easy to see that $h(\underline{P}) > 0$, and $h(1) < 0$, which $h'(P) > 0$ for any positive constant $P \geq 1$. Hence, for any $\theta, \lambda \in (0, 1]$ there exists a unique constant $P^*_\Delta \in (1, (\frac{\eta_1}{\eta_2})^{\frac{1}{\tau}})$ such that $h(P^*_\Delta) = 0$. Taking $P = P^*_\Delta = e^{\gamma_\Delta(\theta)}$ in (3.36) we get following inequality

$$\mathbb{E}|Y_k|^2 \le C(\psi)e^{-\gamma_\Delta(\theta)k\Delta},$$

which implies (3.24). Moreover

$$\begin{aligned} h(e^{\gamma_\Delta(\theta)}) =& -\eta_1\theta + \eta_3\lambda e^{\gamma_\Delta(\theta)\Delta} + (1+\varepsilon_0)(\frac{e^{\gamma_\Delta(\theta)\Delta}-1}{\Delta}) \\ &+ \left[\eta_2\theta + \eta_4\lambda e^{\gamma_\Delta(\theta)\Delta} + \kappa^2(1+\frac{1}{\varepsilon_0})(e^{\gamma_\Delta(\theta)\Delta}-1)/\Delta\right] e^{\gamma_\Delta(\theta)\tau} = 0, \end{aligned}$$

which implies (3.26). The limitation (3.27) follows from (3.26), directly. Now by similar the arguments used in the proof of Theorem 3.1, we can obtain the relation (3.25). □

Theorem 3.3 shows that the SIE method for $\theta, \lambda \in (0, 1]$, can recovers the exponential mean-square stability unconditionally.

4. Numerical illustrations

By the numerical test, we show the influence of θ and λ on exponential mean-square stability of the SSBE and SIE methods.

Example 4.1. Consider the following nonlinear NSDDE:

$$d\left[x(t)-\frac{1}{4}\sin(x(t-1))\right]=\big(-6x(t)+x(t-1)\big)dt+x(t)\cos(x(t-1))dW(t),\ \ t>0, \tag{4.1}$$

with the initial data $x(t)=1$ for $t\in[-1,0]$, where $W(t)$ is a scalar Brownian motion. It is easy to see that the drift and diffusion coefficients satisfy the linear growth conditions (3.1). We can deduce that for any $\rho\in(0,\frac{11}{5})$,

$$\begin{aligned}2[x-N(y)]^T f(x,y)+|g(x,y)|^2&=-12x^2+2xy+3x\sin(y)-\frac{1}{2}y\sin(y)+x^2\\&\le -11x^2+5|xy|+\frac{1}{2}y^2\\&\le(-11+5\rho)x^2+(\frac{4}{5\rho}+\frac{1}{2})y^2.\end{aligned}\tag{4.2}$$

Let $\mu=11-5\rho$ and $\sigma=\frac{4}{5\rho}+\frac{1}{2}$, then we conclude that $\mu>\sigma$. For example by setting $\rho=1.2$, we obtain $\mu=5$ and $\sigma=\frac{7}{6}$, and applying Theorem 2.1 it yields

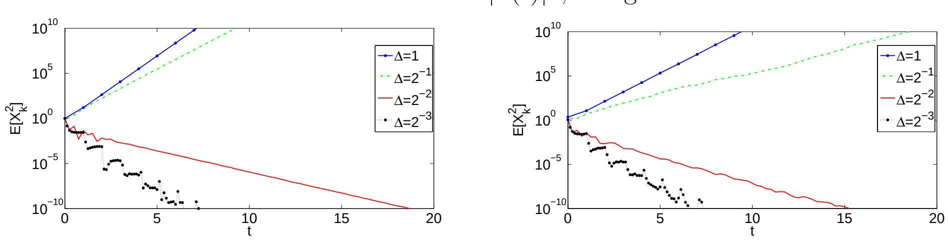
$$\mathbb{E}|x(t)-N(x(t-1))|^2\le C(\psi)e^{-0.7754t},$$

and

$$\mathbb{E}|x(t)|^2\le C(\psi)e^{-0.7754t}.$$

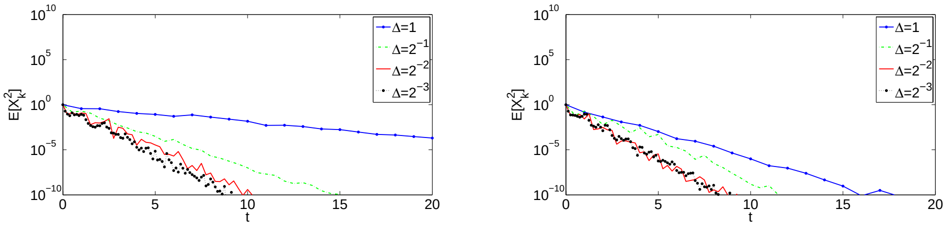
That is, the trivial solution to equation (4.1) is exponentially mean-square stable with the Lyapunov exponent less than -0.7754.

FIGURE 1. Simulation of $\mathbb{E}|x(t)|^2$, using SSBE method.



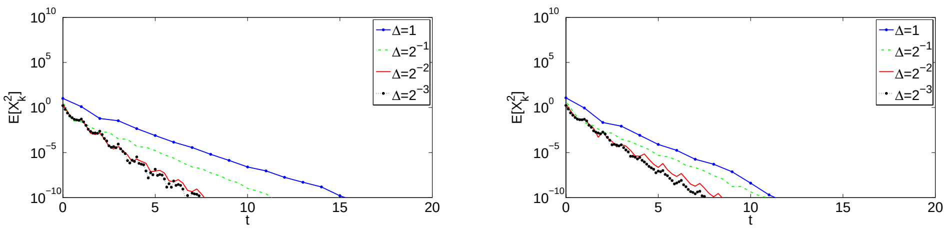
(A) Unstable and stable tests with $\theta=0.1, \lambda=0.1$. (B) Unstable and stable tests with $\theta=0.1, \lambda=0.5$.

FIGURE 2. Simulation of $\mathbb{E}|x(t)|^2$, using SSBE method.



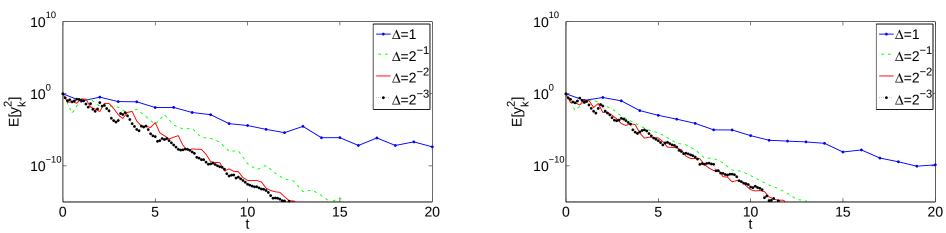
(A) Stable test with $\theta = 0.6, \lambda = 0.1$. (B) Stable test with $\theta = 0.6, \lambda = 0.5$.

FIGURE 3. Simulation of $\mathbb{E}|x(t)|^2$, using SSBE method.



(A) Stable test with $\theta = 0.8, \lambda = 0.5$. (B) Stable test with $\theta = 0.8, \lambda = 0.8$.

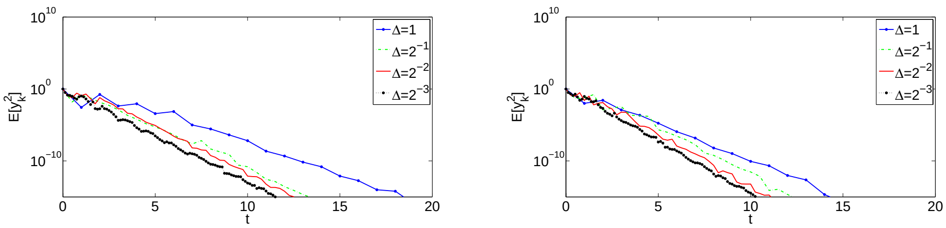
FIGURE 4. Simulation of $\mathbb{E}|y(t)|^2$, using SIE method.



(A) Stable test with $\theta = 0.1, \lambda = 0.1$. (B) Stable test with $\theta = 0.1, \lambda = 0.5$.

Choosing the stepsizes $\Delta = 1, 2^{-1}, 2^{-2}, 2^{-3}$ and taking the average of 10^3 sample paths, we obtain the stability analysis of the SSBE and SIE methods numerically, which are shown in Figures 1-6. In Figure 1, we can see that, the stepsize bound will be needed for the SSBE method to preserve the exponential mean-square stability of the exact solution, but Figures 2 and 3 show that the SSBE method with $(\theta = 0.6, \lambda = 0.1)$, $(\theta = 0.6, \lambda = 0.5)$, $(\theta = 0.8, \lambda = 0.5)$ and $(\theta = 0.8, \lambda = 0.8)$ can share the exponential mean-square stability even for large stepsize. In Figures 3-6 we can see that, the SIE method for any different choices of θ and λ can reproduce the exponential mean-square stability unconditionally.

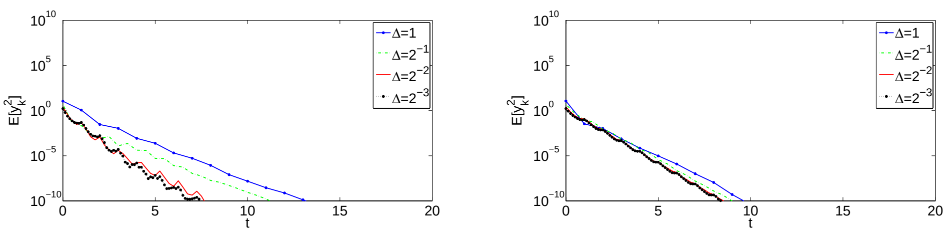
FIGURE 5. Simulation of $\mathbb{E}|y(t)|^2$, using SIE method.



(A) Stable test with $\theta = 0.6, \lambda = 0.1$.

(B) Stable test with $\theta = 0.6, \lambda = 0.5$.

FIGURE 6. Simulation of $\mathbb{E}|y(t)|^2$, using SIE method.



(A) Stable test with $\theta = 0.8, \lambda = 0.5$.

(B) Stable test with $\theta = 0.8, \lambda = 0.8$.

Conclusion

In this paper, we have investigated two classes improved backward Euler methods for NSDDEs under a coupled monotone condition on drift and diffusion coefficients. In this regard we examined the exponential mean-square stability for these kind of equations. The parameters θ and λ can extend the values of stepsize Δ in the exponential mean-square stability for SSBE method. We obtained the stability results of the SSBE and SIE methods numerically, which is shown in Figures 1-6.

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