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Extremal Positive Solutions For The Distributed Order Fractional Hybrid Differential Equations

Hossein Noroozi
Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P. O. Box 115, Shahrekord, Iran.
E-mail: hono1458@yahoo.com

Alireza Ansari
Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P. O. Box 115, Shahrekord, Iran.
E-mail: alireza_1038@yahoo.com

Abstract In this article, we prove the existence of extremal positive solution for the distributed order fractional hybrid differential equation

$$\int_0^1 b(q)D^q[\frac{x(t)}{f(t,x(t))}]dq = g(t,x(t)),$$

using a fixed point theorem in the Banach algebras. This proof is given in two cases of the continuous and discontinuous function g, under the generalized Lipschitz and Caratheodory conditions.

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1. Introduction

In recent years, the hybrid differential equations have attracted much attention to many researchers [7, 9, 10, 12]. For example, Dhage and Lakshmikantham established the existence and uniqueness results for the first order hybrid differential equation [11]

$$\begin{cases} \frac{d}{dt}[\frac{x(t)}{f(t,x(t))}] = g(t,x(t)), \quad t \in J, \\ \qquad x(0) = 0, \end{cases} \tag{1.1}$$

where $J = [0,T]$ is bounded in $\mathbb{R}$ for some $T \in \mathbb{R}$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R})$. Later, Zhao et al. developed the following fractional hybrid

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differential equations involving the Riemann-Liouville differential operators of order $0 < q < 1$ and found the existence and uniqueness results for these type equations [15, 17]

$$\begin{cases} D^q[\frac{x(t)}{f(t,x(t))}] = g(t, x(t)), \quad t \in J, \\ \qquad\qquad x(0) = 0. \end{cases} \tag{1.2}$$

Now, in view of the idea of fractional derivative of distributed order stated by Caputo [3, 4], we develop a new class of distributed order fractional hybrid differential equations (DOFHDEs) with respect to a nonnegative density function. For this purpose, we prove the existence of maximal and minimal positive solutions for the DOFHDEs between the given upper and lower solutions on $J = [0, T]$. We use two fixed point theorems in order Banach spaces for establishing our results under two cases of the Caratheodory and discontinuous function g.

In this regard, in Section 2, we recall some basic definition, theorem and fixed point theorem in the Banach algebras. In section 3, we introduce the distributed order fractional hybrid differential equation and their properties and in Section 4, we prove the existence of extremal solutions for DOFHDEs in the Caratheodory case. In Section 5, we prove the existence of extremal solutions for DOFHDEs in discontinuous case.

2. Elementary Definitions and Theorems in The Banach Algebras

In this section, we consider the main definitions and theorems. Also, we recall two important fixed point theorems in Banach algebra which can be used to prove the existence theorems. First, we introduce a cone in Banach Algebras in what follows.

Definition 2.1. A non-empty closed set K in a Banach Algebra X is called a cone with vertex 0, if the following statements hold:

- **(i):** $K + K \subseteq K$,
- **(ii):** $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda \geq 0$,
- **(iii):** $(-K) \bigcap K = 0$, where 0 is the zero element of X,
- **(iv):** a cone K is called to be positive if $K \circ K \subseteq K$, where $\circ$ is a multiplication composition in X.

Definition 2.2. A cone K is said to be normal if the norm $\|.\|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded.

The details of cones and their properties can be founded in Heikkila and Lakshmikantham [13].

Lemma 2.3. *([7]). Let K be a positive cone in a real Banach algebra X and let u_1, u_2, v_1, $v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1 u_2 \leq v_1 v_2$.*

Definition 2.4. A mapping $Q : [a, b] \to X$ is said to be nondecreasing or monotone increasing if $Qx \leq Qy$ for all $x, y \in [a, b]$ with $x \leq y$ where the interval $[a, b]$ is given by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

Theorem 2.5. *([8]). Let K be a cone in a Banach algebra X and let $a, b \in X$ such that $a \leq b$. Also, suppose that $A, B : [a, b] \to K$ are two nondecreasing operators such that*

- **(a):** *A is Lipschitzian with a Lipschitz constant α,*
- **(b):** *B is completely continuous,*
- **(c):** *$AxBx \in [a, b]$ for each $x \in [a, b]$.*

Further, if cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solution in $[a, b]$, whenever $\alpha M_1 < 1$, where $M_1 = \|B([a, b])\| = \sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 2.6. *([7]). Let K be a cone in a Banach algebra X and let $a, b \in X$ such that $a \leq b$. Also, suppose that $A, B : [a, b] \to K$ are two nondecreasing operators such that*

- **(a):** *A is completely continuous,*
- **(b):** *B is totally bounded,*
- **(c):** *$AxBy \in [a, b]$ for each $x, y \in [a, b]$.*

Further, if cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solution in $[a, b]$.

3. The Fractional Hybrid Differential Equation of Distributed Order

Definition 3.1. The distributed order fractional hybrid differential equation (DOFHDEs), involving the Riemann-Liouville differential operator of order $0 < q < 1$ with respect to the nonnegative density function $b(q) > 0$, is defined as

$$\begin{cases} \int_0^1 b(q) D^q [\frac{x(t)}{f(t,x(t))}] dq = g(t, x(t)), \quad t \in J, \quad \int_0^1 b(q) dq = 1, \\ \qquad\qquad x(0) = 0. \end{cases} \tag{3.1}$$

Moreover, the function $t \mapsto \frac{x}{f(t,x)}$ is continuous for each $x \in \mathbb{R}$, where $J = [0, T]$ is bounded in $\mathbb{R}$ for some $T \in \mathbb{R}$. Also, $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : J \times \mathbb{R} \to \mathbb{R}$.

Remark 3.2. Suppose that

$$b(q) = a_0 \delta(q - q_0) + a_1 \delta(q - q_1) + a_2 \delta(q - q_2) + ... + a_n \delta(q - q_n), \tag{3.2}$$

which $1 > q_n > q_{n-1} > ... > q_0 > 0$ and a_i for $i = 0, 1, 2, ..., n$ is nonnegative constant coefficients and δ is the Dirac delta function. For this case, the DOFHDE (3.1) is

$$a_0 D^{q_0}[\frac{x(t)}{f(t,x(t))}] + a_1 D^{q_1}[\frac{x(t)}{f(t,x(t))}] + ... + a_n D^{q_n}[\frac{x(t)}{f(t,x(t))}] = g(t,x(t)),$$
$$x(0) = 0,$$

where $t \in J$.

We apply the following lemma from [14] to prove the main existence extremal solution theorem for the DOFHDE (3.1).

Lemma 3.3. *Assume that the function $x \to \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ for $t \in J$, then for any $h \in L^1(J,\mathbb{R})$ and $0 < q < 1$, the function $x \in C(J,\mathbb{R})$ is a solution of the DOFHDE (3.1) if and only if x satisfies the following equation*

$$x(t) = \frac{f(t,x(t))}{\pi} \int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\} g(\tau, x(\tau)) d\tau, \tag{3.3}$$

such that $0 \le \tau \le t \le T$ and

$$B(s) = \int_0^1 b(q) s^q dq. \tag{3.4}$$

4. Existence of Extremal Solution in The Caratheodory Case

In this section, for the continuous function g on $J \times \mathbb{R}$, we prove the existence of extremal solutions for the DOFHDE (3.1). We need the following definitions in what follows.

Definition 4.1. A mapping $g : J \times \mathbb{R} \to \mathbb{R}$ is said to be Caratheodory if

- **(i):** the map $t \to g(t,x)$ is measurable for each $x \in \mathbb{R}$,
- **(ii):** the map $x \to g(t,x)$ is continuous almost everywhere for $t \in J$.

Definition 4.2. A Caratheodory function $g(t,x)$ is called L^1-Caratheodory if for each number $r > 0$ and $x \in \mathbb{R}$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$\mid g(t,x) \mid \le h_r(t), \quad t \in J, \quad |x| \le r.$$

Also, a Caratheodory function $g(t,x)$ is called L^1_X-Caratheodory if for each $x \in \mathbb{R}$ there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\mid g(t,x) \mid \le h(t), \quad t \in J.$$

Definition 4.3. A function $a \in C(J, \mathbb{R})$ is called a lower solution of the DOFHDE (3.1) defined on J if

$$\int_0^1 b(q)D^q[\frac{a(t)}{f(t,a(t))}]dq \le g(t,a(t)), \quad t \in J, \quad \int_0^1 b(q)dq = 1, \tag{4.1}$$

Similarly, a function $a \in C(J, \mathbb{R})$ is called a upper solution of the DOFHDE (3.1) defined on J if

$$\int_0^1 b(q)D^q[\frac{a(t)}{f(t,a(t))}]dq \ge g(t,a(t)), \quad t \in J, \quad \int_0^1 b(q)dq = 1. \tag{4.2}$$

We consider the certain monotonicity conditions in what follows:

(A_1): $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$, $g : J \times \mathbb{R} \to \mathbb{R}^+$.

(A_2): There exists a constant $L > 0$ such that

$$| f(t,x) - f(t,y) | \le L | x - y |,$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(A_3): The DOFHDE (3.1) has a lower solution a and an upper solution b defined on J with $a \le b$.

(A_4): The function $x \to \frac{x}{f(t,x)}$ is increasing in the interval $[\min_{t\in J} a(t), \max_{t\in J} b(t)]$ almost everywhere for $t \in J$.

(A_5): The functions $f(t,x)$ and $g(t,x)$ are nondecreasing in x almost everywhere for $t \in J$.

(A_6): There exists a function $k \in L^1(J, \mathbb{R})$ such that $g(t,b(t)) \le k(t) \le k^*$ where a real positive number k^* is upper bound of the function $k(t)$ for $t \in J$.

We consider that hypotheses (A_6) holds in particular if g is L^1-Caratheodory and f is continuous on $J \times \mathbb{R}$. Our main existence theorem for extremal solutions of the DOFHDE (3.1) in this section is given by the following theorem.

Theorem 4.4. *Assume that hypotheses (A_1)-(A_6) hold. Moreover, if*

$$\frac{LM}{\pi}\|k\|_{L^1} < 1, \quad M > 0, \tag{4.3}$$

then the DOFHDE (3.1) has a minimal and a maximal positive solution defined on $J = [0,T]$.

Proof: We set $X = C(J, \mathbb{R})$ as a Banach algebra. By Lemma 3.3, the DOFHDE (3.1) is equivalent to equation

$$x(t) = \frac{f(t,x(t))}{\pi} \int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\} g(\tau, x(\tau))d\tau \tag{4.4}$$

Define operators $A : X \longrightarrow X$ and $B : X \longrightarrow X$ by

$$Ax(t) = f(t,x(t)), \quad t \in J, \tag{4.5}$$

and

$$Bx(t)=\frac{1}{\pi}\int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}g(\tau,x(\tau))d\tau,\quad t\in J, \tag{4.6}$$

thus, from the equation (3.3), we obtain the operator equation as follows:

$$Ax(t)Bx(t)=x(t),\quad t\in J. \tag{4.7}$$

We equip the space $C(J,\mathbb{R})$ by cone K given by

$$K=\{x\in C(J,\mathbb{R})\ :\ x(t)\geq 0,\forall t\in J\}, \tag{4.8}$$

when the cone K is positive and normal in $C(J,\mathbb{R})$. Then the interval $[a,b]$ is a norm-bounded set in X. Also, by hypotheses (A_1) we have $A,B:[a,b]\to K$. If operators A and B satisfy all the conditions of Theorem 2.5, then the operator equation (4.7) has a solution in S. To see this, let $x,y\in X$ which by hypothesis (A_1) we have

$$|Ax(t)-Ay(t)|=|f(t,x(t))-f(t,y(t))|\leq L|x(t)-y(t)|\leq L\|x-y\|, t\in J,$$

and if for all $x,y\in X$ take a supremum over t, then we have

$$\|Ax-Ay\|\leq L\|x-y\|. \tag{4.9}$$

Therefore, A is a Lipschitz operator on X with the Lipschitz constant $L>0$, and the condition (a) from Theorem 2.5 is held. Now, for checking the condition (b) from this theorem, let $\{x_n\}$ be a sequence in S such that

$$\lim_{n\to\infty}x_n=x, \tag{4.10}$$

with $x\in S$. Now,

$$\begin{aligned}\lim_{n\to\infty}Bx_n(t) &= \lim_{n\to\infty}\frac{1}{\pi}\int_0^t\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}(g(\tau,x_n(\tau))+\epsilon)d\tau\\ &= \frac{1}{\pi}\int_0^t\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}\lim_{n\to\infty}(g(\tau,x_n(\tau))+\epsilon)d\tau\\ &= \frac{1}{\pi}\int_0^t\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}(g(\tau,x(\tau))+\epsilon)d\tau\\ &= Bx(t).\end{aligned}$$

This shows that B is pointwise continuous on J. It can be shown as in the following part that the sequence $\{Bx_n\}$ is an equicontinuous set in $C(J,\mathbb{R})$. So the convergence $Bx_n\to Bx$ is uniform. As a result, B is continuous on $C(J,\mathbb{R})$. In next stage, we shall show that B is a compact operator on S. To see this we shall show that $B(S)$ is a uniformly bounded and eqicontinuous set in X. Let $x \in S$, then by hypothesis (A_2) for all $t \in J$ we have

$$\begin{aligned} |Bx(t)| &= \left| \frac{1}{\pi} \int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\} g(\tau, x(\tau)) d\tau \right| \\ &\leq \frac{1}{\pi} \int_0^t \left| \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\} \right| |k(\tau)| \, d\tau. \end{aligned} \tag{4.11}$$

If we set $s = t - \tau$ such that $0 \leq \tau \leq t \leq T$, then by the existence Laplace transform theorem [6], there exist a constant $M' > 0$ such that for a constant c that $s > c$,

$$\left| \Im\{\frac{1}{B(re^{-i\pi})}\} \right| \leq M' e^{cr}. \tag{4.12}$$

Hence, we find an upper bound for the integral of (4.11)

$$\begin{aligned} \left| \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\} \right| &= \left| \int_0^\infty e^{-sr} \Im\{\frac{1}{B(re^{-i\pi})}\} dr \right| \\ &\leq \int_0^\infty e^{-sr} \left| \Im\{\frac{1}{B(re^{-i\pi})}\} \right| dr \\ &\leq \int_0^\infty M' e^{(c-s)r} dr \leq \frac{M'}{|s-c|} \leq M, \end{aligned} \tag{4.13}$$

such that

$$M = \sup_{0 \leq \tau \leq t \leq T} \frac{M'}{|t - \tau - c|}. \tag{4.14}$$

Finally, with respect to the inequality (4.11) we obtain

$$|Bx(t)| \leq \frac{M \, \|k\|_{L^1}}{\pi},$$

which by applying supremum over t, we get for all $x \in S$

$$\|Bx\| \leq \frac{M}{\pi} \|k\|_{L^1}. \tag{4.15}$$

Thus, B is uniformly bounded on S. In this stage, we show that $B(S)$ is an equicontinuous set in X. Let $t_1, t_2 \in J$, with $t_1 < t_2$. In this sense we have for all $x \in S$

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= |\frac{1}{\pi}\int_0^{t_1} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_1 - \tau\} g(\tau, x(\tau)) d\tau \\
&- \frac{1}{\pi}\int_0^{t_2} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_2 - \tau\} g(\tau, x(\tau)) d\tau| \\
&\leq |\frac{1}{\pi}\int_0^{t_1} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_1 - \tau\} g(\tau, x(\tau)) d\tau \\
&- \frac{1}{\pi}\int_0^{t_1} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_2 - \tau\} g(\tau, x(\tau)) d\tau| \\
&+ |\frac{1}{\pi}\int_0^{t_1} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_2 - \tau\} g(\tau, x(\tau)) d\tau \\
&- \frac{1}{\pi}\int_0^{t_2} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_2 - \tau\} g(\tau, x(\tau)) d\tau|.
\end{aligned} \tag{4.16}$$

If we set $s_1 = t_1 - \tau$ and $s_2 = t_2 - \tau$, then by Laplace transform definition and equation (4-19), for $s_1 > c$ and $s_2 > c$ we can write

$$\begin{aligned}
&\left|\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; s_1\} - \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; s_2\}\right| \\
&= \left|\int_0^{\infty} e^{-s_1 r}\Im\{\frac{1}{B(re^{-i\pi})}\})dr - \int_0^{\infty} e^{-s_2 r}\Im\{\frac{1}{B(re^{-i\pi})}\}dr\right| \\
&\leq \int_0^{\infty} \left|e^{-s_1 r} - e^{-s_2 r}\right| \left|\Im\{\frac{1}{B(re^{-i\pi})}\}\right| dr \\
&\leq M' \int_0^{\infty} (e^{(c-s_1)r} - e^{(c-s_2)r}) dr = M'(\frac{1}{s_1 - c} - \frac{1}{s_2 - c}).
\end{aligned} \tag{4.17}$$

Therefore, we have

$$\begin{aligned}
&\left|\frac{1}{\pi}\int_0^{t_1} (\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; s_1\} - \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; s_2\}) g(\tau, x(\tau)) d\tau\right| \\
&\leq \frac{k^*}{\pi}\int_0^{t_1} M'(\frac{1}{t_1 - \tau - c} - \frac{1}{t_2 - \tau - c}) d\tau \\
&= \frac{M'k^*}{\pi}\ln(\frac{(c + t_1 - t_2)(c - t_1)}{c(c - t_2)}).
\end{aligned} \tag{4.18}$$

Also, by equation (4.13) we have

$$\left|\frac{1}{\pi}\int_{t_2}^{t_1}\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t_2-\tau\}g(\tau,x(\tau))d\tau\right|\leq\frac{k^*}{\pi}\int_{t_2}^{t_1}\frac{M'}{t_2-\tau-c}d\tau$$
$$=\frac{M'k^*}{\pi}\ln(\frac{c}{c+t_1-t_2}).\quad(4.19)$$

Finally, with respect to (4.16), (4.18) and (4.19) we obtain

$$\begin{aligned}|Bx(t_1)-Bx(t_2)| &\leq \frac{M'k^*}{\pi}(\ln(\frac{(c+t_1-t_2)(c-t_1)}{c(c-t_2)})+\ln(\frac{c}{c+t_1-t_2}))\\ &= \frac{M'k^*}{\pi}\ln(\frac{c-t_1}{c-t_2}).\end{aligned}\quad(4.20)$$

Hence, for $\epsilon>0$, there exists $\delta>0$ such that if $|t_1-t_2|<\delta$, then for all $t_1,t_2\in J$ and all $x\in S$ we have

$$|Bx(t_1)-Bx(t_2)|<\epsilon,\quad(4.21)$$

which implies that $B(S)$ is an equicontinuous set in X and according to the Arzela-Ascoli theorem, B is compact. Therefore B is continuous and compact operator on S into X and B is a completely continuous operator on S and the condition (b) from the Theorem 2.5 is held. Also, hypotheses (A_5) implies that the operators A and B are nondecreasing on $[a,b]$. To see this, we consider $x,y\in[a,b]$ with $x\leq y$. Therefore by hypotheses (A_5), for all $t\in J$ we get

$$Ax(t)=f(t,x(t))\leq f(t,y(t))=Ay(t),$$

similarly, we obtain

$$\begin{aligned}Bx(t)&=\frac{1}{\pi}\int_0^t\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}g(\tau,x(\tau))d\tau\\ &=\leq\frac{1}{\pi}\int_0^t\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}g(\tau,y(\tau))d\tau\\ &=By(t),\end{aligned}$$

thus, the operator A and B are nondecreasing on $[a,b]$. By Lemma 2.3 and in view of hypothesis (A_5) we have

$$\begin{aligned}a(t)&\leq\frac{f(t,a(t))}{\pi}\int_0^t\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}g(\tau,x(\tau))d\tau\\ &\leq\frac{f(t,x(t))}{\pi}\int_0^t\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}g(\tau,x(\tau))d\tau\\ &\leq\frac{f(t,b(t))}{\pi}\int_0^t\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\}g(\tau,x(\tau))d\tau\\ &\leq b(t).\end{aligned}$$

Then, for all $t \in J$ and $x \in [a,b]$ we obtain $a(t) \leq Ax(t)Bx(t) \leq b(t)$. Therefore, for all $x \in [a,b]$ we have $AxBx \in [a,b]$. Also, we notice that the bound (4.13) and hypotheses A_6 yields that

$$\begin{aligned} M_1 = \|B([a,b])\| &= \sup\{\|Bx\| \,:\, x \in [a,b]\} \\ &= \sup\{\sup_{t\in J} |Bx(t)| \,:\, x \in [a,b]\} \\ &\leq \sup_{t\in J}\{\frac{1}{\pi}\int_0^t \left|\mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\}\right| |k(\tau)|\, d\tau\} \\ &\leq \frac{M}{\pi}\, \|k\|_{L^1}, \end{aligned}$$

such that the condition (4.3) implies that

$$\alpha M_1 \leq \frac{LM}{\pi}\, \|k\|_{L^1} < 1.$$

Finally, all the conditions of Theorem 2.5 are satisfied and hence the operator equation $AxBx = x$ has a least and a greatest positive solution in $[a,b]$. As a result, the DOFHDE (3.1) has a minimal and a maximal positive solution in $[a,b]$ defined on J and the proof is completed.

5. Existence of Extremal Solution in Discontinuous Case

In this section, for the discontinuous function g on $J \times \mathbb{R}$, we prove the existence of extremal solutions for DOFHDE (3.1). We need the following definitions in what follows.

Definition 5.1. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be Chandrabhan if

- **(i):** $t \to g(t,x)$ is measurable for each $x \in \mathbb{R}$,
- **(ii):** $x \to g(t,x)$ is nondecreasing almost everywhere for $t \in J$.

Definition 5.2. A Chandrabhan function $g(t,x)$ is called L^1-Chandrabhan if for each number $r > 0$, there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$|g(t,x)| \leq h_r(t), \quad t \in J,$$

with $|x| \leq r$ for all $x \in \mathbb{R}$.
Also, a Chandrabhan function $g(t,x)$ is called L^1_X-Chandrabhan if there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|g(t,x)| \leq h(t), \quad t \in J,$$

for all $x \in \mathbb{R}$.

We consider the following hypotheses in the sequel:

- **(B_1):** The function g is Chandrabhan.
- **(B_2):** The function f is continuous on $J \times \mathbb{R}$.

Now, we prove the existence theorem for extremal solutions of the DOFHDE (3.1) in this section.

Theorem 5.3. *Assume that the hypotheses* (A_1)*-*(A_6) *and* (B_1)*-*(B_2) *hold. Then DOFHDE* (3.1) *has a minimal and a maximal positive solution on* J*.*

Proof: According to Lemma 3.3, the DOFHDE (3.1) is equivalent to equation (3.3). We set $X = C(J, \mathbb{R})$ as a Banach algebra and define an order relation $\leq$ by the cone given by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0, \forall t \in J\}.$$

Define operators $A : X \longrightarrow X$ and $B : X \longrightarrow X$ by (4.5) and (4.6) respectively. Thus, from the equation (3.3), we obtain the operator equation as follows:

$$Ax(t)Bx(t) = x(t), \quad t \in J. \tag{5.1}$$

The cone K is positive and normal in $C(J, \mathbb{R})$. Then the interval $[a, b]$ is a norm-bounded set in Banach algebra X and there exists a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in [a, b]$. Therefore by hypotheses (B_2), f is continuous on compact $J \times [-r, r]$ and it has a maximum. Also, by hypothesis (A_1) we have $A, B : [a, b] \to K$. If operators A and B satisfy all the conditions of Theorem 2.6, then the operator equation (5.1) has a solution in S. To see this, first we show that A is completely continuous on $[a, b]$. For any subset S of $[a, b]$ we get

$$\begin{aligned}
\|A(S)\|_{\mathcal{P}} &= \sup\{\|Ax\| : x \in S\} \\
&= \sup\{\sup_{t \in J} |f(t, x(t))| : x \in S\} \\
&\leq \sup\{\sup_{t \in J} |f(t, x)| : x \in [-r, r]\} \\
&\leq M,
\end{aligned}$$

such that $M = \max_{x \in [-r,r]} |f(t, x)|$ for all $t \in J$. Thus, $A(S)$ is a uniformly bounded subset on X.
Since, the function $f(t, x)$ is continuous on compact $J \times [-r, r]$, hence it is uniformly continuous on $J \times [-r, r]$. Thus, for any $t_1, t_2 \in [0, T]$ as $t_1 \to t_2$, we obtain

$$|f(t_1, x) - f(t_2, x)| \to 0, \qquad x \in [-r, r].$$

Also, for any $x, y \in [-r, r]$ as $x \to y$, we get

$$|f(t_1, x) - f(t_1, y)| \to 0, \qquad t \in [0, T].$$

Therefore, $t_1 \to t_2$ yields that

$$\begin{aligned}
|Ax(t_1) - Ax(t_2)| &= |f(t_1, x(t_1)) - f(t_2, x(t_2))| \\
&\leq |f(t_1, x(t_1)) - f(t_2, x(t_1))| + |f(t_2, x(t_1)) - f(t_2, x(t_2))| \\
&\to 0.
\end{aligned}$$

Then, $A(S)$ is an equi-continuous set in X and by the Arzela-Ascoli theorem, A is a completely continuous operator on $[a, b]$. Now, we show that the operator B is totally bounded on $[a, b]$. To see this, for any subset S of $[a, b]$ we show that $B(S)$ is uniformly bounded and equi-continuous set in X. Suppose that $y \in B(S)$. Therefore, for some $x \in S$ we have

$$y(t) = \frac{1}{\pi} \int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t - \tau\} g(\tau, x(\tau)) d\tau, \quad t \in J.$$

By hypothesis (A_6) and the bound (4.13) we obtain

$$\begin{aligned}
|y(t)| &= \left| \frac{1}{\pi} \int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t - \tau\} g(\tau, x(\tau)) d\tau \right| \\
&\leq \frac{1}{\pi} \int_0^t \left| \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t - \tau\} \right| |k(\tau)| \, d\tau \\
&\leq \frac{M\|k\|_{L^1}}{\pi},
\end{aligned}$$

and taking a supremum over t yields that

$$\|y\| \leq \frac{M\|k\|_{L^1}}{\pi}. \tag{5.2}$$

which shows that $B(S)$ is a uniformly bounded set in X. Thus by similar way to the proof of Theorem 4.4 and using equations (4.16), (4.17), (4.18), (4.19) and hypothesis (A_6), we have for $t_1, t_2 \in J$

$$\begin{aligned}
|y(t_1) - y(t_2)| &= |\frac{1}{\pi} \int_0^{t_1} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_1 - \tau\} g(\tau, x(\tau)) d\tau \\
&- \frac{1}{\pi} \int_0^{t_2} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_2 - \tau\} g(\tau, x(\tau)) d\tau| \\
&\leq \frac{M'k^*}{\pi} \ln(\frac{c - t_1}{c - t_2}).
\end{aligned} \tag{5.3}$$

Hence, for $\epsilon > 0$, there exists $\delta > 0$ such that if $|t_1 - t_2| < \delta$, then for all $t_1, t_2 \in J$ and all $y \in B(S)$ we have

$$|y(t_1) - y(t_2)| < \epsilon, \tag{5.4}$$

which implies that $B(S)$ is an equicontinuous set in X and according to the Arzela-Ascoli theorem, B is totally bounded. Also, similar to proof of Theorem

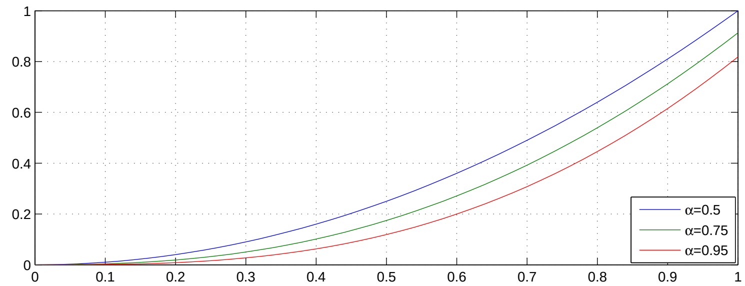

FIGURE 1. The Solution of FDE (6.1) for $\alpha = 0.5, 0.75, 0.95$.

4.4 it is easy to show that $AxBy \in [a,b]$ for each $x,y \in [a,b]$. Finally, all the conditions of Theorem 2.6 are satisfied and hence the operator equation $AxBx = x$ has a least and a greatest positive solution in $[a,b]$. As a result, the DOFHDE (3.1) has a minimal and a maximal positive solution in $[a,b]$ defined on J and the proof is completed.

6. Numerical Examples

In this section, as showing the constructive theorems and lemmas in previous sections, we state the following examples.

Example 6.1. *We consider the following initial value problem in the fractional Riemann-Liouville derivative on* $[0,1]$

$$D_t^{\alpha} x^{\frac{1}{2}}(t) + x^{\frac{1}{2}}(t) = t + \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}, \quad x(0) = 0, \tag{6.1}$$

which we set $f(t,x(t)) = x^{\frac{1}{2}}(t)$ *and* $g(t,x(t)) = t + \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}$*. Now, we consider the approximated solution of* (6.1) *by the shifted Legendre polynomials* $x(t) = \sum_{i=0}^{n} c_i P_i(x)$ *and use the following relation for the fractional derivative of* $x(t)$

$$D_t^{\alpha} x(t) = \sum_{i=\lceil \alpha \rceil}^{n} \sum_{k=\lceil \alpha \rceil}^{i} c_i b_{i,k}^{(\alpha)} x^{k-\alpha}, \tag{6.2}$$

$$b_{i,k}^{(\alpha)} = \frac{(-1)^{k+i}(i+k)!}{k!(i-k)!\Gamma(k+1-\alpha)}, \tag{6.3}$$

where $\lceil \alpha \rceil$ *is the largest integer less than or equal to* α*. If we collocate the equation* (6.1) *at* n *points on* $[0,1]$*, then we can get the solution with respect to* n *unknown coefficients* $c_i, i = 1, 2, \cdots, n$*. We show this for* $n = 10$ *in Figure 1 for different values of* α*. For more details of this method see for example* [1, 16].

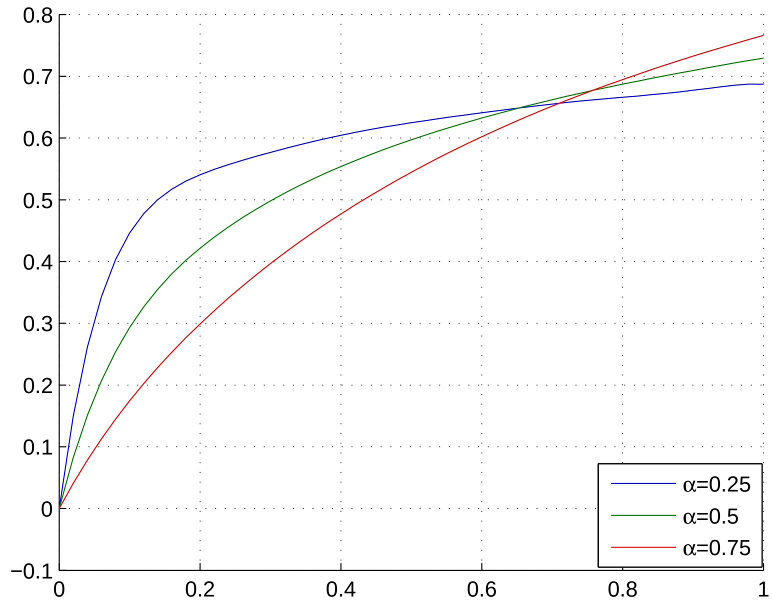

FIGURE 2. The Solution of FDE (6.4) for $\alpha = 0.25, 0.5, 0.75$.

Example 6.2. *For following initial value problem in the fractional Riemann-Liouville derivative on* $[0, 1]$

$$D_t^{\alpha} x(t) = \cos^2 x(t), \quad x(0) = 0, \tag{6.4}$$

we apply the the Legendre collocation method expressed in Example 6.1 for $n = 5$. *The approximated solution has been shown in Figure 2.*

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