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A new approach on studying the stability of evolutionary game dynamics for financial systems

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Abstract Financial market modeling and prediction is a difficult problem and drastic changes of the price causes nonlinear dynamic that makes the price prediction one of the most challenging tasks for economists. Since markets always have been interesting for traders, many traders with various beliefs are highly active in a market. The competition among two agents of traders, namely trend followers and rational agents, to gain the highest profit in market is formulated as a dynamic evolutionary game, where, the evolutionary equilibrium is considered to be the solution to this game. The evolutionarily stablity of the equilibrium points is investigated inspite of the prediction error of the expectation.

Keywords. Heterogeneous Agent Model, Adaptive Belief System, Evolutionary Game Theory, Rational Agent, Evolutionary Stable Strategies.

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1. Introduction

The evolution of financial market is a complicated phenomenon which is one of the most difficult modeling and prediction issues. The main reason for this difficulty is the complex nonlinearity caused by on-normality in returns, particularly fat tails, heterogeneity of expectations among traders and complex evolving nature of the financial markets [13]. Although traditional finance is built on the rationality paradigm and perfect Efficient Market Hypothesis (EMH), laboratory experiments with

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human subjects have shown that individuals often do not behave fully rational but tend to use heuristics, in making economic decisions under uncertainty [25]. Rational economic models have two remarkable features. First, rational traders boost some objective function based on perceived constraints. Second, traders should have perfect knowledge about economic system such as precise equations of the environment and expectations of other traders, namely Rational Expectations (REs). However, REs are unlikely to be satisfied in most economic environments. In an efficient market where all traders are rational and it is common knowledge, there will be no trade since rational traders would buy (sell) an underpriced (overpriced) asset, thus driving its price back to the fundamental value; i.e., there would be no deviation from fundamental value in asset price [1]. On the other hand, considering a heterogeneous financial market consisting of rational traders, any rational agent has to know the beliefs of all other non-rational agents, which seems highly unrealistic. Besides, laboratory experiments have shown that individuals often do not behave rationally. Also, bounded rational agents try to maximize their outcome by altering their forecasts to eliminate systematic forecast errors.Furthermore, there is no need for complicated structure of the whole economy. Moreover, a rational agent model has difficulty in explaining fat tail and volatility facts. It is worth mentioning that, if agents were all behaved rational the market price would quickly converge to its fundamental value [24].

Nowadays, financial markets are viewed as evolutionary systems with boundedly rational interacting agents. Agents base their investment decisions on their forecast of market price and tend to choose strategies which were successful in the past and driven by evolutionary selection. However, the simple models that could be studied with mixture of analytical and computational tools are preferred. Heterogeneous Agent Models (HAMs) are class of models with interacting agents which adapt their heterogeneous beliefs in response to the arrival of new information, and therefore switch between different trading strategies [7,12]. HAMs are basically built on Evolutionary game theory (EGT). EGT was originally developed for biology [9,19] reveals strategic interactions with dynamic adjustment process of players that can switch between strategies. A typical evolutionary game has two main components which are payoff matrix and the dynamic rule of the agents. Note that, payoff matrix indicates the outcome of corresponded strategy. Over time, under dynamic rule of evolutionary strategies, lower associated payoffs will be replaced by strategies with higher payoffs, till the strategies converge towards evolutionarily stable strategy (ESS). ESS is from the set of available strategies that is robust to evolutionary pressures and uninvadable by any other strategy [23]. Simply put, in evolutionary games, players change their strategies slowly to achieve the solution eventually [18]. When the solution to an evolutionary game has more than one equilibriums, a refined solution is required which ensures that the equilibrium is stable [19]. In fact, in ESS no player can increase his payoff by choosing a different action, given other players actions [22]. In society the strategies which perform better than average are the dominant ones in the long run. These dominant strategies will become the set of rules that are adopted by the majority of the population [26]. Thereupon, considering the mentioned specific features of the evolutionary game theory, in the last few years it has been extensively used to model economic issues such as studying the dynamics of the labor market [2,3], studying the interaction between firms and workers [10], macroeconomic monetary policy [8] and neuro-economics [20].

In a HAM model for a typical financial market, heterogeneity easily generates large trading volume consistent with empirical observations which is an immediate advantage of a HAM compared to a representative rational agent model [12]. HAMs also can explain nonlinearity factors such as excess volatility, high trading volume, temporary bubbles, sudden crashes, clustered volatility and fat tails in the returns distribution [7, 16]. Other important stylized facts of financial markets that have motivated more work on HAMs are: (i) asset prices follow a near unit root process, (ii) asset returns are unpredictable with almost no autocorrelations, (iii) the returns distribution has fat tails, and (iv) financial returns exhibit long range volatility clustering, i.e. slow decay of autocorrelations of squared returns and absolute returns [24].

Despite HAM's theoretically appealing features, there are many different alternative model specifications available in the literature; each producing potentially different results. Adaptive Belief Systems (ABSs) of Brock and Hommes [5] are a class of HAMs which are nonlinear dynamic asset-pricing models with evolutionary strategy switching. Two important features of the ABS are that agents are boundedly rational and that they have heterogeneous expectations. An ABS is in fact a standard discounted value asset-pricing model derived from mean-variance maximization, extended to the case of heterogeneous beliefs. Strategy choice is thus based on evolutionary selection or reinforcement learning, with agents switching to more successful (i.e., profitable) rules. An ABS model is an evolutionary competition between trading strategies. Different groups of traders have different expectations about future prices and future dividends. Agents can either invest in a risk free asset or in a risky asset. A feature of an ABS is that the model can be formulated in terms of deviations from a benchmark fundamental [13].

A number of researches have investigated the stability of evolutionary dynamics and they emphasized that the large fraction of fundamentalists tends to stabilize price, whereas, a large fraction of chartists tends to destabilize price. Brock, et al. [6] investigated whether a fully rational agent can employ additional hedging instruments to stabilize markets. It turns out that the composition of the population on irrational traders and the information gathering costs for rationality may affect the answer. In this paper, the stability analyze of the dynamic is being reconsidered. We emphasize the role of heterogeneous beliefs in a market with two groups of traders having different expectations about future price. The traders of the first agent are rational that could predict the future price with neglectable error. The traders of the second agent are technical analysts who believe that asset prices could be predicted by technical trading rules, extrapolation of trends and other patterns observed in past prices. In this paper, an important question in heterogeneous agents modeling is investigated. The question is whether irrational traders can survive in the market or they would be driven out of the market by rational investors and lose their wealth. In this regard, in section 2, a new approach of modeling heterogeneous evolutionary dynamic of asset pricing models with fully rational agents is proposed and the stability condition of the model is studied in section 3. In section 4, a numerical analyze is applied to the model to clarify stability of ESS.

2. Evolutionary dynamic formulation of the asset pricing model in heterogeneous market

In this paper, the asset pricing model with heterogeneous beliefs using evolutionary selection of expectation (BH model) as introduced by Brock and Hommes is used [12]. The BH model consists of several agents where they could either invest in a risk free asset or engage risky asset investment. The risk free investment pays a fixed rate of return r; on the contrary, the risky asset pays an uncertain dividend. Equation (2.1) depicts the wealth dynamic, wherein P_t stands for the price of share of the risky asset and y_t is the stochastic dividend process of the risky asset at time t.

$$W_{t+1} = RW_t + (P_{t+1} + y_{t+1} - RP_t)Z_t, \tag{2.1}$$

where, $R = 1+r$ denotes the gross rate of risk free return and Z_t denotes the number of shares of the risky asset purchased at time t. In a multi agent system with H different agents of traders, each agent tries to maximize the mean-variance equation (2.2) with respect to $Z_{h,t}$ which is the number of shares purchased by agent type h.

$$\underset{Z_{h,t}}{max}\, E_{h,t}[W_{t+1}] - \frac{a}{2}V_{h,t}[W_{t+1}], \tag{2.2}$$

where $E_{h,t}$ and $V_{h,t}$ stand for belief or forecast of trader of agent h about conditional expectation and conditional variance respectively and a is risk-aversion parameter. Consider, Z^s, shows the constant supply of outside risky shares per investor and $n_{h,t}$ denotes the fraction of agent type h at time t. Assuming a constant conditional variance for all trader types as $V_{h,t} = \sigma^2$, the equilibrium of demand and supply yields (2.3).

$$RP_t = \sum_{h=1}^{H} n_{h,t}\ E_{h,t}[P_{t+1} + y_{t+1}] - Z^s a\sigma^2. \tag{2.3}$$

The term $Z^s a\sigma^2$ is the risk premium for traders to hold risky assets. Suppose, p^* denotes the common belief about the fundamental price which is equal for all trader types and x_t, the deviation from the fundamental price, defined as $x_t = P_t - p^*$. In case of $E[y_t] = \overline{y}$, we assume that for all trader types we have $E_{h,t}[y_{t+1}] = E[y_{t+1}] = \overline{y}$ and all conditional believes $E_{h,t}[P_{t+1}]$ are in the form of (2.4),

$$E_{h,t}[P_{t+1}] = E_{h,t}[p^*] + E_{h,t}[x_{t+1}] = p^* + f_{h,t}(x_{t-1}, x_{t-2}, \dots, x_{t-L}). \tag{2.4}$$

The term, $f_{h,t}(x_{t-1}, x_{t-2}, \dots, x_{t-L})$, which is the heterogeneous part of the conditional expectation, is called forecasting rule which differs agents. Now we could re-evaluate the equilibrium of supply and demand equation knowing that $Rp^* = E_t[p^* + y_{t+1}]$, for the special case of zero supply of outside shares, i.e. $Z^s = 0$, which yields (2.5).

$$Rx_t = \sum_{h=1}^{H} n_{h,t}E_{h,t}[x_{t+1}] \equiv \sum_{h=1}^{H} n_{h,t}f_{h,t}. \tag{2.5}$$

Evolving of $n_{h,t}$ describes how believes are updated over time, which is evaluated through the multi-nominal logit model of (2.6) called Gibbs probabilities based on the discrete choice models.

$$n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}, \tag{2.6}$$

where, β shows the intensity of choice which depends on the sensitivity of traders to select the optimal prediction strategy and $U_{h,t}$ is the realized profit of trader type h which is a natural candidate for evolutionary fitness. Let $0 \leq \eta \leq 1$ be memory parameter of fitness function which shows the impact of past realized fitness on strategy selection. Then, the fitness function can be written as

$$U_{h,t} = (x_t - Rx_{t-1})\left(\frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}\right) + \eta.U_{h,t-1}. \tag{2.7}$$

2.1. **Evolutionary model with fully rational agents.** After studying the evolutionary dynamic formulation of Heterogenous market, an evolutionary dynamic with a rational agent will be investigated. Suppose that the traders of the first agent are rational with perfect foresight that have perfect knowledge of heterogeneous market equilibrium equation and beliefs of all other traders and they try to have a perfect prediction of future price. This agent's forecasting rule is obtained by

$$f_{1,t} = x_{t+1}. \tag{2.8}$$

The second agent is trend follower which believes that price varies in a very simple manner with respect to previous data. They use linear forecasting rule that is given by

$$f_{2,t} = g.x_{t-1}. \tag{2.9}$$

Substituting the beliefs of the agents with perfect foresight (2.8) and trend followers (2.9) into (2.5) would establish

$$R.x_t = n_{1,t-1}.x_{t+1} + n_{2,t-1}.g.x_{t-1}, \tag{2.10}$$

where, $n_{h,t-1}$ is the fraction of agent type h at time $t-1$. Let m be the difference of n_1 and n_2 ($m = n_1 - n_2$). Knowing that $n_1 + n_2 = 1$ and $n_{1,t} = \frac{1+m_t}{2}$ and $n_{2,t} = \frac{1-m_t}{2}$, (2.10) could be rewritten as

$$Rx_t = \frac{1+m_{t-1}}{2}.x_{t+1} + \frac{1-m_{t-1}}{2}.g.x_{t-1}. \tag{2.11}$$

Carrying x_{t+1} to the other side of the equation, (2.11) leads to

$$x_t = \frac{2R}{1+m_{t-2}}.x_{t-1} + \frac{m_{t-2}-1}{1+m_{t-2}}.g.x_{t-2}. \tag{2.12}$$

Evaluating fitness function for both agents through (2.7) and substituting them in (2.6), the dynamic of m is

$$\begin{aligned} m = n_1 - n_2 = & \frac{e^{\beta U_{1,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}} - \frac{e^{\beta U_{2,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}} \\ = & \frac{e^{\beta U_{1,t-1}} - e^{\beta U_{2,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}} \\ = & \tanh\left[\frac{1}{2}\beta\left(U_{1,t-1} - U_{2,t-1}\right)\right]. \end{aligned} \tag{2.13}$$

Consequently, the dynamic of m can be written as

$$\begin{aligned} m = & n_1 - n_2 \\ = & \tanh\left[\frac{1}{2}\beta\left((x_t - Rx_{t-1})\left(\frac{x_{t+1} - Rx_{t-1}}{a\sigma^2}\right)\right.\right. \\ & \left.\left. -(x_t - Rx_{t-1})\left(\frac{g.x_{t-1} - Rx_{t-1}}{a\sigma^2}\right)\right)\right] \\ = & \tanh\left[\frac{\beta}{2}\left\{\frac{1}{a\sigma^2}\left[\left(\frac{2R}{1+m_{t-2}} - R\right).x_{t-1} + \frac{m_{t-2}-1}{m_{t-2}+1}.g.x_{t-2}\right]\times\right.\right. \\ & \left.\left.\left[\frac{2R}{1+m_{t-2}}.x_{t-1} + \frac{m_{t-2}-1}{1+m_{t-2}}.g.x_{t-2}\right] - C\right\}\right]. \end{aligned} \tag{2.14}$$

Note that, (2.12) and (2.14) represent dynamics of a nonlinear system and if the states of the system are considered as in (2.15), the dynamic of the system could be easily analyzed.

$$X(k) = \begin{bmatrix} X_1(k) \\ X_1(k) \\ X_1(k) \\ X_1(k) \end{bmatrix} = \begin{bmatrix} m_{t-2} \\ m_{t-1} \\ x_{t-2} \\ x_{t-1} \end{bmatrix}. \tag{2.15}$$

Accordingly, the nonlinear state space equations can be written as

$$X_1(k+1) = X_2(k), \tag{2.16a}$$

$$\begin{aligned} X_2(k+1) = \tanh & \left[\frac{\beta}{2}\left\{\frac{1}{a\sigma^2}\left[\left(\frac{2R}{1+X_1(k)} - R\right).X_4(k) + \frac{X_1(k)-1}{1+X_1(k)}.g.X_3(k)\right]\times\right.\right. \\ & \left.\left.\left[\frac{2R}{1+X_1(k)}.X_4(k) + \left(\frac{X_1(k)-1}{1+X_1(k)} - 1\right).g.X_3(k)\right] - C\right\}\right], \end{aligned} \tag{2.16b}$$

$$X_3(k+1) = X_4(k), \tag{2.16c}$$

$$X_4(k+1) = \frac{2R}{1+X_1(k)}.X_4(k) + \frac{m_{t-2}-1}{1+X_1(k)}.g.X_3(k). \tag{2.16d}$$

Equilibrium and stability analysis.
Economic equilibrium is a condition or state in which economic forces are balanced. It may also be defined as the point at which supply equals demand for a product, with the equilibrium price existing where the hypothetical supply and demand curves intersect. In regards to product pricing, equilibrium exists when the price for a product reaches a point at which the demand for the product at that price equals the level of production or the associated current supply. This point does not suggest that all who may want the product have the ability to purchase it. Instead, it is the point at which all those who would like the product, and can afford to purchase the item, have the opportunity to do so [27]. To find an equilibrium of a pricing model, the fixed points of the EGT model in (2.16) should be evaluated.

Definition 2.1. An equilibrium point E is evolutionary stable state (ESS) of a system if the system in that state cannot be invaded by any new mutant strategies [21].

Theorem 2.2. *If E is an ESS then it is strictly stable equilibrium point of the discrete dynamical system [23].*

For evaluating the fixed points (equilibrium points) of the system, the equality $\overline{X}(k+1) = \overline{X}(k)$ ought to be solved, with $\overline{X} = [\overline{X}_1; \overline{X}_2; \overline{X}_3; \overline{X}_4]$ as

$$\overline{X}_1 = \overline{X}_2, \tag{2.17a}$$

$$\begin{aligned}\overline{X}_2 = \tanh\Bigg[\frac{\beta}{2}\Bigg\{\frac{1}{a\sigma^2}\Big[\big(\frac{2R}{1+\overline{X}_1} - R\big).\overline{X}_4 + \frac{\overline{X}_1 - 1}{1+\overline{X}_1}.g.\overline{X}_3\Big]\times \\ \Big[\frac{2R}{1+\overline{X}_1}.\overline{X}_4 + \big(\frac{\overline{X}_1 - 1}{1+\overline{X}_1} - 1\big).g.\overline{X}_3\Big] - C\Bigg\}\Bigg],\end{aligned} \tag{2.17b}$$

$$\overline{X}_3 = \overline{X}_4, \tag{2.17c}$$

$$\overline{X}_4 = \frac{2R}{1+\overline{X}_1}.\overline{X}_4 + \frac{\overline{m} - 1}{1+\overline{X}_1}.g.\overline{X}_3. \tag{2.17d}$$

Now, considering $\overline{X}_1 = \overline{X}_2 = \overline{m}$ and $\overline{X}_3 = \overline{X}_4 = \overline{x}$, and solving (2.17), we obtain

$$\begin{aligned}\overline{m} = \tanh\Bigg[\frac{\beta}{2}\Bigg\{\frac{1}{a\sigma^2}\Big[\big(\frac{2R}{1+\overline{m}} - R\big).\overline{x} + \frac{\overline{m} - 1}{1+\overline{m}}.g.\overline{x}\Big]\times \\ \Big[\frac{2R}{1+\overline{m}}.\overline{x} + \big(\frac{\overline{m} - 1}{1+\overline{m}} - 1\big).g.\overline{x}\Big] - C\Bigg\}\Bigg],\end{aligned} \tag{2.18a}$$

$$\overline{x} = \frac{2R}{1+\overline{m}}.\overline{x} + \frac{\overline{m} - 1}{1+\overline{m}}.g.\overline{x}. \tag{2.18b}$$

Let $m^{eq} = \tanh(-\beta C/2)$, $m^* = 1 - 2\frac{R-1}{g-1}$, $x^{eq} = 0$ and $x^* = \frac{\sqrt{2C+\ln(\frac{g-R}{R-1})}}{\sqrt{\beta D(R-1)(g-1)}}$ be the positive solution of $1 - 2\frac{R-1}{g-1} = \tanh\left\{\frac{\beta}{2}\left[D\,(g-1)\,(R-1)\,(x^*)^2 - C\right]\right\}$. Suppose,

$$E_1 = \begin{bmatrix} m^{eq} \\ m^{eq} \\ x^{eq} \\ x^{eq} \end{bmatrix}, \quad E_2 = \begin{bmatrix} m^* \\ m^* \\ x^* \\ x^* \end{bmatrix}, \quad E_3 = \begin{bmatrix} m^* \\ m^* \\ -x^* \\ -x^* \end{bmatrix}. \tag{2.19}$$

- For $g < R$, E_1 is the unique equilibrium.
- For $g > 2R - 1$, there are three equilibrium E_1, E_2 and E_3.
- For $R < g < 2R - 1$:
  - If $m^* < m^{eq}$, E_1 is the unique equilibrium.
  - If $m^* < m^{eq}$, there are three equilibrium E_1, E_2 and E_3.

There are many evidences witnesses the fact that at equilibrium, price is equal to fundamental value and it could be concluded that the fixed point E_1 is the only stable equilibrium of the model. To be specific, it could be concluded that the fundamental value of a security is the equilibrium risk adjusted price of the security. Since there is still some disagreement among economists about the correct model to adjust for risk, the fundamental value of a security is conditional on the investigators choice of an asset-pricing model [4]. According to another point of view, at equilibrium, price is equal to fundamental value, that bubbles are a temporary departure from this equilibrium and that their collapse represents a correction, a necessary or inevitable return to equilibrium [17]. Above all, Walras theory certificates the fact that at equilibrium, price is equal to fundamental value [11].

Consider a discrete time nonlinear system of the form

$$x\,(k+1) = f(x\,(k)\,). \tag{2.20}$$

Lemma 2.3. *System in (2.20) is asymptotically stable in fixed point E if the eigenvalues of the Jacobian matrix stays inside unit disk (See Appendix A for proof).*

Here, the Jacobian matrix of the nonlinear dynamic in (2.16) is formed at the fixed point E_1.

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial X_1[k+1]}{\partial X_1[k]} & \cdots & \frac{\partial X_1[k+1]}{\partial X_4[k]} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_4[k+1]}{\partial X_1[k]} & \cdots & \frac{\partial X_4[k+1]}{\partial X_4[k]} \end{bmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g(1+\tanh(\beta C/2)\ )}{\tanh(\beta C/2)\ -1} & \frac{2R}{\tanh(\beta C/2)\ -1} \end{pmatrix}. \end{aligned} \tag{2.21}$$

The eigenvalues of the Jacobian matrix of (2.21) are as

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ \frac{-R+\sqrt{R^2-g+g.\tanh^2(\frac{C\beta}{2})}}{\tanh(\frac{C\beta}{2})-1}\\ \frac{-R-\sqrt{R^2-g+g.\tanh^2(\frac{C\beta}{2})}}{\tanh(\frac{C\beta}{2})-1}\end{pmatrix}, \tag{2.22}$$

where, $1 \leq R \leq 2$, $\beta \geq 0$, $C \geq 0$ and $0 \leq \tanh\left(\frac{C\beta}{2}\right) \leq 1$. To establish stable equilibrium, g has to be evaluated in such way that eigenvalues stay inside unit circle.

2.2. **Evolutionary model with partly rational agents.** A heterogeneous evolutionary model of pricing, including an agent with perfect foresight has been investigated earlier. However, many researchers think the perfect forecast assumption is unrealistic. Possessing rational forecast under homogeneous expectations would require knowledge of the law of motion. But it is even more demanding in the heterogeneous world, where one should also know what others expect. In other words, a perfect forecaster would has to know the whole dynamic of the system and the expectations of other agents about future price to make a precise two step ahead (2SA) predict of future price. Besides, a mistaken 2SA predict could affect the stability of equilibrium points of the model. Here, it is being assumed that the future price has been estimated with a reliable method and the robustness of the model with respect to estimation error is discussed. Suppose, 2SA price is forecasted with error ε which means (2.8) changes into

$$f_{1,t} = x_{t+1} + \boldsymbol{\varepsilon_t}. \tag{2.23}$$

This error parameter causes some variations in state space equations as

$$X_1(k+1) = X_2(k), \tag{2.24a}$$

$$X_2(k+1) = \tanh\left[\frac{\beta}{2}\left\{\frac{1}{a\sigma^2}\left[\Big(\frac{2R}{1+X_1(k)} - R\Big).X_4(k) + \frac{X_1(k)-1}{1+X_1(k)}.g.X_3(k) + \boldsymbol{\varepsilon}[\boldsymbol{k}]\right]\times \left[\frac{2R}{1+X_1(k)}.X_4(k) + \Big(\frac{X_1(k)-1}{1+X_1(k)} - 1\Big).g.X_3(k) + \boldsymbol{\varepsilon}[\boldsymbol{k}]\right] - C\right\}\right], \tag{2.24b}$$

$$X_3(k+1) = X_4(k), \tag{2.24c}$$

$$X_4(k+1) = \frac{2R}{1+X_1(k)}.X_4(k) + \frac{m_{t-2}-1}{1+X_1(k)}.g.X_3(k) + \boldsymbol{\varepsilon}[\boldsymbol{k}]. \tag{2.24d}$$

Suppose $\boldsymbol{\varepsilon}[\boldsymbol{k}]$ is small enough, such that the fixed points of the system stay unchanged. Consider a discrete time nonlinear system of the form

$$X[k+1] = f(X[k], d[k]), \tag{2.25}$$

where, $d(k)$ is the disturbance or time varying parameter.

Lemma 2.4. *The system in (2.25) is stable in equilibrium point E if the eigenvalues of the Jacobian matrix stays inside unit disk (See Appendix B for proof).*

The eigenvalues of the Jacobian matrix of the system (2.24) in E_1 iscalculated.

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} = \begin{bmatrix} 0\\ 0\\ \dfrac{R-R.\Gamma+A+R.\Delta-R.\Delta.\Gamma-\tanh\left(\frac{\beta\left(C-D\varepsilon^2\right)}{2}\right).A}{(\Delta+1).(\Gamma-1).\left(\tanh\left(\frac{\beta(C-D\varepsilon^2)}{2}\right)-1\right)}\\ \dfrac{R-R.\Gamma-A+R.\Delta-R.\Delta.\Gamma+\tanh\left(\frac{\beta\left(C-D\varepsilon^2\right)}{2}\right).A}{(\Delta+1).(\Gamma-1).\left(\tanh\left(\frac{\beta(C-D\varepsilon^2)}{2}\right)-1\right)}\end{bmatrix}, \tag{2.26}$$

where, $A = \sqrt{R^2.\Gamma^2.\Delta^2 - 2R^2.\Delta.\Gamma + R^2 - g.\Delta^2\Delta.\Gamma^2 + g.\Delta^2 + g.\Gamma^2 - g}$ and $\Gamma = \tanh\left(\frac{C\beta}{2}\right)$ and $\Delta = \tanh\left(\frac{\beta D\varepsilon^2}{2}\right)$. As it is shown in (2.26), the J matrix, has four eigenvalues which contains two zero eigenvalues that are permanently inside unit disk.

3. Numerical Analysis

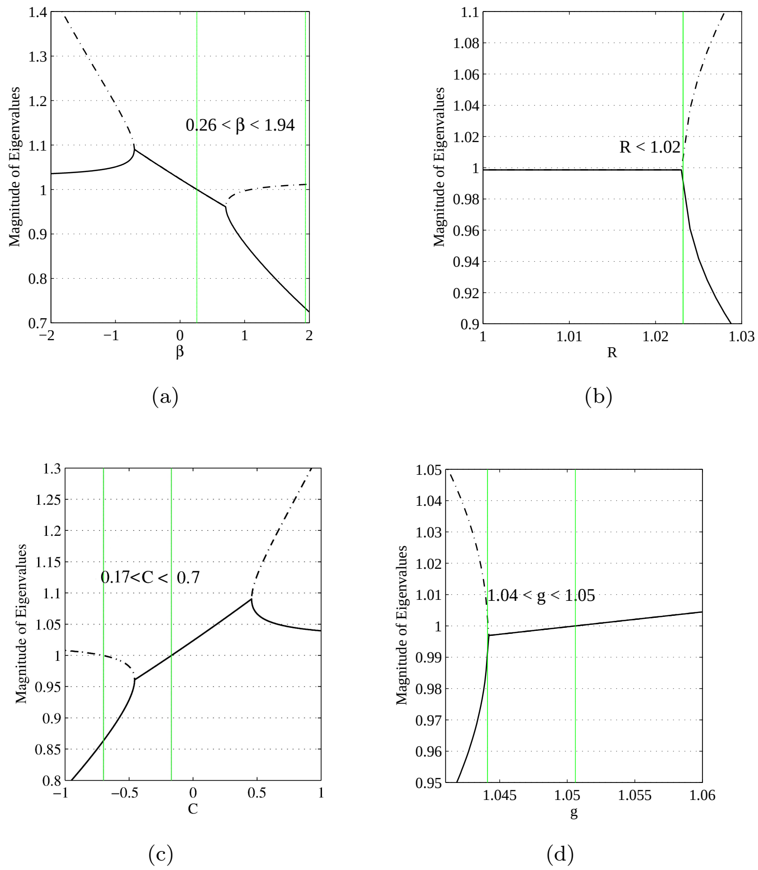
To model behavior of an economical system based on its time series data set, the set of unknown parameters β, R, g, D, C and p^* should be evaluated. For a set of time series like gold market price or asset price, considering (2.22), the condition in (3.1) should be satisfied.

$$Min\ P_T = \frac{1}{2}\sum_{t=1}^{T}\left(d\left(t\right) - o\left(t\right)\right)^2 \quad \text{Subject to}:\ |\lambda_3| < 1\&\,|\lambda_4| < 1, \tag{3.1}$$

where, $d(t)$ and $o(t)$ denote the predicted and observed value of price at time t respectively and P_T indicates the estimation error over time. For numerical analysis, the parameters β, R , g , D , C and p^* are assumed to be $\beta = 0.27$, R= 1.02, g= 1.047, D= 1717.8, C= 0.18 and p^*= 0.73. Assuming these values, the stability condition of three fixed points of the dynamic will be studied. For equilibrium point E_1, the characteristic polynomial of the Jacobian matrix would be $d\left(\lambda\right) = \lambda^4 - 1.9938\lambda^3 + 0.9972\lambda^2$ which means the eigenvalues of the matrix are λ_1= 0, λ_2= 0, λ_3= 0.9969 − 0.0581i and λ_4= 0.9969 + 0.0581i. It is clear that all eigenvalues stay inside the unit circle that proves the stability of the system in E_1 and therefore, E_1 is an ESS of the model. For E_2 and E_3, the solutions of the characteristic polynomial would be λ_1= 1.067+i0.027, λ_2= 1.067−i0.027, λ_3=0.14+i 0.22 and λ_4= 0.14−i 0.22, which means two of four eigenvalues of the matrix are not inside the unit circle that proves the instability of the system in E_2 and E_3. Thus, E_1 is the only stable equilibrium and a regular ESS of the system.

3.1. **Evolutionary equilibrium with respect to varying parameters.** In this section, the stability of E_1 will be discussed whereas parameters vary. As shown in previous section, two of eigenvalues are always constant and equal to zero. Therefore studying λ_3 and λ_4 is adequate. As a first step, parameter β changes while, g, D, C and p^* are constant. As depicted in Figure 2(a), if β remains in [0.26 1.94] area, system stays stable. It means that for large values of β, i.e. high intensity of choice,

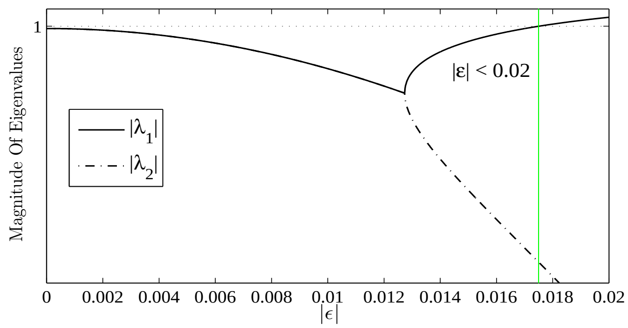
FIGURE 1. Magnitude of eigenvalues with respect to varying parameters. Numerical analysis: Case $\beta = 0.27$, $R = 1.02$, $g = 1.047$, $D = 1717.8$, $C = 0.18$ and $p^* = 0.73$.



system becoms unstable. Figure 2(b) shows that for small amounts of risk free return, $1 \leq R \leq 1.02$ the system stays stable. But as the amount of risk free return grows, investors would rather to invest on risk free asset which makes the system unstable. Figure 2(c) demonstrates the variation of eigenvalues with respect to C. This parameter shows the benefit that rational agents get. Note that, a rational agent would incur furthur expense to get information for being rational. The stable area for parameter C is $0.17 \leq C \leq 0.71$. According to Figure 2(d), if the parameter g, which is affected by the belief of technical traders, remains in the range [1.04 1.05], system stays stable.

3.2. **Model Robustness to Prediction Error.** According to (2.22), it is quite clear that the Jacobian matrix of an evolutionary model with partly rational agents

Figure 2. Magnitude of the eigenvalues of system with partly rational agent.



has only two nonzero eigenvalues. Analyzing the eigenvalues with respect to varying parameter ε, one could conclude that these two eigenvalues will stay inside unit circle for small ε. That means one could claim that if any agent's prediction error is less than 0.02, the equilibrium remains stable; which is depicted in Figure 2.

4. Conclusion

In this paper a new approach of modeling an evolutionary dynamic consisting of rational agent has been proposed. This approach has been used to solve the problem with future dependency of modelling evolutionary dynamics with rational agent. Furthermore, the stability of the dynamic with partly rational agent which has imprecise prediction, has been analyzed. The analysis of stability of the equilibrium leads into finding the maximum value for prediction error that does not affect the stability. Moreover, proposed approach could be used in a case that an agent has a complicated belief that may cause complexity in dynamic and stability analysis of the model. This approach could be applied to management and decision making problems consisting of fully rational agents with complicated dynamics.

Appendix A.

Recall that, if a function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing and $\psi(0) = 0$ then it is a $\mathcal{K}$-function. Furthermore, it is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and also $\psi(s) \to \infty$ as $s \to \infty$. If $\psi(s) > 0$ for all $s > 0$, and $\psi(0) = 0$, it is a positive definite function. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if for each fixed $s \geq 0$, the function $\beta(s, .)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$, and for each fixed $t \geq 0$, the function $\beta(., t)$ is a $\mathcal{K}$-function. Note that in a stable system, every state trajectory remains bounded; and no matter what the initial state is, the state trajectory eventually becomes small.

Definition A.1. A nonlinear dynamic system of (2.20) is asymptotically stable (AS) if there exist a $\mathcal{KL}$-function, $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that, for each $\xi \in \mathbb{R}^n$, it holds $|x(k,\xi)| \leq \beta\left(|\xi|, k\right)$ for each $k \in \mathbb{Z}_+$.

Definition A.2. A continuous function $V{:}\mathbb{R}^n{\rightarrow}\mathbb{R}_{\geq 0}$ is called a *lyapunov function* for a nonlinear dynamic system if the following holds: There exist $\mathcal{K}_\infty$-function α_1 and α_2 such that

$$\alpha_1(\,|\xi|\,){\leq}V(\xi){\leq}\alpha_2(\,|\xi|\,), \quad \forall\xi{\in}\mathbb{R}^n. \tag{A.1}$$

There exist a $\mathcal{K}_\infty$-function α_3 such that

$$V\left(f(\xi)\right) - V\left(\xi\right) \leq \alpha_3\left(|\xi|\right), \quad \forall\xi{\in}\mathbb{R}^n. \tag{A.2}$$

Theorem A.3. *The linear discrete-time system is considered as*

$$x\left(k{+}1\right) = Ax\left(k\right), \tag{A.3}$$

where, the eigenvalues of the matrix A are located strictly inside the unit disk. For a symmetric and positive-definite matrix Q, $P > 0$ is the unique solution to the matrix $A^TPA - A = -Q$. The matrix $V(x) = x^TPx$ is positive-definite and radially unbounded function which satisfies the condition of Definition A.2 with $\alpha_1(r) = \lambda_{min}(P)r^2$, $\alpha_2(r) = \lambda_{max}(P)r^2$ and $\alpha_3(r) = \frac{1}{2}\lambda_{min}(Q)r^2$. Therefore, V is a Lyapunov function for the system in (2.20) [14].

Appendix B.

It is proven that a discrete-time system with disturbances or time-varying parameters, taking values in a compact set, is uniformly asymptotically stable (UAS) with respect to a closed, not necessarily compact, invariant set $\mathcal{A}$ if and only if there exists a smooth *Lyapunov function* V with respect to the set $\mathcal{A}$ [15]. The system presented in (2.25) could be presented as a form of (2.26) assuming $d(k) = \boldsymbol{\varepsilon}\left[\boldsymbol{k}\right]$. The mentioned system is stable in fixed point E if the eigenvalues of the Jacobian matrix stays inside unit disk based on Theorem A.3 and the closed and invariant set $\mathcal{A}$ is the Region of Attraction (ROA) of the system.

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