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Fractional-order Legendre wavelets and their applications for solving fractional-order differential equations with initial/boundary conditions

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Abstract In this manuscript a new method is introduced for solving fractional differential equations. The fractional derivative is described in the Caputo sense. The main idea is to use fractional-order Legendre wavelets and operational matrix of fractional-order integration. First the fractional-order Legendre wavelets (FLWs) are presented. Then a family of piecewise functions is proposed, based on which the fractional order integration of FLWs are easy to calculate. The approach is used this operational matrix with the collocation points to reduce the under study problem to a system of algebraic equations. Convergence of the fractional-order Legendre wavelet basis is demonstrate. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords. Fractional-order Legendre wavelets, Fractional differential equations, Collocation method, Caputo derivative, Operational matrix.

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1. Introduction

Fractional differential equations (FDEs) are generalizations of ordinary differential equations to an arbitrary (non-integer) order. A history of the development of fractional differential operators can be found in [27, 31].
FDEs appear in several problems in science and engineering such as

- Viscoelasticity [2].
- Bioengineering [23].
- Dynamics of interfaces between nanoparticles and substrates [7].

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- Fluid-dynamic traffic [17].
- Signal processing [33].
- Control theory [5].
- Solid mechanics [40].
- Statistical mechanics [24].
- Economics [3].
- Nonlinear oscillations of earthquakes [16].

In general, most of fractional differential equations do not have exact solution, in this case, finding numerical solutions has become the destination of many mathematicians, some of these methods include: Fourier transforms [14], spectral Tau method [11], eigenvector expansion [46], Laplace transforms [34], Adomian decomposition method [28], variational iteration method [29], power series method [30], finite difference method [25], fractional differential transform method [1], collocation method [38], Tau method [43], homotopy analysis method [10], Bernstein operational matrix method [41], Legendre operational matrix method [42] and fractional-order Bernoulli wavelets [36].

The available sets of orthogonal functions can be divided into three classes. The first class includes sets of piecewise constant basis functions (e.g., block-pulse, Haar and Walsh [32]). The second class consists of sets of orthogonal polynomials (e.g., Chebyshev, Laguerre and Legendre). The third class is the set of sine-cosine functions in the Fourier series. Orthogonal functions have been used when dealing with various problems of the dynamical systems. The most frequently used orthogonal functions are sine-cosine functions, block-pulse functions, Legendre, Laguerre and Chebyshev orthonormal sets of functions. The remarkable orthonormal wavelets provide basis for many important spaces.

Wavelets theory is a relatively new and an emerging area in the field of applied science and engineering. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representation and segmentation, time-frequency analysis and fast algorithms for easy implementation [8]. Wavelets have many attractive features: orthogonality, arbitrary regularity and good localization. The study of wavelets attained its present growth after the mathematical analysis of wavelets by Stromberg [45], Grossmann and Morlet [15] and Meyer [26]. Wavelet basis can be used to reduce the underlying problem to a system of algebraic equations by estimating the integrals using operational matrices. Recently the operational matrices for fractional order integration for Haar wavelets [49], Chebyshev wavelets [48], CAS wavelet [44], Legendre wavelet [39] and Bernoulli wavelet [19, 35] have been developed to solve fractional order differential equations.

Recently, in [18] Kazem et al. defined new orthogonal functions based on the shifted Legendre polynomials to obtain the numerical solution of fractional-order differential equations. Yin et al. [47] extended this definition and presented the operational matrix of fractional derivative and integration for such functions to construct a new Tau technique for solving fractional partial differential equations (FPDEs). Bhrawy et al. [4] proposed the fractional-order generalized Laguerre functions based on the generalized Laguerre polynomials. They used these functions to find numerical solution of systems of fractional differential equations. In [51] Yüzbasi presented a collocation method based on the fractional-order Bernstein functions for solving the fractional Riccati type differential equations. Chen et al. [6] expanded the fractional Legendre functions to interval $[0, h]$ and to acquire numerical solution of the FPDEs. In [21], Krishnasamy and Razzaghi defined the fractional Taylor vector approximation for solving the Bagley-Torvik equation. Darani and Nasiri [9] introduced a type of fractional-order polynomials based on the classical Chebyshev polynomials of the second kind for solving linear fractional differential equations. Rahimkhani et al. [36] constructed the fractional-order Bernoulli wavelets for solving FDEs and system of FDEs. Moreover, Rahimkhani et al. [37] expanded the fractional-order Bernoulli wavelets to interval $[0, h]$ and to acquire numerical solution of the fractional pantograph differential equations.

In this paper, a new numerical method for solving the initial and boundary value problems for the fractional differential equations is presented. The method is based upon fractional-order Legendre wavelets (FLWs) approximation. First, the fractional-order Legendre wavelets are constructed. Then, we obtain the operational matrix of fractional order integration for FLWs by expanding these wavelets into a family of piecewise functions where are introduced. Finally, this matrix is utilized to reduce the solution of the fractional differential equations with initial and boundary conditions to the solution of algebraic equations.

The rest part of this paper is organized as follows. In Section 2, some necessary mathematical preliminaries and notations of fractional calculus are given. Section 3 is devoted to the basic formulation of wavelets and the fractional-order Legendre wavelets. In Section 4, we derive the fractional-order Legendre wavelets operational matrix of fractional integration. Section 5 is devoted to problem statement. In Section 6, the numerical method for solving the initial and boundary value problems for FDEs and systems of FDEs is presented. In Section 7, convergence of the fractional-order Legendre wavelet bases is demonstrated. In Section 8 we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering seven numerical examples. The conclusion is included in Section 9.

2. Preliminaries and notations

2.1. **The fractional derivative and integral.** There are various definitions of fractional derivative and integration. The widely used definition of a fractional derivative is the Caputo definition, and a fractional integration is the Riemann-Liouville definition.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\nu \geq 0$ is defined as [20, 36]

$$I^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{f(s)}{(t-s)^{1-\nu}} ds, \qquad \nu > 0, \quad t > 0, \tag{2.1}$$
$$I^0 f(t) = f(t),$$

where

$$\Gamma(\nu) = \int_0^{\infty} t^{\nu-1} e^{-t} dt.$$

For the Riemann-Liouville fractional integral we have [13]

$$\begin{aligned} I^{\nu}t^{\beta} &= \tfrac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)}t^{\nu+\beta}, \qquad \beta > -1, \\ I^{\nu}(\lambda f(t) + \mu g(t)) &= \lambda I^{\nu}f(t) + \mu I^{\nu}g(t), \\ I^{\nu}I^{\mu}f(t) &= I^{\nu+\mu}f(t), \\ I^{\nu}I^{\mu}f(t) &= I^{\mu}I^{\nu}f(t). \end{aligned}$$

Definition 2.2. Caputo's fractional derivative of order ν is defined as [20, 36]

$$D^{\nu}f(t) = \frac{1}{\Gamma(n-\nu)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\nu+1-n}}ds, \qquad n-1 < \nu \leq n,\ n \in \mathbb{N}. \tag{2.2}$$

For the Caputo derivative we have [13]

$$\begin{aligned} D^{\nu}c &= 0, \\ D^{\nu}I^{\nu}f(t) &= f(t), \\ I^{\nu}D^{\nu}f(t) &= f(t) - \textstyle\sum_{i=0}^{n-1} y^{(i)}(0)\frac{t^i}{i!}, \\ D^{\nu}(\lambda f(t) + \mu g(t)) &= \lambda D^{\nu}f(t) + \mu D^{\nu}g(t), \end{aligned}$$

where c is constant.

Definition 2.3. (Generalized Taylor's formula) [18]. Suppose that $D^{i\alpha}f \in C(0,1]$ for $i = 0, 1, ..., m$. Then we have

$$f(t) = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}f(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}D^{m\alpha}f(\xi), \tag{2.3}$$

with $0 < \xi \leq t, \forall t \in (0,1]$. Also, one has

$$|f(t) - \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}f(0^+)| \leq M_{\alpha}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{2.4}$$

where $M_{\alpha} \geq sup_{\xi\in(0,1]}|D^{m\alpha}f(\xi)|$.
In case of $\alpha = 1$, the generalized Taylor's formula (2.3) reduces to the classical Taylor's formula.

2.2. **Fractional-order Legendre functions.** The fractional-order Legendre functions (FLFs) are proposed by Kazem et al. [18]. These functions are defined by transformation $x = t^{\alpha}, (\alpha > 0)$ based on the shifted Legendre polynomials and are denoted by $Fl_i^{\alpha}(t),\ i = 0, 1, 2, \ldots$. They are particular solution of the normalized eigenfunctions of the singular Sturm-Liouville problem [18]

$$((t - t^{1+\alpha})Fl_i'^{\alpha}(t))' + \alpha^2 i(i+1)t^{\alpha-1}Fl_i^{\alpha}(t) = 0, \qquad t \in [0,1].$$

The fractional-order Legendre functions $Fl_i^{\alpha}(t)$ satisfy the following recursive formula

$$\begin{aligned} Fl_{i+1}^{\alpha}(t) &= \tfrac{(2i+1)(2t^{\alpha}-1)}{i+1}Fl_i^{\alpha}(t) - \tfrac{i}{i+1}Fl_{i-1}^{\alpha}(t), \qquad i = 1, 2, \ldots, \\ Fl_0^{\alpha}(t) &= 1, \qquad Fl_1^{\alpha}(t) = 2t^{\alpha} - 1. \end{aligned}$$

The analytic form of $Fl_i^{\alpha}(t)$ of degree $i\alpha$ given by

$$Fl_i^{\alpha}(t) = \sum_{s=0}^{i} b_{s,i} t^{s\alpha}, \qquad i = 0, 1, 2, \dots, \tag{2.5}$$

where $b_{s,i} = \frac{(-1)^{i+s}(i+s)!}{(i-s)!(s!)^2}$, and $Fl_i^{\alpha}(0) = (-1)^i$, $Fl_i^{\alpha}(1) = 1$.
The FLFs are orthogonal with respect to the weight function $\omega(t) = t^{\alpha-1}$ on the interval $[0, 1]$, then the orthogonal condition is [18]

$$\int_0^1 Fl_n^{\alpha}(t) Fl_m^{\alpha}(t) t^{\alpha-1} dt = \frac{1}{(2m+1)\alpha} \delta_{nm}, \qquad m \geq n, \tag{2.6}$$

where δ_{nm} is the Kronecker function.

3. Fractional-order Legendre wavelets and their properties

3.1. Wavelets and fractional-order Legendre wavelets. Wavelets have been very successfully used in many scientific and engineering fields. They constitute a family of functions constructed from dilation and translation of a signal function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [36]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R}, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, where n and k are positive integers, the family of discrete wavelets are defined as

$$\psi_{k,n}(t) = |a_0|^{k/2} \psi(a_0^k t - nb_0),$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(\mathbb{R})$.
Fractional-order Legendre wavelets $\psi_{n,m}^{\alpha}(t) = \psi^{\alpha}(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = n - 1, n = 1, 2, \dots, 2^{k-1}, k$ can assume any positive integer, m is the order for fractional-order Legendre functions and t is the normalized time. We define them on the interval $[0, 1)$ as follows

$$\psi_{n,m}^{\alpha}(t) = \begin{cases} ((2m+1)\alpha)^{\frac{1}{2}} 2^{\frac{k-1}{2}} Fl_m^{\alpha}(2^{k-1}t - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $m = 0, 1, 2, \dots, M-1$ and $n = 1, 2, \dots, 2^{k-1}$. The coefficient $((2m+1)\alpha)^{\frac{1}{2}}$ is for normality, the dilation parameter is $a = 2^{-(k-1)}$ and the translation parameter $b = \hat{n}2^{-(k-1)}$. Here, $Fl_m^{\alpha}(t)$ are the fractional-order Legendre functions of order $m\alpha$ which are defined in the Section 2.2.

3.2. Function approximation. A function f, defined over $[0, 1)$, may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}^{\alpha}(t), \tag{3.2}$$

where the coefficients c_{nm} are given by

$$c_{nm} = < f, \psi_{nm}^{\alpha} > = \int_0^1 f(t) \psi_{nm}^{\alpha}(t) t^{\alpha-1} dt,$$

where $<,>$ denotes the inner product in $L^2[0,1]$. If the infinite series in Eq. (3.2) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}^{\alpha}(t) = C^T \Psi^{\alpha}(t),$$

where C and $\Psi^{\alpha}(t)$ are $2^{k-1}M \times 1$, vectors given by

$$\begin{aligned} C &= [c_{10}, c_{11}, \dots, c_{1(M-1)}, c_{20}, \dots, c_{2(M-1)}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}(M-1)}]^T \\ &= [c_1, c_2, c_3, \dots, c_{\hat{m}}]^T, \qquad \hat{m} = 2^{k-1}M, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \Psi^{\alpha}(t) &= [\psi_{10}^{\alpha}(t), \psi_{11}^{\alpha}(t), \dots, \psi_{1(M-1)}^{\alpha}(t), \dots, \psi_{2^{k-1}0}^{\alpha}(t), \dots, \psi_{2^{k-1}(M-1)}^{\alpha}(t)]^T \\ &= [\psi_1^{\alpha}(t), \psi_2^{\alpha}(t), \psi_3^{\alpha}(t), \dots, \psi_{\hat{m}}^{\alpha}(t)]^T. \end{aligned} \tag{3.4}$$

3.3. **Piecewise fractional-order Taylor functions.** In this section, a new family of piecewise functions is introduced to facilitate the calculation of the fractional-order Legendre wavelets operational matrix of fractional integration.
Since fractional-order Legendre wavelets are formulated based on fractional-order functions, a family of functions based on fractional-order Taylor defined on $[0,1)$ is constructed as

$$\phi_{n,m}^{\alpha}(t) = \begin{cases} t^{m\alpha}, & \frac{\hat{n}}{2^{k-1}} \le t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

where $\hat{n} = n-1, n = 1, 2, \dots, 2^{k-1}$, $m = 0, 1, 2, \dots, M-1$ and α is real constant. Unlike fractional-order Legendre wavelets, this family of functions is not normalized. We let

$$\begin{aligned} \Phi^{\alpha}(t) &= [\phi_{10}^{\alpha}(t), \phi_{11}^{\alpha}(t), \dots, \phi_{1(M-1)}^{\alpha}(t), \dots, \phi_{2^{k-1}0}^{\alpha}(t), \dots, \phi_{2^{k-1}(M-1)}^{\alpha}(t)]^T \\ &= [\phi_1^{\alpha}(t), \phi_2^{\alpha}(t), \phi_3^{\alpha}(t), \dots, \phi_{\hat{m}}^{\alpha}(t)]^T, \end{aligned} \tag{3.6}$$

and

$$T^{\alpha}(t) = [1, t^{\alpha}, t^{2\alpha}, \dots, t^{(M-1)\alpha}]^T, \tag{3.7}$$

where $T^{\alpha}(t)$ is the $M \times 1$ vector of fractional-order Taylor functions [21].

4. Operational matrix of fractional integration

In this section, we derive the fractional-order Legendre wavelets operational matrix of fractional integration by first transforming these wavelets to piecewise fractional-order Taylor functions introduced in previous section. Then, we derive the piecewise fractional-order Taylor functions operational matrix of fractional integration, and finally we derive the fractional-order Legendre wavelet operational matrix of fractional integration.

4.1. **Transformation matrix of the fractional-order Legendre wavelets to the piecewise fractional-order Taylor functions.** The fractional-order Legendre wavelets $\Psi^{\alpha}(t)$ can be expanded into $\hat{m}-$set of piecewise fractional-order Taylor functions $\Phi^{\alpha}(t)$ as

$$\Psi^{\alpha}_{\hat{m}\times 1}(t) = \Theta^{-1}_{\hat{m}\times\hat{m}}\Phi^{\alpha}_{\hat{m}\times 1}(t), \tag{4.1}$$

where Θ^{-1} is the transformation matrix of fractional-order Legendre wavelets to the piecewise fractional-order Taylor functions. Also, Θ^{-1} denotes inverse matrix of Θ.
The connection between the piecewise fractional-order Taylor functions and the fractional-order Legendre wavelets can be demonstrated as follows

$$\phi^{\alpha}_{i}(t) = \sum_{j=1}^{\hat{m}} \theta_{ij}\psi^{\alpha}_{j}(t), \qquad i = 1, 2, \ldots, \hat{m}, \tag{4.2}$$

where

$$\theta_{ij} = \langle \phi^{\alpha}_{i}, \psi^{\alpha}_{j} \rangle = \int_0^1 \phi^{\alpha}_{i}(t)\psi^{\alpha}_{j}(t)t^{\alpha-1}dt, \qquad i = j = 1, 2, \ldots, \hat{m},$$

and we let

$$\Theta = [\theta_{ij}],$$

where Θ is a matrix of order $\hat{m} \times \hat{m}$.
Then, the following result is obtained

$$\Phi^{\alpha}(t) = \Theta\Psi^{\alpha}(t). \tag{4.3}$$

Also, we can write

$$\Psi^{\alpha}(t) = \Theta^{-1}\Phi^{\alpha}(t), \tag{4.4}$$

therefore, Θ^{-1} is the transformation matrix of the fractional-order Legendre wavelets to the piecewise fractional-order Taylor functions.
For example, for $k = 2, M = 3$ and $\alpha = 1$ the transformation matrix can be expressed as

$$\Theta = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4\sqrt{2}} & \frac{1}{4\sqrt{6}} & 0 & 0 & 0 & 0 \\ \frac{1}{12\sqrt{2}} & \frac{1}{8\sqrt{6}} & \frac{1}{24\sqrt{10}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4\sqrt{2}} & \frac{1}{4\sqrt{6}} & 0 \\ 0 & 0 & 0 & \frac{7}{12\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{8} & \frac{1}{24\sqrt{10}} \end{bmatrix}.$$

Also, for $k = 2, M = 3$ and $\alpha = 2$ the transformation matrix can be expressed as

$$\Theta = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{32} & \frac{1}{32\sqrt{3}} & 0 & 0 & 0 & 0 \\ \frac{1}{192} & \frac{1}{128\sqrt{3}} & \frac{1}{384\sqrt{5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{5}} \\ 0 & 0 & 0 & \frac{15}{32} & -\frac{\sqrt{3}}{160} & \frac{3}{28\sqrt{5}} \\ 0 & 0 & 0 & \frac{21}{64} & \frac{47\sqrt{3}}{896} & \frac{71}{2688\sqrt{5}} \end{bmatrix}.$$

4.2. **Piecewise fractional-order Taylor functions operational matrix of fractional integration.** The Riemann-Liouville fractional integration of the vector $\Phi^{\alpha}(t)$ given in Eq. (3.6) can be expressed by

$$I^{\nu}\Phi^{\alpha}(t) = F^{\alpha}(t,\nu)\Phi^{\alpha}(t), \tag{4.5}$$

where $F^{\alpha}(t,\nu)$ is the $\hat{m}\times\hat{m}$ operational matrix of fractional integration of order ν.

From Eq. (3.7), and the properties of the operator I^{ν} we have

$$\begin{aligned} I^{\nu}T^{\alpha}(t) &= [\frac{1}{\Gamma(\alpha+1)}, \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)}t^{\alpha+\nu}, \ldots, \frac{\Gamma((M-1)\alpha+1)}{\Gamma((M-1)\alpha+\nu+1)}t^{(M-1)\alpha+\nu}]^{T} \\ &= t^{\nu}G^{(\alpha,\nu)}T^{\alpha}(t) = H^{\alpha}(t,\nu)T^{\alpha}(t), \end{aligned} \tag{4.6}$$

where

$$G^{(\alpha,\nu)} = diag[\tfrac{1}{\Gamma(\alpha+1)}, \tfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)}, \tfrac{\Gamma(2\alpha+1)}{\Gamma(2\alpha+\nu+1)}, \ldots, \tfrac{(M-1)\alpha+1)}{\Gamma((M-1)\alpha+\nu+1)}].$$

Using Eqs. (4.5) and (4.6), we get

$$F^{\alpha}(t,\nu) = diag[H^{\alpha}(t,\nu), H^{\alpha}(t,\nu), \ldots, H^{\alpha}(t,\nu)]. \tag{4.7}$$

4.3. **Fractional-order Legendre wavelets operational matrix of fractional integration.** We now derive the fractional-order Legendre wavelets operational matrix of fractional integration. Let

$$I^{\nu}\Psi^{\alpha}(t) = P^{\alpha}(t,\nu)\Psi^{\alpha}(t), \tag{4.8}$$

where the matrix $P^{\alpha}(t,\nu)$ is called fractional-order Legendre wavelets operational matrix of fractional integration of order ν. Using Eqs. (4.1) and (4.5) we get

$$I^{\nu}\Psi^{\alpha}(t) = I^{\nu}\Theta^{-1}\Phi^{\alpha}(t) = \Theta^{-1}F^{\alpha}(t,\nu)\Phi^{\alpha}(t). \tag{4.9}$$

From Eqs. (4.8) and (4.9) we have

$$P^{\alpha}(t,\nu)\Psi^{\alpha}(t) = P^{\alpha}(t,\nu)\Theta^{-1}\Phi^{\alpha}(t) = \Theta^{-1}F^{\alpha}(t,\nu)\Phi^{\alpha}(t). \tag{4.10}$$

Then, the fractional-order Legendre wavelets operational matrix of fractional integration $P^{\alpha}(t,\nu)$ is given by

$$P^{\alpha}(t,\nu) = \Theta^{-1}F^{\alpha}(t,\nu)\Theta. \tag{4.11}$$

Therefore, we have found the operational matrix of fractional integration for the fractional-order Legendre wavelets.

For example, for $k = 2, M = 3$, and $\nu = \alpha = 1$ the operational matrix of fractional integration at $t = 0.1$ can be expressed as

$$P^1(0.1,1) = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 & 0 \\ -0.0866 & 0.05 & 0 & 0 & 0 & 0 \\ 0.0373 & -0.0645 & 0.0333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & -0.2598 & 0.05 & 0 \\ 0 & 0 & 0 & 0.9317 & -0.193649 & 0.0333 \end{bmatrix}.$$

5. Problem statement

In this paper, we focus on the following problems:

5.1. **Problem a.** Caputo fractional differential equation with initial conditions as

$$\begin{cases} D^{\nu_1}y(t) = F(t, y(t), D^{\nu_2}y(t)), & 0 \le t < 1,\ m-1 < \nu_1 \le m, \\ & 0 \le \nu_2 < \nu_1, \\ y^{(i)}(0) = y_{i0}, \quad i = 0, 1, \dots, m-1. \end{cases} \tag{5.1}$$

5.2. **Problem b.** Caputo fractional differential equation with boundary conditions as

$$\begin{cases} D^{\nu_1}y(t) = F(t, y(t), D^{\nu_2}y(t), D^{\nu_3}y(t)), & 0 \le t < 1,\ 1 < \nu_2 < \nu_1 \le 2, \\ & 0 \le \nu_3 \le 1, \\ y(0) = y_0, \quad y(1) = y_1. \end{cases} \tag{5.2}$$

5.3. **Problem c.** System of Caputo fractional differential equations with initial conditions as

$$\begin{cases} D^{\nu_i}y_i(t) = F(t, y_1(t), y_2(t), \dots, y_n(t)), & 0 \le t < 1,\ 0 < \nu_i \le 1, \\ & i = 1, 2, \dots, n, \\ y_i(0) = y_{i0}, \quad i = 1, 2, \dots, n, \end{cases} \tag{5.3}$$

where F is a continuous linear or nonlinear function.

6. Numerical method

In this section, we use the fractional-order Legendre wavelets for solving problem (a) given in Eq. (5.1), problem (b) given in Eq. (5.2) and problem (c) given in Eq. (5.3).

6.1. **Problem a.** For problem (a), we expand $D^{\nu_1}y(t)$ by the fractional-order Legendre wavelets as

$$D^{\nu_1}y(t) \simeq C^T\Psi^{\alpha}(t). \tag{6.1}$$

By using Eqs. (4.8), (5.1) and (6.1) we have

$$y(t) \simeq C^T P^{\alpha}(t,\nu_1)\Psi^{\alpha}(t) + \sum_{i=0}^{m-1} y_{i0}\frac{t^i}{i!}. \tag{6.2}$$

From Eqs. (4.8), (6.2), we get

$$D^{\nu_2}y(t) \simeq C^T P^{\alpha}(t,\nu_1-\nu_2)\Psi^{\alpha}(t) + D^{\nu_2}\Big(\sum_{i=0}^{m-1} y_{i0}\frac{t^i}{i!}\Big). \tag{6.3}$$

Substituting Eqs. (6.1) - (6.3) in Eq. (5.1), we get

$$\begin{aligned} C^T\Psi^{\alpha}(t) &= F\Big(t, C^T P^{\alpha}(t,\nu_1)\Psi^{\alpha}(t) + \sum_{i=0}^{m-1} y_{i0}\frac{t^i}{i!}, C^T P^{\alpha}(t,\nu_1-\nu_2)\Psi^{\alpha}(t) \\ &+ D^{\nu_2}\big(\sum_{i=0}^{m-1} y_{i0}\frac{t^i}{i!}\big)\Big). \end{aligned} \tag{6.4}$$

Next, we collocate Eq. (6.4) at the $\hat{m}$ zeros of shifted Legendre polynomials $P_{\hat{m}}(t)$. These equations give $\hat{m}$ nonlinear algebraic equations, which can be solved for the unknown vector C using Newton's iterative method.

6.2. **Problem b.** For problem b, we expand $D^{\nu_1}y(t)$ by the fractional-order Legendre wavelets as

$$D^{\nu_1}y(t) \simeq C^T\Psi^{\alpha}(t). \tag{6.5}$$

By using Eqs. (4.8), (5.2) and (6.5) we have

$$y(t) \simeq C^T P^{\alpha}(t,\nu_1)\Psi^{\alpha}(t) + y_0 + y_0' t. \tag{6.6}$$

From Eqs. (4.8), (6.6), we get

$$D^{\nu_2}y(t) \simeq C^T P^{\alpha}(t,\nu_1-\nu_2)\Psi^{\alpha}(t). \tag{6.7}$$

$$D^{\nu_3}y(t) \simeq C^T P^{\alpha}(t,\nu_1-\nu_3)\Psi^{\alpha}(t) + D^{\nu_3}(y_0 + y_0' t). \tag{6.8}$$

Substituting Eqs. (6.5) - (6.8) in Eq. (5.2), we get

$$\begin{aligned} C^T\Psi^{\alpha}(t) &= F\big(t, C^T P^{\alpha}(t,\nu_1)\Psi^{\alpha}(t) + y_0 + y_0' t, C^T P^{\alpha}(t,\nu_1-\nu_2)\Psi^{\alpha}(t), \\ &\quad C^T P^{\alpha}(t,\nu_1-\nu_3)\Psi^{\alpha}(t) + D^{\nu_3}(y_0 + y_0' t)\big). \end{aligned} \tag{6.9}$$

Also, by using Eqs. (5.2) and (6.6), we have

$$y(1) = C^T P^{\alpha}(1,\nu_1)\Psi^{\alpha}(1) + y_0 + y_0' = y_1. \tag{6.10}$$

Next, we collocate Eq. (6.9) at the $\hat{m}$ zeros of shifted Legendre polynomials $P_{\hat{m}}(t)$. The resulting equations together with Eq. (6.10) give $\hat{m}+1$ algebraic equations, which can be solved for the unknown vector C and y'_0.

6.3. **Problem c.** For problem (c), we expand $D^{\nu_i}y_i(t), i = 1, 2, ..., n$, by the fractional-order Legendre wavelets as

$$D^{\nu_i}y_i(t) \simeq C_i^T \Psi^{\alpha}(t), \qquad i = 1, 2, \dots, n. \tag{6.11}$$

From Eqs. (4.8), (5.3) and (6.11), we obtain

$$y_i(t) \simeq C_i^T P^{\alpha}(t, \nu_i)\Psi^{\alpha}(t) + y_{i0}, \qquad i = 1, 2, \dots, n. \tag{6.12}$$

Substituting Eqs. (6.11) and (6.12) in Eq. (5.3), we obtain a system of algebraic equations with $n\hat{m}$ unknowns. Next, we collocate this system at the $n\hat{m}$ zeros of shifted Legendre polynomials $P_{n\hat{m}}(t)$. This system give $n\hat{m}$ nonlinear algebraic equations, which can be solved for the unknown vectors $C_i, i = 1, 2, \dots, n$ using Newton's iterative method.

7. Convergence of the fractional-order Legendre wavelet bases

Now, we derive the convergence of approximation of a function with respect the fractional-order Legendre bases. In fact, we will show that $f_{\hat{m}}$ converges to f as k or m approaches ∞. Also, we obtain an upper bound for its error by the following theorem.

Theorem 7.1. *Suppose $D^{k\alpha}f \in C[0,1]$ for $k = 0, 1, \dots, M-1$ and $Y_m^{\alpha} = span\{Fl_0^{\alpha}, Fl_1^{\alpha}, \dots Fl_{M-1}^{\alpha}\}$. If $f_m = \sum_{i=0}^{M-1} a_i Fl_i^{\alpha} = A^T FL^{\alpha}$ is the best approximation of f out of Y_m^{α} on the interval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$, then*

$$\|f - f_{\hat{m}}\|_2 \le \frac{sup_{t\in[0,1]}|D^{M\alpha}f(0^+)|}{\Gamma(M\alpha+1)}\sqrt{\frac{1}{(2M+1)\alpha}}. \tag{7.1}$$

Proof. Considering the Generalized Taylors formula, we let

$$y_1(t) = \textstyle\sum_{i=0}^{M-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha}f(0^+),$$

then, we have

$$|f(t) - y_1(t)| \le \tfrac{t^{M\alpha}}{\Gamma(M\alpha+1)} sup_{t\in I_{k,n}}|D^{M\alpha}f(0^+)|,$$

where $I_{k,n} = [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$.
Because the interval $[0, 1]$ is divided into 2^{k-1} subintervals $I_{k,n} = [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$ that the function f is approximated on them by using fractional-order Legendre functions as a function of the $(M-1)\alpha$ order at most with the least-square property, therefore

would be as

$$\begin{aligned}\|f-f_{\hat{m}}\|_2^2 &= \int_0^1 w(t)[f(t)-C^T\Psi^\alpha(t)]^2dt\\ &= \sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} w_n(t)(f(t)-C^T\Psi^\alpha(t))^2dt\\ &= \sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} w_n(t)(f(t)-A^TFL^\alpha(t))^2dt, \qquad (7.2)\end{aligned}$$

where A^TFL^α is any function of order $(M-1)\alpha$ that interpolates f on $I_{k,n}$ with the following error bound for interpolating

$$|f(t)-A^TFL^\alpha(t)| \le \tfrac{t^{M\alpha}}{\Gamma(M\alpha+1)} sup_{t\in I_{k,n}}|D^{M\alpha}f(0^+)|.$$

Therefore, using the above relations, we obtain

$$\begin{aligned}\|f-f_{\hat{m}}\|_2^2 &= \sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} w_n(t)\Big(f(t)-A^TFL^\alpha(t)\Big)^2dt\\ &\le \sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} w_n(t)\Big(\frac{t^{M\alpha}}{\Gamma(M\alpha+1)}sup_{t\in I_{k,n}}|D^{M\alpha}f(0^+)|\Big)^2dt\\ &= \int_0^1 w(t)\Big(\frac{t^{M\alpha}}{\Gamma(M\alpha+1)}sup_{t\in I_{k,n}}|D^{M\alpha}f(0^+)|\Big)^2dt\\ &= \frac{1}{\Gamma(M\alpha+1)^2(2M+1)\alpha}\Big(sup_{t\in[0,1]}|D^{M\alpha}f(0^+)|\Big)^2. \qquad (7.3)\end{aligned}$$

□

By taking the square roots of both sides, we obtain the upper bound of the error. Using the fractional-order Legendre wavelet basis we will have two degrees of freedom which increase the accuracy of the method. One of these parameters is the dilation argument k and the other is M corresponding to the number of elements of the basis in every subinterval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$.
When M is fixed and k approaches ∞, we have

$$|I_{k,n}| = \left|\tfrac{n}{2^{k-1}} - \tfrac{n-1}{2^{k-1}}\right| \to 0, \qquad \int_{I_{k,n}}[f(t)-f_m(t)]^2dt \to 0.$$

By using Eq. (7.2), we have

$$\lim_{k\to\infty}\|f-f_{\hat{m}}\|_2 = 0.$$

When k is fixed and M approaches ∞, by (7.1) we get

$$\lim_{M\to\infty}\|f-f_{\hat{m}}\|_2 = 0.$$

This completes the convergence of approximation by the fractional-order Legendre wavelets bases.

8. Illustrative examples

In this section, seven examples are given to demonstrate the applicability and accuracy of our method. Examples 1-3, are initial value problems, Examples 4-5, are boundary value problems, and Examples 6 and 7 are systems of fractional differential equations.

Example 1. Consider the following fractional differential equation with initial condition [22]

$$\begin{cases} D^{0.5}y(t) + y(t) = \sqrt{t} + \frac{\sqrt{\pi}}{2}, & 0 \le t < 1, \\ y(0) = 0. \end{cases} \tag{8.1}$$

The exact solution of this problem is

$$y(t) = \sqrt{t}.$$

Here we solve this problem by using the fractional-order Legendre wavelet approach. Let

$$D^{0.5}y(t) \simeq C^T \Psi^{\alpha}(t). \tag{8.2}$$

Then using Eqs. (4.8), (8.1) and (8.2) we have

$$y(t) \simeq C^T P^{\alpha}(t, 0.5)\Psi^{\alpha}(t). \tag{8.3}$$

Substituting Eqs. (8.2) and (8.3) into Eq. (8.1), we have

$$C^T \Psi^{\alpha}(t) + C^T P^{\alpha}(t, 0.5)\Psi^{\alpha}(t) = \sqrt{t} + \frac{\sqrt{\pi}}{2}. \tag{8.4}$$

Now, we collocate Eq. (8.4) at the zeros of shifted Legendre polynomials, which can be solved for the unknown vector C, using Newton's iterative method. It is well known that the initial guesses for Newton's iterative method are very important. For this problem, by using $y(0)$, and Eq. (8.3), we choose the initial guesses such that $C^T P^{\alpha}(0, 0.5)\Psi^{\alpha}(0) = 0$.

In Tables 1 and 2 we compare the L_∞ and L_2 errors of the present method for $k = 2, M = 2, \alpha = 1$ with the B-spline functions method of [22]. Also, Table 3 shows the absolute errors between the exact and approximate solutions for various values of α with $k = 1, M = 3$. The comparisons show that the best value of α for this problem is $\alpha = \frac{1}{3}$.

Table 1. Comparison of L_∞ errors for $y(t)$ for Example 1.

t	Ref. [22]			*Our method*
	$\hat{m} = 17$	$\hat{m} = 65$	$\hat{m} = 257$	$\hat{m} = 4 (k = 2, M = 2)$
0	7.8×10^{-3}	7.7×10^{-4}	7.8×10^{-5}	1.11×10^{-16}

Example 2. Consider the fractional nonlinear differential equation with initial conditions [12]

$$\begin{cases} D^{\nu}y(t) - y^2(t) = 1, & p - 1 < \nu \le p, \ \ 0 \le t < 1, \\ y^{(i)}(0) = 0, & i = 0, 1, \dots, p - 1. \end{cases} \tag{8.5}$$

TABLE 2. Comparison of L_2 errors for $y(t)$ for Example 1.

t	Ref. [22]			*Our method*
	$\hat{m} = 17$	$\hat{m} = 65$	$\hat{m} = 257$	$\hat{m} = 4(k = 2, M = 2)$
0	3.9×10^{-3}	4.0×10^{-4}	4.2×10^{-5}	9.54×10^{-17}

TABLE 3. Comparison of absolute error of $y(t)$ with $k = 1, M = 3$ for different values of α for Example 1.

t	$\alpha = \frac{1}{5}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = 1$	$\alpha = 2$
0	0	0	0	0	0
0.1	5.55×10^{-17}	0	0	5.55×10^{-17}	2.66×10^{-15}
0.2	1.11×10^{-16}	0	0	5.55×10^{-17}	3.61×10^{-15}
0.3	1.11×10^{-16}	0	1.11×10^{-16}	0	4.22×10^{-15}
0.4	1.11×10^{-16}	0	2.22×10^{-16}	1.11×10^{-16}	4.66×10^{-15}
0.5	1.11×10^{-16}	0	1.11×10^{-16}	1.11×10^{-16}	4.77×10^{-15}
0.6	1.11×10^{-16}	0	1.11×10^{-16}	1.11×10^{-16}	4.77×10^{-15}
0.7	2.22×10^{-16}	0	0	0	4.88×10^{-15}
0.8	1.11×10^{-16}	0	1.11×10^{-16}	1.11×10^{-16}	5.11×10^{-15}
0.9	1.11×10^{-16}	0	1.11×10^{-16}	1.11×10^{-16}	5.33×10^{-15}

The exact solution of the initial value problem (8.5) for $\nu = 1$, is

$$y(t) = tan(t).$$

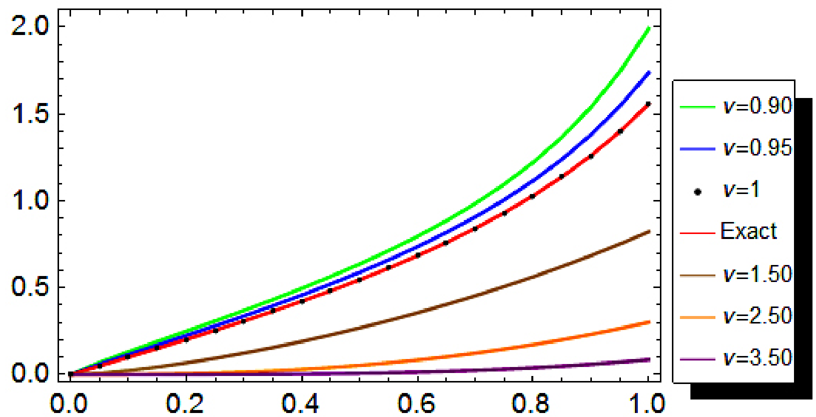
Similar to the method of Example 1, we approximate the solutions to the problem in Eq. (8.5).

Table 4 shows the absolute errors between the exact and approximate solutions for various values of α with $k = 1, M = 5, \nu = 1$. Also, the approximate solutions by the present method at various values of ν with $k = 1, M = 5$ and $\alpha = 2$ are plotted in Figure 1. Again we see that as ν approaches 1, the solution of the fractional differential equation approaches to the integer-order differential equations.

TABLE 4. Comparison of absolute error of $y(t)$ with $k = 1, M = 5, \nu = 1$ for different values of α for Example 2.

t	$\alpha = \frac{1}{2}$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
0.1	1.73×10^{-2}	9.78×10^{-5}	1.94×10^{-7}	4.64×10^{-5}
0.3	1.08×10^{-2}	3.43×10^{-4}	4.71×10^{-7}	1.92×10^{-4}
0.5	1.95×10^{-2}	7.37×10^{-4}	2.50×10^{-5}	4.96×10^{-3}
0.7	2.10×10^{-2}	4.78×10^{-4}	7.64×10^{-5}	1.44×10^{-2}
0.9	3.37×10^{-2}	1.76×10^{-4}	8.07×10^{-5}	1.40×10^{-2}

FIGURE 1. Comparison of $y(t)$ for $k = 1, M = 5, \alpha = 2$ with $\nu = 0.90, 0.95, 1, 1.5, 2.5, 3.5$ and the exact solution at $\nu = 1$, for Example 2.



Example 3. Consider the fractional differential equation with initial conditions [12]

$$\begin{cases} D^2y(t) + D^{\frac{3}{4}}y(t) + y(t) = t^3 + 6t + \frac{8.533333333}{\Gamma(0.25)}t^{2.25}, & 0 \leq t < 1, \\ y(0) = 0, \qquad y'(0) = 0. \end{cases} \tag{8.6}$$

The exact solution of this problem is

$$y(t) = t^3.$$

We solve this problem by using the proposed method in section 6. In Table 5, we compare the maximum absolute error of the present method for $k = 2, M = 2, \alpha = 1$ with the Chebyshev spectral method [12]. Also, Table 6 shows the absolute errors between the exact and approximate solutions for $k = 1, M = 6$ and different values of α.

TABLE 5. Comparison of the maximum absolute error for Example 3.

t	Ref. [12]			*Our method*
	$\hat{m} = 5$	$\hat{m} = 9$	$\hat{m} = 17$	$\hat{m} = 4(k = 2, M = 2)$
0	8.82×10^{-6}	1.91×10^{-7}	2.52×10^{-9}	2.11×10^{-12}

Example 4. Consider the fractional differential equation with boundary conditions [50]

$$\begin{cases} MD^2y(t) + 2S\sqrt{\mu\rho}D^{\frac{3}{2}}y(t) + Ky(t) = f(t), & 0 \leq t < 1, \\ y(0) = 1, \qquad y(1) = 2. \end{cases} \tag{8.7}$$

Table 6. Comparison of absolute error of $y(t)$ with $k = 1, M = 6$ for different values of α for Example 3.

t	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = 1$	$\alpha = 2$
0	0	0	0	0	0
0.1	4.23×10^{-14}	1.50×10^{-14}	7.00×10^{-17}	1.54×10^{-9}	1.88×10^{-4}
0.2	8.62×10^{-14}	3.39×10^{-14}	4.83×10^{-15}	5.58×10^{-9}	7.24×10^{-5}
0.3	1.52×10^{-13}	7.70×10^{-14}	3.02×10^{-14}	1.13×10^{-8}	4.98×10^{-4}
0.4	2.74×10^{-13}	1.78×10^{-13}	1.15×10^{-13}	1.78×10^{-8}	2.63×10^{-3}
0.5	5.00×10^{-13}	3.87×10^{-13}	3.19×10^{-13}	2.42×10^{-8}	4.10×10^{-3}
0.6	8.97×10^{-13}	7.70×10^{-13}	7.11×10^{-13}	2.99×10^{-8}	2.51×10^{-3}
0.7	1.54×10^{-12}	1.40×10^{-12}	1.37×10^{-12}	3.41×10^{-8}	4.95×10^{-4}
0.8	2.53×10^{-12}	2.39×10^{-12}	2.35×10^{-12}	3.62×10^{-8}	4.45×10^{-3}
0.9	3.97×10^{-12}	3.82×10^{-12}	3.74×10^{-12}	3.56×10^{-8}	1.15×10^{-2}

This problem describes the motion of a large plate of the surface S and mass M in a Newtonian fluid with viscosity μ and density ρ. The plate is hanging on a massless spring of stiffness K. The function f represents the loading force.
In this problem, we choose $M = 2S\sqrt{\mu\rho} = K = 1$ and $f(t) = t + 1$ in Eq. (8.7) to obtain

$$\begin{cases} D^2y(t) + D^{\frac{3}{2}}y(t) + y(t) = f(t), & 0 \leq t < 1, \\ y(0) = 1, \quad y(1) = 2. \end{cases} \tag{8.8}$$

The exact solution of this problem is

$$y(t) = t + 1.$$

For solving this problem, we expand $D^2y(t)$ as

$$D^2y(t) \simeq c_{10}\psi_{10}^{\alpha}(t) + c_{20}\psi_{20}^{\alpha}(t) = C^T\Psi^{\alpha}(t). \tag{8.9}$$

Using Eqs. (4.8) and (8.9) we have

$$y(t) \simeq C^T P^{\alpha}(t,2)\Psi^{\alpha}(t) + 1 + y_0' t, \qquad D^{\frac{3}{2}}y(t) \simeq C^T P^{\alpha}(t, \frac{1}{2})\Psi^{\alpha}(t). \tag{8.10}$$

Substituting Eqs. (8.9) and (8.10) into Eq. (8.8), we have

$$C^T\Psi^{\alpha}(t) + C^T P^{\alpha}(t, \frac{1}{2})\Psi^{\alpha}(t) + C^T P^{\alpha}(t,2)\Psi^{\alpha}(t) + 1 + y_0' t = f(t). \tag{8.11}$$

Also by using the boundary condition $y(1) = 2$, we have

$$C^T P^{\alpha}(1,2)\Psi^{\alpha}(1) + 1 + y_0' = 2. \tag{8.12}$$

Now, we collocate Eq. (8.11) at $t_0 = 0.211325$ and $t_1 = 0.788675$. Then, by solving these equations we find

$$c_{10} = c_{20} = 0, \qquad y_0' = 1.$$

By using Eq. (8.10), we find the exact solution.
Example 5. Consider the fractional differential equation with boundary conditions

[19]

$$\begin{cases} D^{\nu}y(t) - y^2(t) = 2\pi^2 cos(2\pi t) - sin^4(t), & 1 < \nu \leq 2, \ \ 0 \leq t < 1, \\ y(0) = 0, \quad y(1) = 0. \end{cases} \tag{8.13}$$

The exact solution, when $\nu = 2$ is

$$y(t) = sin^2(\pi t).$$

Similar to the method of Example 4, we approximate the solutions to the problem in Eq. (8.13). The computational results for $k = 1, M = 6, \alpha = 1$ and different values of ν are given in Figure 2. My result is in good agreement with the numerical results obtained by [19]. Also, Table 7 shows the absolute errors between the exact and approximate solutions for various values of α with $k = 1, M = 11$.

FIGURE 2. Comparison of $y(t)$ for $k = 1, M = 6, \alpha = 1$ with $\nu =$ 1.70, 1.80, 1.90, 1.95, 2 and the exact solution, for Example 5.

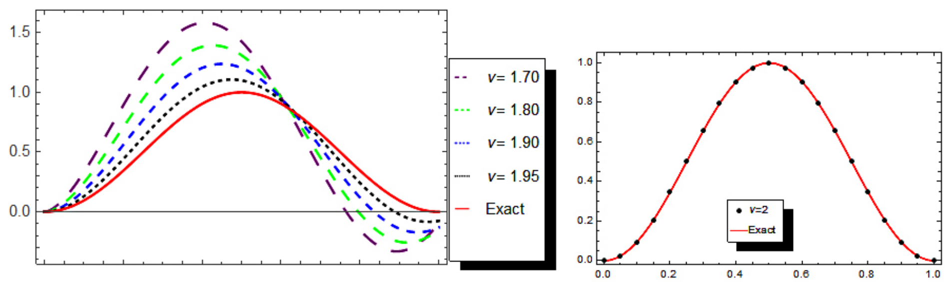


TABLE 7. Comparison of absolute error of $y(t)$ with $k = 1, M = 11$ for different values of α for Example 5.

t	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = \frac{2}{3}$	$\alpha = \frac{4}{3}$	$\alpha = \frac{3}{2}$
0.1	8.11×10^{-4}	1.81×10^{-4}	1.43×10^{-6}	1.05×10^{-5}	1.56×10^{-5}
0.2	3.20×10^{-4}	2.43×10^{-4}	3.20×10^{-6}	2.00×10^{-5}	4.11×10^{-5}
0.3	3.81×10^{-4}	1.71×10^{-4}	2.21×10^{-6}	4.18×10^{-5}	1.34×10^{-4}
0.4	2.87×10^{-4}	1.53×10^{-4}	1.66×10^{-6}	4.98×10^{-5}	2.33×10^{-4}
0.5	2.30×10^{-4}	1.23×10^{-4}	1.52×10^{-6}	1.13×10^{-5}	1.16×10^{-4}
0.6	1.88×10^{-4}	9.34×10^{-5}	1.33×10^{-6}	5.51×10^{-5}	2.83×10^{-4}
0.7	1.34×10^{-4}	7.16×10^{-5}	6.66×10^{-7}	8.84×10^{-5}	6.28×10^{-4}
0.8	9.11×10^{-5}	4.64×10^{-5}	6.93×10^{-7}	7.46×10^{-5}	6.48×10^{-4}
0.9	4.47×10^{-5}	2.35×10^{-5}	2.10×10^{-7}	5.59×10^{-5}	5.46×10^{-4}

Example 6. Consider the fractional order system of nonlinear differential equations

[52]

$$\begin{cases} D^{\nu_1}y_1(t) = \frac{1}{2}y_1(t), & 0 < \nu_1, \nu_2 \leq 1, \quad 0 \leq t < 1, \\ D^{\nu_2}y_2(t) = y_2(t) + y_1^2(t), & \\ y_1(0) = 1, \quad y_2(0) = 0. & \end{cases} \tag{8.14}$$

The exact solution of this system, when $\nu_1 = \nu_2 = 1$ is

$$y_1(t) = e^{\frac{t}{2}}, \qquad y_2(t) = te^t.$$

For solving this problem, we approximate $D^{\nu_1}y_1(t)$ and $D^{\nu_2}y_2(t)$ as

$$D^{\nu_1}y_1(t) \simeq C_1^T\Psi^\alpha(t), \qquad D^{\nu_2}y_2(t) \simeq C_2^T\Psi^\alpha(t). \tag{8.15}$$

From Eqs. (4.8) and (8.15) we get

$$y_1(t) \simeq C_1^T P^\alpha(t,\nu_1)\Psi^\alpha(t) + 1, \qquad y_2(t) \simeq C_2^T P^\alpha(t,\nu_2)\Psi^\alpha(t). \tag{8.16}$$

Substituting Eqs. (8.15) and (8.16) into Eq. (8.14), we obtain

$$\begin{cases} C_1^T\Psi^\alpha(t) = \frac{1}{2}(C_1^T P^\alpha(t,\nu_1)\Psi^\alpha(t) + 1), \\ \\ C_2^T\Psi^\alpha(t) = C_2^T P^\alpha(t,\nu_2)\Psi^\alpha(t) + (C_1^T P^\alpha(t,\nu_1)\Psi^\alpha(t) + 1)^2. \end{cases} \tag{8.17}$$

Then, by collocation system (8.17) at the zeros of shifted Legendre polynomials and using Newton's iterative method, we can obtain the unknown vectors C_1 and C_2. Table 8 shows the absolute errors between the exact and approximate solutions obtained for $y_1(t)$ and $y_2(t)$ by using the present method for $k = 2, M = 10, \nu_1 = \nu_2 = \alpha = 1$. Also, Table 9 displays the absolute errors between the exact and approximate solutions at $k = 1, M = 14$ with various values of α for $y_1(t)$ and $y_2(t)$. The numerical results for $y_1(t)$ and $y_2(t)$ with $k = 2, M = 10, \alpha = 1$ and $\nu = 0.6, 0.7, 0.8, 0.9$ and the exact solutions are plotted in Figures 3(a) and 3(b) respectively.

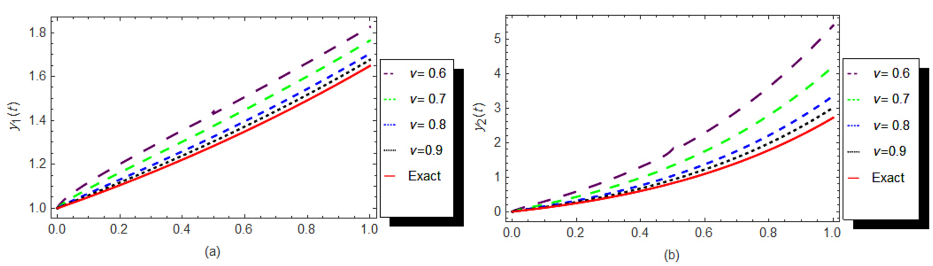
TABLE 8. Absolute errors of $y_1(t)$ and $y_2(t)$ at $k = 2, M = 10$ and $\nu_1 = \nu_2 = \alpha = 1$ for Example 6.

t	$y_1(t)$	$y_2(t)$
0	0	0
0.1	0	2.78×10^{-17}
0.2	2.22×10^{-16}	0
0.3	0	1.67×10^{-16}
0.4	0	8.88×10^{-16}
0.5	5.91×10^{-10}	8.41×10^{-8}
0.6	6.21×10^{-10}	9.31×10^{-8}
0.7	6.53×10^{-10}	1.03×10^{-7}
0.8	6.87×10^{-10}	1.14×10^{-7}
0.9	7.22×10^{-10}	1.26×10^{-7}

TABLE 9. Comparison of absolute error of $y_1(t)$ and $y_2(t)$ with $k = 1, M = 14$ for different values of α for Example 6.

t	$y_1(t)$			
	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.1	3.34×10^{-9}	1.39×10^{-7}	2.33×10^{-7}	0
0.2	1.74×10^{-9}	1.58×10^{-7}	2.90×10^{-7}	2.22×10^{-16}
0.3	4.55×10^{-9}	1.44×10^{-7}	2.45×10^{-7}	0
0.4	2.99×10^{-9}	1.68×10^{-7}	4.38×10^{-7}	2.22×10^{-16}
0.5	6.60×10^{-10}	1.95×10^{-7}	7.82×10^{-8}	0
0.6	5.32×10^{-9}	1.49×10^{-7}	6.08×10^{-7}	2.22×10^{-16}
0.7	6.61×10^{-9}	2.01×10^{-7}	1.38×10^{-7}	0
0.8	1.06×10^{-9}	2.35×10^{-7}	5.30×10^{-7}	2.22×10^{-16}
0.9	6.54×10^{-9}	1.49×10^{-7}	2.70×10^{-8}	2.22×10^{-16}
t	$y_2(t)$			
	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.1	6.98×10^{-8}	1.23×10^{-6}	2.32×10^{-6}	0
0.2	8.04×10^{-8}	1.33×10^{-6}	2.64×10^{-6}	1.67×10^{-16}
0.3	8.14×10^{-8}	1.59×10^{-6}	2.81×10^{-6}	3.89×10^{-16}
0.4	9.93×10^{-8}	1.71×10^{-6}	4.19×10^{-6}	2.22×10^{-16}
0.5	1.05×10^{-7}	1.93×10^{-6}	2.27×10^{-6}	2.22×10^{-16}
0.6	1.06×10^{-7}	2.33×10^{-6}	6.12×10^{-6}	2.22×10^{-16}
0.7	1.27×10^{-7}	2.46×10^{-6}	3.38×10^{-6}	4.44×10^{-16}
0.8	1.41×10^{-7}	2.78×10^{-6}	6.62×10^{-6}	2.22×10^{-16}
0.9	1.42×10^{-7}	1.35×10^{-6}	3.71×10^{-6}	8.88×10^{-16}

FIGURE 3. Comparison of $y_1(t)$ and $y_2(t)$ for $k = 2, M = 10, \alpha = 1$ with $\nu = \nu_1 = \nu_2 = 0.6, 0.7, 0.8, 0.9$ and the exact solution, for Example 6.



Example 7. Consider the fractional order system of nonlinear differential equations [52]

$$\begin{cases} D^{\nu_1} y_1(t) = y_1(t), & 0 < \nu_1, \nu_2, \nu_3 \le 1, \quad 0 \le t < 1, \\ D^{\nu_2} y_2(t) = 2y_1^2(t), \\ D^{\nu_3} y_3(t) = 3y_1(t)y_2(t), \\ y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 0. \end{cases} \tag{8.18}$$

The exact solution of this system, when $\nu_1 = \nu_2 = \nu_3 = 1$ is

$$y_1(t) = e^t, \qquad y_2(t) = e^{2t}, \qquad y_3(t) = e^{3t} - 1.$$

Similar to the method of Example 6, we approximate the solutions to the problem in Eq. (8.18). Table 10, shows the absolute errors between the exact and approximate solutions obtained for $y_1(t), y_2(t)$ and $y_3(t)$ by using the present method for $k = 2, M = 10, \nu_1 = \nu_2 = \nu_3 = \alpha = 1$. Also, Table 11 displays the absolute errors between the exact and approximate solutions at $k = 1, M = 14$ with various values of α for $y_1(t), y_2(t)$ and $y_3(t)$. The numerical results for $y_1(t), y_2(t)$ and $y_3(t)$ with $k = 2, M = 10, \alpha = 1$ and $\nu_1 = \nu_2 = \nu_3 = 0.6, 0.7, 0.8, 0.9$ and the exact solutions are plotted in Figures 4(a), 4(b) and 4(c) respectively. From these figures, it is seen that the approximate solutions converge to the exact solutions. My results are in good agreement with the numerical results obtained by [52]. This demonstrates the importance of my numerical scheme in solving system of nonlinear fractional differential equations.

Table 10. Absolute errors of $y_1(t), y_2(t)$ and $y_3(t)$ at $k = 2, M = 10$ and $\nu_1 = \nu_2 = \nu_3 = \alpha = 1$ for Example 7.

t	$y_1(t)$	$y_2(t)$	$y_3(t)$
0	0	0	0
0.1	0	0	5.44×10^{-15}
0.2	0	1.11×10^{-15}	1.20×10^{-13}
0.3	2.22×10^{-16}	2.91×10^{-14}	3.05×10^{-12}
0.4	0	1.39×10^{-13}	1.47×10^{-11}
0.8	7.62×10^{-9}	1.90×10^{-5}	3.57×10^{-3}

9. Conclusion

The aim of this work is to develop an efficient and accurate method for solving fractional-order differential equations with initial and boundary conditions. We derive a general formulation for the fractional-order Legendre wavelets operational matrix of fractional order integration. This operator and collocation method are used to reduce the problem to the solution of a system of algebraic equations. Illustrative examples are given to demonstrate the applicability and accuracy of the proposed method. Some of the advantages of the present approach are summarized as

- Fractional-order functions can well reflect the properties of fractional-order differential equations.
- We can increase the accuracy of numerical solutions without increase $\hat{m}$ and CPU time.
- Original Legendre wavelets are special case of fractional-order Legendre wavelets.
- It is shown that only a small value of FLWs is needed to achieve high accuracy and satisfactory results.

TABLE 11. Comparison of absolute error of $y_1(t), y_2(t)$ and $y_3(t)$ with $k = 1, M = 14$ for different values of α for Example 7.

t	$y_1(t)$			
	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.1	3.89×10^{-8}	5.94×10^{-7}	1.07×10^{-6}	0
0.2	4.68×10^{-8}	6.65×10^{-7}	1.27×10^{-6}	2.22×10^{-16}
0.3	4.57×10^{-8}	7.12×10^{-7}	1.24×10^{-6}	2.22×10^{-16}
0.4	5.52×10^{-8}	7.93×10^{-7}	1.98×10^{-6}	2.22×10^{-16}
0.5	6.53×10^{-8}	9.20×10^{-7}	8.16×10^{-7}	2.22×10^{-16}
0.6	6.21×10^{-8}	9.34×10^{-7}	2.84×10^{-6}	2.22×10^{-16}
0.7	6.80×10^{-8}	1.08×10^{-6}	1.25×10^{-6}	0
0.8	9.10×10^{-8}	1.25×10^{-6}	2.86×10^{-6}	0
0.9	8.41×10^{-8}	1.22×10^{-6}	1.16×10^{-6}	0
t	$y_2(t)$			
	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.1	7.13×10^{-5}	1.53×10^{-6}	5.43×10^{-6}	0
0.2	7.36×10^{-5}	8.93×10^{-6}	4.94×10^{-6}	0
0.3	7.09×10^{-5}	2.14×10^{-6}	6.22×10^{-6}	0
0.4	6.35×10^{-5}	6.96×10^{-6}	7.57×10^{-6}	4.44×10^{-16}
0.5	8.31×10^{-5}	1.69×10^{-5}	5.96×10^{-6}	0
0.6	7.38×10^{-5}	1.17×10^{-5}	1.17×10^{-5}	4.44×10^{-16}
0.7	4.87×10^{-5}	1.21×10^{-5}	8.28×10^{-6}	0
0.8	9.00×10^{-5}	2.58×10^{-5}	1.43×10^{-5}	8.88×10^{-16}
0.9	6.97×10^{-5}	2.46×10^{-5}	1.10×10^{-5}	0
t	$y_3(t)$			
	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.1	6.33×10^{-2}	2.57×10^{-3}	1.56×10^{-5}	7.26×10^{-14}
0.2	6.53×10^{-2}	1.87×10^{-3}	1.24×10^{-5}	1.59×10^{-14}
0.3	6.30×10^{-2}	1.53×10^{-3}	1.95×10^{-5}	4.06×10^{-14}
0.4	5.68×10^{-2}	1.07×10^{-3}	2.03×10^{-5}	2.49×10^{-14}
0.5	7.34×10^{-2}	3.83×10^{-3}	2.46×10^{-5}	8.44×10^{-15}
0.6	6.55×10^{-2}	4.83×10^{-3}	3.47×10^{-5}	3.02×10^{-14}
0.7	4.42×10^{-2}	2.01×10^{-3}	3.61×10^{-5}	5.24×10^{-14}
0.8	7.92×10^{-2}	5.79×10^{-3}	5.29×10^{-5}	1.07×10^{-14}
0.9	6.21×10^{-2}	9.49×10^{-3}	5.83×10^{-5}	1.12×10^{-13}

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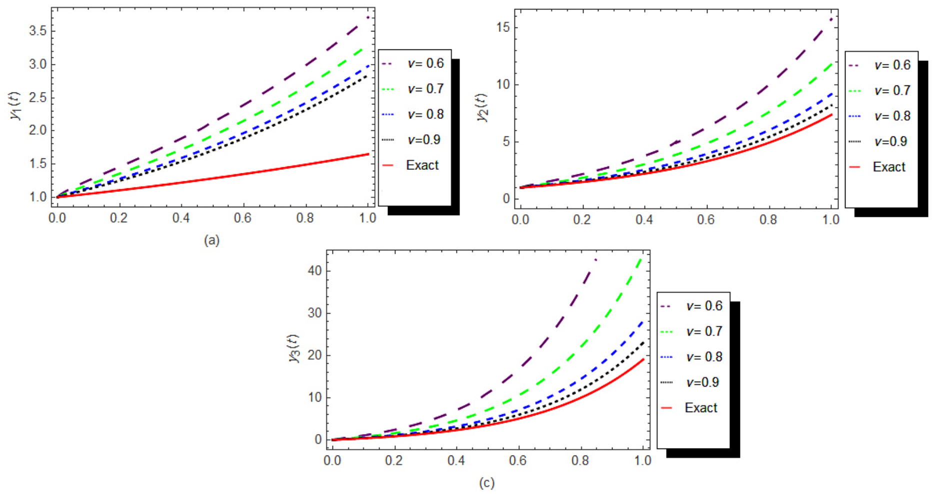
FIGURE 4. Comparison of $y_1(t)$, $y_2(t)$ and $y_3(t)$ for $k = 2, M = 10, \alpha = 1$ with $\nu = \nu_1 = \nu_2 = \nu_3 = 0.6,\ 0.7,\ 0.8,\ 0.9$ and the exact solution, for Example 7.



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