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A fractional type of the Chebyshev polynomials for approximation of solution of linear fractional differential equations

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Abstract In this paper we introduce a type of fractional-order polynomials based on the classical Chebyshev polynomials of the second kind (FCSs). Also we construct the operational matrix of fractional derivative of order γ in the Caputo for FCSs and show that this matrix with the Tau method are utilized to reduce the solution of some fractional -order differential equations.

Keywords. Chebyshev polynomials, Orthogonal system, Fractional differential equation, Fractional-order Chebyshev functions, Operational matrix.

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1. Introduction

The Chebyshev polynomials are one of the most useful polynomials which are suitable in numerical analysis including polynomial approximation, integral and differential equations and spectral methods for partial differential equations [1, 2, 3, 4]. One of the attractive concepts in the initial and boundary value problems is differentiation and integration of fractional order [5, 6, 7, 8]. Many researchers extend classical methods in studies of differential and integral equations of integer order to fractional type of these problems [10, 11]. One of the wide classes of researches focuses to constructing the operational matrix of derivative in some spectral methods. Recently, a lot of attention has been devoted to construct operational matrix of fractional derivative [1, 12, 13]. For example the fractional type of Legendre polynomials are used to solving some fractional differential equations [14]. In [15] operational matrix of fractional Jacobi functions is investigated.

In this paper we use shifted Chebyshev polynomials of second kind and recall some important properties. Next we extend these polynomials to fractional type and obtain the operational matrix of fractional derivative. This matrix is introduced and applied with the Galerkin method for solving linear fractional differential equations [16]. For this purpose, organization of paper is expressed as follows. In section 2, we define some necessary definitions and initial preliminaries of fractional calculus. In section 3, we obtain fractional Chebyshev function and their properties. We make a new operational matrix for fractional derivative by fractional Chebyshev function in section 4 . We show applications of the operational matrix in section 5. Finally in section 6, we solve some numerical examples.

2. Essential Preliminaries

2.1. **Overview on Chebyshev polynomials of second kind.** Chebyshev polynomials of the second kind of degree n are defined on the interval $[-1,1]$ as

$$U_n(t)=\frac{\sin(n+1)\theta}{\sin\theta}, \tag{2.1}$$

where $\theta=\arccos(t)$. From above, for example we get $U_0(t)=1, U_1(t)=2t$. These polynomials satisfy the following recurrence relation:

$$U_n(t)=2tU_{n-1}(t)-U_{n-2}(t), \qquad n=2,3,\cdots .$$

For using this polynomials on the interval $(0,1)$, which called shifted Chebyshev polynomials by introducing the change of variable $2t-1$ we derive

$$U_n^*(t)=U_n(2t-1). \tag{2.2}$$

The shifted Chebyshev polynomials satisfy to the following recurrence formula:

$$U_n^*(t)=(4t-2)U_{n-1}^*(t)-U_{n-2}^*(t), \qquad n=2,3,\cdots , \tag{2.3}$$

with the initial conditions $U_0^*(t)=1$ and $U_1^*(t)=4t-2$. The analytic form of the shifted Chebyshev polynomials $U_n^*(t)$ of degree n is given by

$$U_n^*(t)=\sum_{r=0}^{n+1} r(-1)^{(n+1-r)}\frac{(n+r)!2^{2r-1}}{(n+1-r)!(2r)!}t^{r-1}, \tag{2.4}$$

obviously $U_n^*(0)=(-1)^n$ and $U_n^*(1)=2$. The orthogonality condition is

$$\int_0^1 U_m^*(t)U_n^*(t)w(t)dt=\tau\delta_{mn}, \tag{2.5}$$

where $w(t)=\sqrt{t-t^2}$, $\tau=\frac{1}{8}\pi$ and δ_{mn} is the Kronecker function. According to [3] we can obtain the following equation for these polynomials:

$$A\cdot\mathbf{U}=\mathbf{T}, \tag{2.6}$$

where $\mathbf{T} = [1, t, t^2, \cdots, t^N]^T$, $\mathbf{U} = [U_0^*(t), U_1^*(t), \cdots, U_N^*(t)]^T$ and A is $(N+1) \times (N+1)$ lower triangular matrix with following entries:

$$A[i,j] = \frac{1}{4^{i-1}} \frac{j}{i} \binom{2i}{i-j},$$
$$i, j = 1, 2, \cdots, N+1.$$

2.2. **The fractional derivative in the Caputo sense.** We recall some preliminaries for our discussion in this paper. First we start by fractional calculus. There are various definitions of differentiation of fractional order $\gamma > 0$, and not necessarily equivalent to each other,(see, e.g. [1,5]). The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer other derivative, which have some physical interpretations.

Definition 2.1. *The The Caputo fractional derivative is defined as* [9]

$$D^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\gamma+1-n}} dt, \quad n-1 < \gamma < n, \quad n \in \mathbb{N}, \tag{2.7}$$

where $\Gamma(.)$ denotes the Gamma function and $n = [\gamma] + 1$, which $[\gamma]$ denotes the integer part of γ.

We know for $\gamma \in \mathbb{N}$, the Caputo differential operator equals with the usual differential operator of integer order. Similar to integer-order differentiation, Caputo's fractional differential has the linear property:

$$D^\gamma(\lambda f(x) + \mu g(x)) = \lambda D^\gamma f(x) + \mu D^\gamma g(x), \tag{2.8}$$

where λ and μ are constants. Also, for the Caputo's derivative we have [5],

$$D^\gamma C = 0, \quad (C \text{ is a constant}), \tag{2.9}$$

$$D^\gamma x^\alpha = \begin{cases} 0, & \text{for } \alpha \in \mathbb{N}_0 \text{ and } \alpha < \lceil \gamma \rceil, \\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\gamma)} x^{\alpha-\gamma}, & \text{for } \alpha \in \mathbb{N}_0 \text{ and } \alpha \geq \lceil \gamma \rceil \\ & \text{or } \alpha \notin \mathbb{N} \text{ and } \alpha > \lfloor \gamma \rfloor, \end{cases} \tag{2.10}$$

which $\lfloor \gamma \rfloor$ and $\lceil \gamma \rceil$ are the floor and ceiling functions respectively. Also $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We need the generalized Taylor's formula that involves Caputo fractional derivatives which is presented in [17].

Theorem 2.2. *(Generalized Taylor formula) Let* $D^{i\alpha} f(x) \in (0, 1]$ *for* $i = 0, 1, \ldots, N$ *and* $0 < \alpha \leqslant 1$. *Then*

$$f(x) = \sum_{i=0}^{N-1} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + R_N^\alpha(x)$$

where

$$R_N^\alpha(x) = \frac{x^{N\alpha}}{\Gamma(N\alpha+1)} D^{N\alpha} f(\xi_x), \quad \xi_x \in (0,x], \forall x \in (0,1].$$

3. Fractional-order Chebyshev functions

Same as [14, 18], we introduce the fractional Chebyshev polynomials of second kind ($FCSs$) by changing of variable $t = x^\alpha$ which $\alpha > 0$. Also we denote the FCSs $U_i^*(x^\alpha)$ as $\overline{U}_i^\alpha(x)$. From the recurrence relation of the shifted Chebyshev polynomials (2.3), we find that $\overline{U}_i^\alpha(x)$ can be obtained with the following recurrence formula:

$$\overline{U}_i^\alpha(x) = (4x^\alpha - 2)\overline{U}_{i-1}^\alpha(x) - \overline{U}_{i-2}^\alpha(x), \quad i = 1, 2, \cdots. \tag{3.1}$$

Clearly $\overline{U}_0^\alpha(x) = 1$ and $\overline{U}_1^\alpha(x) = 4x^\alpha - 2$. Also, according to Eq. (2.4), the analytic form of $\overline{U}_i^\alpha(x)$ of degree $i\alpha$ given by

$$\overline{U}_i^\alpha(x) = \frac{1}{2}\sum_{r=0}^{i+1} r(-1)^{(n+1-r)} \frac{(n+r)!2^{2r}}{(n+1-r)!(2r)!} x^{\alpha(r-1)}, \tag{3.2}$$

Lemma 3.1. *$\overline{U}_i^\alpha(x)$s are orthogonal with respect to the weight function $w_\alpha(x) = x^{2\alpha-1}\sqrt{x^{-\alpha}-1}$ and we have:*

$$\int_0^1 \overline{U}_i^\alpha(x)\overline{U}_j^\alpha(x) w_\alpha(x) dx = \frac{1}{\alpha}\tau\delta_{ij},$$

Proof. Taking $t = x^\alpha$ in (2.5), we have $dt = \alpha x^{\alpha-1}dx$. Substituting these values in (2.5), we get

$$\begin{aligned}
\tau\delta_{ij} &= \int_0^1 U_i^*(x^\alpha)U_j^*(x^\alpha)w(x^\alpha)\alpha x^{\alpha-1}dx \\
&= \int_0^1 \overline{U}_i^\alpha(x)\overline{U}_j^\alpha(x) x^\alpha\sqrt{x^{-\alpha}-1}\alpha x^{\alpha-1}dx \\
&= \int_0^1 \overline{U}_i^\alpha(x)\overline{U}_j^\alpha(x)\alpha x^{2\alpha-1}\sqrt{x^{-\alpha}-1}dx \\
&= \alpha\int_0^1 \overline{U}_i^\alpha(x)\overline{U}_j^\alpha(x) w_\alpha(x)dx.
\end{aligned}$$

□

Proposition 3.2. *The fractional-order chebyshev function $\overline{U}_i^\alpha(t)$, has precisely i zeros in the form*

$$t_j = \left(\frac{1 + \left(cos\frac{j\pi}{i+1}\right)}{2}\right)^{\frac{1}{\alpha}}, \quad j = 1, 2, \cdots, i. \tag{3.3}$$

Proof. . The shifted Chebyshev polynomial $U_i^*(x)$ has i zeros

$$x_j = \frac{1 + (cos\frac{j\pi}{i+1})}{2}, \quad j = 1, \cdots, i. \tag{3.4}$$

Then $U_i^*(x)$ can be written as

$$U_i^*(x) = (x - x_1)(x - x_2)\cdots(x - x_i).$$

Changing of variable $x = t^\alpha$ yields

$$\overline{U}_i^\alpha(t) = (t^\alpha - x_1)(t^\alpha - x_2)\cdots(t^\alpha - x_i),$$

so, the zeros of $\overline{U}_i^\alpha(t)$ are

$$t_j = \left(x_j\right)^{\frac{1}{\alpha}}, \quad j = 1, 2, \cdots, i.$$

□

For any function $f \in L^2_{w_\alpha}$ we write

$$f = \sum_{k=0}^{\infty} f_k \overline{U}_k^\alpha(x), \tag{3.5}$$

with

$$f_k = \frac{\langle f, \overline{U}_k^\alpha \rangle_{w_\alpha}}{\|\overline{U}_k^\alpha\|^2_{w_\alpha}}, \tag{3.6}$$

Where $f_k^{,}s$ are the expansion coefficients associated with the family $\{\overline{U}_k^\alpha\}$.

4. The operational matrix of fractional derivative

Let

$$U_\alpha(x) = [\overline{U}_0^\alpha(x), \overline{U}_1^\alpha(x), \cdots, \overline{U}_N^\alpha(x)]^T,$$
$$X_\alpha(x) = [1, x^\alpha, x^{2\alpha}, \cdots, x^{N\alpha}]^T. \tag{4.1}$$

According to 2.6 if we take $\mathbf{F} = A^{-1}$ we obtain:

$$U_\alpha(x) = \mathbf{F} X_\alpha, \tag{4.2}$$

or

$$\overline{U}_i^\alpha(x) = \sum_{j=0}^{N} f_{ij} x^{j\alpha}, \; i = 0, 1, 2, \ldots, N. \tag{4.3}$$

The fractional derivative of order γ of the vector $U_\alpha(x)$ can be expressed by

$$D^\gamma U_\alpha(x) \simeq \mathbf{D}^{(\gamma)} U_\alpha(x),$$

where $\mathbf{D}^{(\gamma)}$ is the $(N + 1) \times (N + 1)$ operational matrix of the fractional derivative. In this section, we want to obtain $\mathbf{D}^{(\gamma)}$.

Lemma 4.1. *Let*

$$k = \begin{cases} \textit{the largest integer such that } k\alpha < \lceil \gamma \rceil, & \textit{for } \alpha \in \mathbb{N}_0, \\ 0, & \textit{for } \alpha \notin \mathbb{N} \textit{ and } \alpha < \lceil \gamma \rceil. \end{cases} \tag{4.4}$$

Then, the we have

$$D^{\gamma} X_{\alpha}(x) = \overline{D}_{\gamma} X_{\alpha}^{\gamma}(x), \tag{4.5}$$

where $\overline{D_{\gamma}}$ is the following $(n+1) \times (n+1)$ diagonal matrix

$$\overline{\boldsymbol{D}}\gamma = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\gamma)} & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\gamma)} \end{bmatrix}. \tag{4.6}$$

$$X_{\alpha}^{\gamma}(x) = [0, \ldots, 0, x^{(k+1)\alpha-\gamma}, x^{(k+2)\alpha-\gamma}, \ldots, x^{N\alpha-\gamma}]^T. \tag{4.7}$$

Proof. Using (2.10) we have

$$\begin{aligned} D^{\gamma} X_{\alpha}(x) &= \begin{bmatrix} 0 \\ D^{\gamma} x^{\alpha} \\ \vdots \\ D^{\gamma} x^{N\alpha} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\gamma)} & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x^{(k+1)\alpha-\gamma} \\ \vdots \\ x^{N\alpha-\gamma} \end{bmatrix} \\ &= \overline{D_{\gamma}} X_{\alpha}^{\gamma}(x) \end{aligned}$$

□

Noting that, in D^{γ}, the first $k+1$ rows, are all zero.

Lemma 4.2. *Under the notations of lemma 4.1 we have*

$$X_{\alpha}^{\gamma}(x) \simeq B U_{\alpha}(x), \tag{4.8}$$

where $B = (b_{ij})$ is the following $(N+1)\times(N+1)$ matrix:

$$b_{ij} = \begin{cases} 0, & \begin{cases} i=0,1,2,\dots,k, \\ j=0,1,2,\dots,N, \end{cases} \\ \frac{\sqrt{\pi}}{\tau}\sum_{l=0}^{j-1} f_{jl}\left(\frac{\Gamma(i-\frac{\gamma}{\alpha}+l+\frac{5}{2})}{\Gamma(i-\frac{\gamma}{\alpha}+l+3)} - \frac{\Gamma(i-\frac{\gamma}{\alpha}+l+\frac{7}{2})}{\Gamma(i-\frac{\gamma}{\alpha}+l+4)}\right), & \begin{cases} i=k+1,k+2,\dots,N, \\ j=0,1,2,\dots,N. \end{cases} \end{cases}$$

Proof. Obviously, for $i = 0, 1, \dots, k$, we have $b_{ij} = 0$. Now, for some $i > k$, approximate $x^{i\alpha-\gamma}$ by $N+1$ terms of fractional-order Chebyshev series, we get

$$x^{i\alpha-\gamma} \simeq \sum_{j=0}^{N} b_{ij}\overline{U}_i^{\alpha}(x).$$

By (4.3) we get

$$\begin{aligned} b_{ij} &= \frac{\alpha}{\tau}\int_0^1 x^{i\alpha-\beta}\overline{U}_j^{\alpha}(x)x^{2\alpha-1}\sqrt{x^{-\alpha}-1}dx \\ &= \frac{\alpha}{\tau}\int_0^1 x^{i\alpha-\beta}\sum_{l=0}^{j-1} f_{jl}x^{2\alpha-1}\sqrt{x^{-\alpha}-1}dx \\ &= \frac{\alpha}{\tau}\sum_{l=0}^{j-1} f_{jl}\int_0^1 x^{i\alpha-\gamma}x^{l\alpha}x^{2\alpha-1}\sqrt{x^{-\alpha}-1}dx \\ &= \frac{\alpha}{\tau}\sum_{l=0}^{j-1} f_{jl}\int_0^1 G(x,i,l,\alpha,\gamma)dx. \end{aligned}$$

Set $u = \sqrt{x^{-\alpha}-1}$, we obtain

$$\begin{aligned} \int_0^{\infty} G(x,i,l,\alpha,\gamma)dx =& \frac{2}{\alpha}\int_0^{\infty}\left(\frac{1}{(u^2+1)^{i-\frac{\gamma}{\alpha}+l+2}} - \frac{1}{(u^2+1)^{i-\frac{\gamma}{\alpha}+l+3}}\right)du \\ &= \frac{2}{\alpha}\frac{\sqrt{\pi}}{2}\left(\frac{\Gamma(i-\frac{\gamma}{\alpha}+l+\frac{5}{2})}{\Gamma(i-\frac{\gamma}{\alpha}+l+3)} - \frac{\Gamma(i-\frac{\gamma}{\alpha}+l+\frac{7}{2})}{\Gamma(i-\frac{\gamma}{\alpha}+l+4)}\right) \end{aligned}$$

Therefore

$$b_{ij} = \frac{\sqrt{\pi}}{\tau}\sum_{l=0}^{j-1} f_{jl}\left(\frac{\Gamma(i-(\frac{\gamma}{\alpha})l+\frac{5}{2})}{\Gamma(i-(\frac{\gamma}{\alpha})+l+3)} - \frac{\Gamma(i-(\frac{\gamma}{\alpha})l+\frac{7}{2})}{\Gamma(i-(\frac{\gamma}{\alpha})+l+4)}\right),$$

Where τ is defined in (2.5). □

Theorem 4.3. *Let $U_{\alpha}(x)$ be FCSs vector,D^{γ} is the $(N+1)\times(N+1)$ operational matrix of fractional derivative of order $\gamma > 0$ in Caputo sense and $\alpha \in \mathbb{N}_0$ or $\alpha > [\gamma]$ when $\alpha \notin \mathbb{N}$ then*

$$D^{\gamma} \simeq F\overline{D}_{\gamma}B,$$

where $\overline{D}_\gamma$, B and F are given in Eqs. (4.6), (4.8) respectively.

Proof. Applying Eq. (2.4), Lemma 4.1 and Lemma 4.2, we can write γth order fractional derivative of $U_\alpha(x)$ as

$$\begin{aligned} D^\gamma U_\alpha(x) &= F D^\gamma X_\alpha(x) = F\overline{D}_\gamma X_\alpha^\gamma(x) \\ &\simeq F\overline{D}_\gamma B U_\alpha(x) = D^{(\gamma)} U_\alpha(x). \end{aligned}$$

□

Proposition 4.4. *The operational matrix of fractional derivative, α, can be computed as*

$$\boldsymbol{D}^{(\alpha)} = \boldsymbol{F}\,\overline{\boldsymbol{D}}_\alpha \boldsymbol{F}^{-1},$$

where $\overline{\boldsymbol{D}}_\alpha$ is a $(n+1)\times(n+1)$ matrix

$$\overline{\overline{\boldsymbol{D}}}\gamma = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\Gamma(n\alpha+1)}{\Gamma\big((N-1)\alpha+1\big)} & 0 \end{bmatrix}. \tag{4.9}$$

Proof. Using (2.10) we have

$$\begin{aligned} D^\alpha x^\alpha &= \frac{\Gamma(\alpha+1)}{\Gamma(1)}, \\ D^\alpha x^{2\alpha} &= \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} x^\alpha, \\ &\vdots \\ D^\alpha x^{N\alpha} &= \frac{\Gamma(N\alpha+1)}{\Gamma\big((N-1)\alpha+1\big)} x^{(N-1)\alpha} \end{aligned} \tag{4.10}$$

Thus, αth order fractional derivative of $X_\alpha(x)$ is written in the matrix form

$$D^\alpha X_\alpha(x) = \overline{\overline{\mathbf{D}}}_\alpha X_\alpha(x), \tag{4.11}$$

According to E.q (4.2), we get

$$X_\alpha(x) = \mathbf{F}^{-1} U_\alpha(x), \tag{4.12}$$

Thus, using Eqs.(4.2), (4.11) and (4.12) we have

$$\begin{aligned} D^\gamma U_\alpha(x) &= \mathbf{F}\; D^\gamma X_\alpha(x) = \mathbf{F}\;\overline{\overline{\mathbf{D}}}_\gamma X_\alpha(x) \\ &= \mathbf{F}\;\overline{\overline{\mathbf{D}}}_\gamma \mathbf{F}^{-1} U_\alpha(x). \end{aligned}$$

□

5. Application of the operational matrix of fractional derivatives

In this section, we apply the above discussions to solve some fractional differential equations. Consider the linear multi-order FDE

$$D^{\gamma}y(x) = a_1 D^{\beta_1}y(x) + \ldots + a_k D^{\beta_k}y(x) + a_{k+1}y(x) + a_{k+2}g(x), \quad (5.1)$$

with the initial conditions

$$y^{(i)}(0) = d_i, \quad i = 0, \cdots, m, \quad (5.2)$$

which $a_j \in \mathbb{R}$, $m < \gamma \leq m+1$, $0 < \beta_1 < \beta_2 < \ldots < \beta_k < \gamma$ and $y(x)$ and $g(x)$ are in $L^2_{w\alpha}$. To solve (5.1) and (5.2) we approximate $y(x)$ and $g(x)$ by FCSs as

$$y(x) \simeq \sum_{i=0}^{N} c_i \overline{U}_i^{\alpha}(x) = C^T U_{\alpha}(x), \quad (5.3)$$

$$g(x) \simeq \sum_{i=0}^{N} g_i \overline{U}_i^{\alpha}(x) = G^T U_{\alpha}(x), \quad (5.4)$$

where vector $G = [g_0, \ldots, g_N]^T$ is known but $C = [c_0, \ldots, c_N]^T$ is an unknown vector. Applying Theorem 4.3 to (5.3), we obtain

$$D^{\gamma}y(x) \simeq C^T D^{\gamma} U_{\alpha}(x) \simeq C^T \mathbf{D}^{\gamma} U_{\alpha}(x), \quad (5.5)$$

$$D^{\beta_j}y(x) \simeq C^T D^{\beta_j} U_{\alpha}(x) \simeq C^T \mathbf{D}^{(\beta_j)} U_{\alpha}(x), \quad j = 1, \ldots, k. \quad (5.6)$$

Employing Eqs. (5.3), (5.4), (5.5) and (5.6) the residual $R_N(x)$ for Eq. (5.1) can be written as

$$R_N(x) \simeq \big(C^T \mathbf{D}^{\gamma} - C^T \sum_{j=1}^{k} a_j \mathbf{D}^{(\beta_j)} - a_{k+1}C^T - a_{k+2}G^T\big) U_{\alpha}(x). \quad (5.7)$$

By Galerkin method [31], by applying

$$\langle R_N(x), \overline{U}_j^{\alpha}(x) \rangle_{w\alpha} = \int_0^1 R_N(x) \overline{U}_j^{\alpha}(x) w_{\alpha}(x) dx = 0, \quad (5.8)$$

where $j = 0, 1, \ldots, N-m-1$, we generate $m-n$ linear equations. Also, according to Theorem 4.3 and by substituting Eq.(5.3) in Eq. (5.2) we get

$$y^{(i)}(0) = C^T \mathbf{D}^{(i)} U_{(\alpha}(0) = d_i. \quad i = 0, 1, \ldots, m \quad (5.9)$$

Equations (5.8) and (5.9) generate $N-m$ and $m+1$ set of linear equations, respectively. These linear equations can be solved to obtain unknown vector C.

6. Numerical Examples

In this section we apply the proposed approximation procedure in two examples.

Example 1. As the second example, consider the fractional differential equation [12]

$$D^{0.5}y(x) + y(x) = \sqrt{x} + \frac{\sqrt{\pi}}{2},$$
$$y(0) = 0.$$

Lakestani et al. [12] solved this example with the best maximum absolute error 7.8×10^{-5}. Regarding example1,we solved this problem, with $\alpha = \frac{1}{2}$ and $n = 1$. For $n = 1$ we have:

$$\mathbf{D}^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 2\sqrt{\pi} & 0 \end{pmatrix}, G = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore, we obtain the following system of algebraic equations:

$$\begin{cases} c_0 - 2c_1 = 0, \\ c_0 + 2\sqrt{\pi}c_1 = \frac{1}{2}(1+\sqrt{\pi}). \end{cases}$$

Solving this equations yields

$$c_0 = \frac{1}{2}, c_1 = \frac{1}{4}$$

Thus

$$y(x) = \left(\frac{1}{2}, \frac{1}{2}\right) \begin{pmatrix} 1 \\ 4x^{\frac{1}{2}} - 2 \end{pmatrix} = \sqrt{x},$$

which is the exact solution.

Example 2. The simple inhomogeneous Bagely-Torvik equation[14]

$$D^2y(x) + D^{3/2}y(x) + y(x) = 1 + x,$$

$$y(0) = 1, \ y'(0) = 1.$$

has the exact solution $y(x) = 1 + x$. With our technique described before with $N = 2$ and $\alpha = 1$, we get

$$y(x) = c_0\overline{U}_0^1(x) + c_1\overline{U}_1^1(x) + c_2\overline{U}_2^1(x) = C^T U_1(x).$$

Here we have:

$$\mathbf{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}^{\frac{3}{2}} = \left(\frac{1}{\sqrt{\pi^3}}\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2048}{15} & \frac{4096}{105} & -\frac{2048}{315} \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Therefore using Eq. (5.8) we obtain

$$c_0 + (32 + \frac{2048}{15}\pi^{-\frac{3}{2}})c_2 = \frac{3}{2} \tag{6.1}$$

Now, by applying Eq. (5.9)we have

$$C^T U_1(0) = c_0 - 2c_1 + 3c_2 = 1, \tag{6.2}$$

$$C^T D^{(1)} U_1(0) = 4c_1 - 16c_2 = 1. \tag{6.3}$$

Finally by solving linear system of three equation, (6.1), (6.2) and (6.3) we get

$$c_0 = \frac{3}{2},\ c_1 = \frac{1}{4},\ c_2 = 0.$$

Thus

$$y(x) = \left(\frac{3}{2}, \frac{1}{2}, 0\right) \begin{pmatrix} 1 \\ 4x - 2 \\ 16x^2 - 16x + 3 \end{pmatrix} = 1 + x,$$

which is the exact solution.

7. Conclusion

In this paper, we introduced fractional-order Chebyshev polynomials of second kind, next we obtained a operational matrix of fractional derivative for these orthogonal polynomials. This matrix can be used to solve the fractional differential equations. The method yields good result. Illustrative examples are included to demonstrate the applicability of the technique.

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