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Numerical solution of nonlinear Fredholm-Volterra integral equations via Bell polynomials

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Abstract In this paper, we propose and analyze an efficient matrix method based on Bell polynomials for solving nonlinear Fredholm-Volterra integral equations, numerically. For this aim, first we calculate operational matrix of integration and product based on Bell polynomials. By using these matrices, nonlinear Fredholm-Volterra integral equations reduce to the system of nonlinear algebraic equations which can be solved by an appropriate numerical method such as Newton's method. Also, we show that the proposed method is convergent. Some examples are provided to illustrate the applicability, efficiency and accuracy of suggested scheme. Comparison of the proposed method with other previous methods shows that this method is very accurate.

Keywords. Fredholm-Volterra integral equation; Bell polynomials; Collocation method; Operational matrix; Error analysis.

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1. Introduction

Integral equations are applied in modelling many phenomena which arise in different science such as mathematical finance, biology, medical, social sciences, etc. For example, the Volterra-Fredholm integral equations occurs from parabolic boundary value problems, from the mathematical modelling of the spatial-temporal development of an epidemic, and from various physical and biological models. Since, in many situation, such equations cannot be solved exactly, it is important to obtain their approximate solutions by using some numerical methods. In this paper, we consider the following nonlinear Fredholm-Volterra integral equation

$$\begin{aligned} f(x) = g(x) + \lambda_1 \int_0^1 k_1(x,y)\mathcal{N}_1(y,f(y))dy \\ + \lambda_2 \int_0^x k_2(x,y)\mathcal{N}_2(y,f(y))dy, \qquad x \in [0,1], \end{aligned} \tag{1.1}$$

where, λ_1 and λ_2 are arbitrary real constant, $g(x), k_1(x,y), k_2(x,y), \mathcal{N}_1(x,f(x))$ and $\mathcal{N}_2(x,f(x))$ are known functions whereas $f(x)$ is an unknown function which should be determined. $\mathcal{N}_1(x,f(x))$ and $\mathcal{N}_2(x,f(x))$ are nonlinear terms. First integral in Eq.

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(1.1) is Fredholm integral term and second integral in Eq. (1.1) is Volterra integral term. Also, We assume that Eq. (1.1) have a unique solution $f(x)$ [11, 7]. Several numerical methods have been applied for solving nonlinear Fredholm-Volterra integral equations. For example, collocation methods [4], Taylor polynomial methods [20], Homotopy perturbation method [10], triangular functions methods [14], rationalized Haar functions methods [17], wavelets methods [21] and many other methods.
In this paper, we suggest an efficient method based on Bell polynomials for solving nonlinear Fredholm-Volterra integral equation (1.1).

2. Bell polynomials

The Bell polynomials were studied extensively by E. T. Bell in 1934 [2]. These polynomials naturally occur from differentiating a composite function several times. Bell polynomials have many application in number theory and classical analysis and there are a vast literature about their applications [5, 6, 19]. They are frequently applied in combinatorial analysis [18] and statistics [12]. Also, these polynomials have been used in many other contexts such as the Blissard problem [18], the representation of Lucas polynomials of the first and second kinds [8], the recurrence relations for a class of Freud-type polynomials [3], the representation of symmetric functions of a countable set of numbers, generalising the classical algebraic Newton-Girard formulas [13].
In the following of this section, we mention some properties of the Bell polynomials which will be used in the next section.
Property 1(Differentiation property [2, 6, 18])

$$\frac{d}{dx}B_n(x) = \frac{B_{n+1}(x)}{x} - B_n(x), \qquad n = 1, 2, \dots.$$

Property 2(Recurrence equation [2, 6, 18])

$$xB_{n+1}(x) = x[B'_n(x) + B_n(x)], \qquad n = 1, 2, \dots.$$

Property 3(A series representation in term of Stirling number of the second kind [2, 6, 18]). The Bell polynomials can be computed as

$$B_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$

where $S(n, k)$ for $k = 0, 1, \dots, n$ are the Stirling numbers of the second kind which can be calculated as follows

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

By using third property of the Bell polynomials, we obtain

$$\mathbf{B}(x) = \mathbf{S}\mathbf{X}(x), \tag{2.1}$$

where

$$\mathbf{B}(x) = [B_0(x), B_1(x), \dots, B_N(x)]^T, \qquad \mathbf{X}(x) = [1, x, \dots, x^N]^T, \tag{2.2}$$

$$\mathbf{S} = \begin{pmatrix} S(0,0) & 0 & \cdots & 0 \\ S(1,0) & S(1,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S(N,0) & S(N,1) & \cdots & S(N,N) \end{pmatrix}. \tag{2.3}$$

Since matrix $\mathbf{S}$ is a lower triangular matrix with nonzero diagonal elements, so this matrix is nonsingular and hence $\mathbf{S}^{-1}$ exists. So, from Eq. (2.1), we have

$$\mathbf{X}(x) = \mathbf{S}^{-1}\mathbf{B}(x). \tag{2.4}$$

3. Function approximation

Consider the set of Bell polynomials

$$\mathbf{B}(x) = [B_1(x), B_2(x), \dots, B_N(x)]^T,$$

and suppose that

$$\mathcal{S} = span\{B_1(x), B_2(x), \dots, B_N(x)\},$$

also, suppose that f be an arbitrary function in $\mathcal{H} = L^2[0,1]$. Because $\mathcal{S}$ is a finite dimensional vector space, so f has the best approximation out of $\mathcal{S}$ such as $f_N \in \mathcal{S}$, that is

$$\forall g \in \mathcal{S} \qquad \|f - f_N\| \le \|f - g\|.$$

Since $f_N \in \mathcal{S}$, there exist unique coefficients $f_1, f_2, \dots, f_N$, such that

$$f(x) \simeq f_N(x) = \sum_{i=0}^{N} f_i B_i(x) = F^T \mathbf{B}(x), \tag{3.1}$$

where $\mathbf{B}(x)$ is the Bell vector defined in Eq. (2.2) and F is the Bell coefficients vector defined as

$$F = [f_0, f_1, \dots, f_N]^T.$$

For computing the coefficients f_i, we let

$$c_j = \int_0^1 f(x) B_j(x) dx, \qquad j = 0, 1, \dots, N.$$

So, by using Eq. (3.1), we have

$$\begin{aligned} c_j &= \int_0^1 \sum_{i=0}^{N} f_i B_i(x) B_j(x) dx = \sum_{i=0}^{N} f_i \int_0^1 B_i(x) B_j(x) dx \\ &= \sum_{i=0}^{N} f_i d_{ij}, \qquad j = 0, 1, \dots, N, \end{aligned} \tag{3.2}$$

where

$$d_{ij} = \int_0^1 B_i(x)B_j(x)dx. \tag{3.3}$$

We let

$$D = [d_{ij}]_{(N+1)\times(N+1)},$$
$$C = [c_0, c_1, \dots, c_N]^T.$$

From Eq. (3.2), we have

$$C^T = F^T D \Longrightarrow F^T = C^T D^{-1}. \tag{3.4}$$

Similarly, we can expand an arbitrary bivariate function in terms of Bell polynomials as follows

$$k(x,y) \simeq k_N(x,y) = \mathbf{B}^T(x)K\mathbf{B}(y) = \mathbf{B}^T(y)K^T\mathbf{B}(x), \tag{3.5}$$

where $K = [k_{ij}]$ is an $(N+1)\times(N+1)$ matrix which k_{ij} can be obtained as follows

$$K = D^{-1}\Big[\int_0^1\int_0^1 k(x,y)\mathbf{B}(x)\mathbf{B}(y)dxdy\Big]D^{-1}.$$

4. Operational matrix

From definition of standard basis, we have

$$\begin{aligned}\int_0^x \mathbf{X}(y)dy &= \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & \frac{1}{2} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{N}\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}}_{M}\begin{pmatrix}1\\ x\\ \vdots\\ x^{N-1}\\ x^N\end{pmatrix} + \underbrace{\begin{pmatrix}0\\0\\ \vdots\\ 0\\ 1\end{pmatrix}}_{I_{N+1}}\frac{x^{N+1}}{N+1}\\ &= M\mathbf{X}(x) + \frac{x^{N+1}}{N+1}I_{N+1}.\end{aligned} \tag{4.1}$$

Now, by omitting the second term in Eq. (4.1), we can approximate the integration of the vector $\mathbf{X}(x)$ as follows

$$\int_0^x \mathbf{X}(y)dy \simeq M\mathbf{X}(x). \tag{4.2}$$

By using Eqs. (2.1), (2.4) and (4.2), the operational matrix of integration based on Bell polynomials obtain as

$$\int_0^x \mathbf{B}(y)dy = \mathbf{S}\int_0^x \mathbf{X}(y)dy = \mathbf{S}M\mathbf{X}(x) = \underbrace{\mathbf{S}M\mathbf{S}^{-1}}_{P}\mathbf{B}(x) = P\mathbf{B}(x). \tag{4.3}$$

The matrix P in Eq. (4.3), is called operational matrix of integration.
The dual matrix of $\mathbf{B}(x)$ is defined by

$$Q = \int_0^1 \mathbf{B}(x)\mathbf{B}^T(x)dx = \int_0^1 \mathbf{S}\mathbf{X}(x)\mathbf{X}^T(x)\mathbf{S}^T = \mathbf{S}H\mathbf{S}^T, \tag{4.4}$$

where $\mathbf{S}$ was defined in Eq. (2.3) and H is the Hilbert matrix of order $N+1$.
It is always necessary to compute the product of $\mathbf{B}(x)$ and $\mathbf{B}^T(x)$ in a arbitrary vector $U=[u_0,u_1,\dots,u_N]^T$, that is called the product matrix of Bell polynomials basis. In the following of this section, we introduce the concept of operational matrix of product. Suppose that U be a column vector and $\mathbf{X}(x)$ be the Bell vector which was defined in Eq. (2.2). The matrix $\hat{U}$ of order $(N+1)\times(N+1)$ which satisfies in the following relation is named the operational matrix of product based on Bell polynomials

$$\mathbf{B}(x)\mathbf{B}^T(x)U \simeq \hat{U}\mathbf{B}(x). \tag{4.5}$$

Now, we try to get an explicit formula for $\hat{U}$. By substituting Eq. (2.1) into Eq. (4.5), we have

$$\begin{aligned}\mathbf{B}(x)\mathbf{B}^T(x)U &= \mathbf{S}\mathbf{X}(x)\mathbf{B}^T(x)U\\ &= \mathbf{S}\Big[\sum_{i=0}^{N}u_iB_i(x),\sum_{i=0}^{N}u_ixB_i(x),\dots,\sum_{i=0}^{N}u_ix^NB_i(x)\Big]^T.\end{aligned} \tag{4.6}$$

For $j=0,1,\dots,N$, we can expand each $x^jB_i(x)$ by applying Bell polynomials as follows

$$x^jB_i(x)\simeq\sum_{r=0}^{N}e_r^{j,i}B_r(x)=e_{j,i}^T\mathbf{B}(x)=\mathbf{B}^T(x)e_{j,i},\qquad i,j=0,1,\dots,N, \tag{4.7}$$

where $e_{j,i}=[e_0^{j,i},e_1^{j,i},\dots,e_N^{j,i}]^T$. From Eq. (4.7), we obtain

$$\begin{aligned}\sum_{i=0}^{N}u_ix^jB_i(x) &= \sum_{i=0}^{N}u_i\sum_{r=0}^{N}e_r^{j,i}B_r(x)=\sum_{r=0}^{N}\sum_{i=0}^{N}u_ie_r^{j,i}B_r(x)\\ &= \sum_{r=0}^{N}B_r(x)\Big(\sum_{i=0}^{N}u_ie_r^{j,i}\Big)=\mathbf{B}^T(x)[e_{j,0},e_{j,1},\dots,e_{j,N}]U\\ &= \mathbf{B}^T(x)E_j,\end{aligned} \tag{4.8}$$

where $E_j=[e_{j,0},e_{j,1},\dots,e_{j,N}]U$. We define matrix E of order $(N+1)\times(N+1)$ as follows

$$E=\big[E_0,E_1,\dots,E_N\big].$$

By using Eqs. (4.6) and (4.8), we conclude

$$\mathbf{B}(x)\mathbf{B}^T(x)U\simeq\mathbf{S}E^T\mathbf{B}(x). \tag{4.9}$$

So, by comparing Eqs. (4.5) and (4.9), we conclude $\hat{U}=\mathbf{S}E^T$.

5. Method of solution

In this section, we consider nonlinear mixed Fredholm-Volterra integral equation (1.1). To solve this equation numerically, let

$$z_1(x)=\mathcal{N}_1\big(x,f(x)\big),\qquad\qquad z_2(x)=\mathcal{N}_2\big(x,f(x)\big). \tag{5.1}$$

So, we have

$$f(x) = g(x) + \lambda_1 \int_0^1 k_1(x,y) z_1(y) dy + \lambda_2 \int_0^x k_2(x,y) z_2(y) dy. \tag{5.2}$$

From Eqs. (5.1) and (5.2), we have

$$\begin{cases} z_1(x) = \mathcal{N}_1\big(x, g(x) + \lambda_1 \int_0^1 k_1(x,y) z_1(y) dy + \lambda_2 \int_0^x k_2(x,y) z_2(y) dy\big), \\ z_2(x) = \mathcal{N}_2\big(x, g(x) + \lambda_1 \int_0^1 k_1(x,y) z_1(y) dy + \lambda_2 \int_0^x k_2(x,y) z_2(y) dy\big). \end{cases} \tag{5.3}$$

Now, we approximate all functions in Eq. (5.3) by using Bell polynomials as follows

$$\begin{aligned} &z_i(x) \simeq Z_i^T \mathbf{B}(x) = \mathbf{B}^T(x) Z_i, \qquad i = 1,2, \\ &k_i(x,y) \simeq \mathbf{B}^T(x) K_i \mathbf{B}(y) = \mathbf{B}^T(y) K_i^T \mathbf{B}(x), \qquad i = 1,2, \\ &g(x) \simeq G^T \mathbf{B}(x) = \mathbf{B}^T(x) G, \end{aligned} \tag{5.4}$$

where $\mathbf{B}(x)$ was defined in Eq. (2.2) and for $i = 1,2$, the vectors Z_i and matrices K_i are Bell polynomials coefficients of $z_i(x)$ and $k_i(x,y)$, respectively.
By substituting Eq. (5.4) into Eq. (5.3) and using Eqs. (4.3), (4.4) and (4.5), we have

$$\begin{cases} Z_1^T \mathbf{B}(x) = \mathcal{N}_1\big(x, G^T \mathbf{B}(x) + \lambda_1 \mathbf{B}^T(x) K_1 Q Z_1 + \lambda_2 \mathbf{B}^T(x) K_2 \hat{Z}_2 \mathbf{B}(x)\big), \\ Z_2^T \mathbf{B}(x) = \mathcal{N}_2\big(x, G^T \mathbf{B}(x) + \lambda_1 \mathbf{B}^T(x) K_1 Q Z_1 + \lambda_2 \mathbf{B}^T(x) K_2 \hat{Z}_2 \mathbf{B}(x)\big). \end{cases} \tag{5.5}$$

By collocating Eq. (5.5) at the following Newton-Cotes points

$$x_i = \frac{2i-1}{2(N+1)}, \qquad i = 1,2,\ldots,N+1, \tag{5.6}$$

the following nonlinear system of $2(N+1)$ algebraic equations and $2(N+1)$ unknowns are concluded

$$\begin{cases} Z_1^T \mathbf{B}(x_i) = \mathcal{N}_1\big(x_i, G^T \mathbf{B}(x_i) + \lambda_1 \mathbf{B}^T(x_i) K_1 Q Z_1 + \lambda_2 \mathbf{B}^T(x_i) K_2 \hat{Z}_2 \mathbf{B}(x_i)\big), \\ Z_2^T \mathbf{B}(x_i) = \mathcal{N}_2\big(x_i, G^T \mathbf{B}(x_i) + \lambda_1 \mathbf{B}^T(x_i) K_1 Q Z_1 + \lambda_2 \mathbf{B}^T(x_i) K_2 \hat{Z}_2 \mathbf{B}(x_i)\big). \end{cases} \tag{5.7}$$

After solving this nonlinear system by a suitable method such as Newton's method, the approximate solution of Eq. (1.1) can be obtained as

$$f(x) = g(x) + \lambda_1 \mathbf{B}^T(x) K_1 Q Z_1 + \lambda_2 \mathbf{B}^T(x) K_2 \hat{Z}_2 \mathbf{B}(x).$$

6. Convergence and error estimation

Theorem 6.1. Suppose that $f(x)$ be a sufficiently smooth function on $[0,1]$ and $P_N(x)$ be the interpolating polynomials of $f(x)$ at points $x_i, i = 0,1,\ldots,N$, which for $i = 0, 1, \dots, N$, the points x_i are the roots of the shifted Chebyshev polynomial of order $N+1$ on the interval $[0, 1]$. Then we have [9]

$$f(x) - P_N(x) = \frac{\partial^{N+1} f(\eta)}{\partial x^{N+1}(N+1)!} \prod_{i=0}^{N} (x - x_i), \tag{6.1}$$

where $\eta \in [0, 1]$. Therefore

$$\big|f(x) - P_N(x)\big| \le \max_{x\in[0,1]} \Big|\frac{\partial^{N+1} f(x)}{\partial x^{N+1}}\Big| \frac{\prod_{i=0}^{N} |x - x_i|}{(N+1)!}. \tag{6.2}$$

Suppose that there is the following upper error bound

$$\max_{x\in[0,1]} \Big|\frac{\partial^{N+1} f(x)}{\partial x^{N+1}}\Big| \le \xi, \tag{6.3}$$

By replacing Eq. (6.3) into Eq. (6.2) and taking into account the estimates for Chebyshev interpolation nodes [15], conclude

$$\big|f(x) - P_N(x)\big| \le \xi \frac{(\frac{1}{2})^{N+1}}{(N+1)!2^N}. \tag{6.4}$$

Theorem 6.2. Let $f_N(x)$ defined in Eq. (3.1), be the best approximation of real sufficiently smooth function $f(x)$ by using Bell polynomials. Then there is real constant ξ such that

$$\|f(x) - f_N(x)\|_2 \le \xi \frac{(\frac{1}{2})^{N+1}}{(N+1)!2^N}. \tag{6.5}$$

Proof. Π_N be the space of polynomials of order N. According to the definition, $f_N(x)$ is the best approximation of $f(x)$ when

$$\forall g(x) \in \Pi_N; \qquad \|f(x) - f_N(x)\|_2 \le \|f(x) - g(x)\|_2. \tag{6.6}$$

In particular, by considering $g(x) = P_N(x)$ and using Eq. (6.4), we have

$$\begin{aligned}\|f(x) - f_N(x)\|_2^2 \le \|f(x) - P_N(x)\|_2^2 &= \int_0^1 |f(x) - P_N(x)|^2 dx \\ &\le \int_0^1 \Big[\xi \frac{(\frac{1}{2})^{N+1}}{(N+1)!2^N}\Big]^2 dx = \Big[\xi \frac{(\frac{1}{2})^{N+1}}{(N+1)!2^N}\Big]^2.\end{aligned} \tag{6.7}$$

From Eq. (6.7), Eq. (6.5) is established. □

Remark 6.1. From Eq. (6.5), we have

$$\|f(x) - f_N(x)\|_2 = \mathcal{O}\Big(\frac{1}{(N+1)!2^{2N+1}}\Big). \tag{6.8}$$

So, if $N \to \infty$ then $\frac{1}{(N+1)!2^{2N+1}} \to 0$, that means $f_N(x) \to f(x)$. Thus the proposed method is convergent.

Theorem 6.3. Suppose that $f(x)$ and $f_N(x)$ be the exact solution and approximate solution of Eq.(5.2), respectively. Also, suppose that the non-linear term satisfies the Lipschitz condition i.e.

$$\|z_i(x) - \hat{z}_{iN}(x)\| \leq L_i \|f(x) - f_N(x)\| + \alpha_i, \qquad i = 1, 2, \tag{6.9}$$

and

$$1 - |\lambda_1| L_1(\gamma_1 + \eta_1) - |\lambda_2| L_2(\gamma_2 + \eta_2) > 0. \tag{6.10}$$

Then, we have the following upper error bound

$$\|f(x) - f_N(x)\| \leq \frac{\theta + |\lambda_1|\Big(\alpha_1(\gamma_1 + \eta_1) + \eta_1\beta_1\Big) + |\lambda_2|\Big(\alpha_2(\gamma_2 + \eta_2) + \eta_2\beta_2\Big)}{1 - |\lambda_1| L_1(\gamma_1 + \eta_1) - |\lambda_2| L_2(\gamma_2 + \eta_2)}, \tag{6.11}$$

where

$$\begin{aligned}
&\max |g(x) - g_N(x)| = \theta,\\
&\max |z_i(x)| = \beta_i, \qquad i = 1, 2,\\
&\max |k_i(x, y)| = \gamma_i, \qquad i = 1, 2,\\
&\max |k_i(x, y) - k_{iN}(x, y)| = \eta_i, \qquad i = 1, 2,\\
&\max |z_{iN}(x) - \hat{z}_{iN}(x)| = \alpha_i, \qquad i = 1, 2.
\end{aligned} \tag{6.12}$$

Proof. The approximate solution of Eq. (5.2) can be written as

$$f_N(x) = g_N(x) + \lambda_1 \int_0^1 k_{1N}(x, y)\hat{z}_{1N}(y)dy + \lambda_2 \int_0^x k_{2N}(x, y)\hat{z}_{2N}(y)dy, \tag{6.13}$$

where $g_N, k_{1N}, k_{2N}, \hat{z}_{1N}$ and $\hat{z}_{2N}$ are the approximate function of $g, k1, k2, \mathcal{N}_1(x, f_N)$ and $\mathcal{N}_2(x, f_N)$ by using Bell polynomials, respectively.
From Eqs. (5.2) and (6.13), we get

$$\begin{aligned}
f(x) - f_N(x) = g(x) - g_N(x) + \lambda_1 \int_0^1 \big(k_1(x, y)z_1(y) - k_{1N}(x, y)\\
\hat{z}_{1N}(y)\big)dy + \lambda_2 \int_0^x \big(k_2(x, y)z_2(y) - k_{2N}(x, y)\hat{z}_{2N}(y)\big)dy.
\end{aligned} \tag{6.14}$$

By using Eq. (6.14), we have

$$\begin{aligned}
\|f(x) - f_N(x)\| \leq{}& \|g(x) - g_N(x)\|\\
&+ |\lambda_1| \|k_1(x, y)z_1(y) - k_{1N}(x, y)\hat{z}_{1N}(y)\|\\
&+ |\lambda_2| \|k_2(x, y)z_2(y) - k_{2N}(x, y)\hat{z}_{2N}(y)\|.
\end{aligned} \tag{6.15}$$

By using Eq. (6.9), for $i = 1, 2$, we have

$$\begin{aligned}\|k_i(x,y) - z_i(y) - k_{iN}(x,y)\hat{z}_{iN}(y)\| &\leq \|k_i(x,y)\|\|z_i(y) - \hat{z}_{iN}(y)\| \\ &+ \|k_i(x,y) - k_{iN}(x,y)\|\Big(\|z_i(y) - \hat{z}_{iN}(y)\| + \|z_i(y)\|\Big) \\ &\leq \|k_i(x,y)\|\Big(L_i\|f(x) - f_N(x)\| + \alpha_i\Big) \\ &+ \|k_i(x,y) - k_{iN}(x,y)\|\Big(L_i\|f(x) - f_N(x)\| + \alpha_i + \|z_i(y)\|\Big).\end{aligned}$$

By using notation (6.12), we get

$$\begin{aligned}\|k_i(x,y) - z_i(y) - k_{iN}(x,y)\hat{z}_{iN}(y)\| \leq \alpha_i(\gamma_i + \eta_i) + \eta_i\beta_i \\ + L_i(\gamma_i + \eta_i)\|f(x) - f_N(x)\|.\end{aligned} \tag{6.16}$$

By using Eqs. (6.15) and (6.16) and notation (6.12), we get

$$\begin{aligned}\Big(1 - |\lambda_1|L_1(\gamma_1 + \eta_1) - |\lambda_2|L_2(\gamma_2 + \eta_2)\Big)\|f(x) - f_N(x)\| \leq \theta \\ + |\lambda_1|\Big(\alpha_1(\gamma_1 + \eta_1) + \eta_1\beta_1\Big) \\ + |\lambda_2|\Big(\alpha_2(\gamma_2 + \eta_2) + \eta_2\beta_2\Big).\end{aligned}$$

So, by using assumption (6.10), Eq. (6.11) is proved. □

7. Numerical Example

In this section, we solve some examples by using proposed method to show accuracy and efficiency of this method. Exact solution of these examples are available. The exact solution and approximate solution at the selected point on the interval $[0, 1]$, are reported in tables. Comparing exact solution with approximate solution shows that this method is very accurate. Also, for comparing proposed method with previous method, we use L^2-norms of errors that is calculated as follows

$$E_2 = \Big(\int_0^1 |f(x) - f_N(x)|^2 dx\Big)^{\frac{1}{2}}, \qquad x \in [0, 1]. \tag{7.1}$$

The following examples have been tested.
For first example, in Eq. (1.1), let $\lambda_2 = 0$ and we get the nonlinear Fredholm integral equation.

Example 7.1. Consider the following nonlinear Fredholm integral equation

$$f(x) = \sin(\pi x) + \frac{1}{5}\int_0^1 \sin(\pi y)\cos(\pi x) f^3(y)dy, \qquad x \in [0, 1]. \tag{7.2}$$

The exact solution of this equation is $f(x) = \sin(\pi x) + \frac{20-\sqrt{391}}{3}\cos(\pi x)$.
Table 1 shows the comparison between the exact solution and approximate solution obtained by proposed method for different values of N. Also, in this table we compare proposed method with modification of hat functions method (MHFs) [16] for different values of m. Moreover, L^2-norm of error obtained by the present method are compared

TABLE 1. Numerical results of Example 7.1.

x_i	Exact solution	MHFs method		Present method	
		m=8	m=16	N=4	N=5
0.0	0.0754267	0.0750345	0.0753865	0.0753307	0.0759277
0.1	0.3807520	0.3803265	0.3807220	0.3798065	0.3798065
0.2	0.6488067	0.6486153	0.6487845	0.6485490	0.6485824
0.3	0.8533517	0.8528618	0.8533130	0.8531290	0.8532638
0.4	0.9743646	0.9743778	0.9743167	0.9738160	0.9739685
0.5	1.0000000	1.0000000	1.0000000	0.9995789	0.9995756
0.6	0.9277484	0.9277352	0.9277963	0.9280825	0.9279098
0.7	0.7646823	0.7651722	0.7647210	0.7656931	0.7655258
0.8	0.5267638	0.5269552	0.5267860	0.5274741	0.5274391
0.9	0.2372820	0.2377075	0.2373120	0.2371867	0.2371673
1.0	-0.0754266	-0.0750350	-0.0753870	-0.0727086	-0.0731506

TABLE 2. Comparison of the errors E_2 of Example 7.1.

Methods	E_2
Hat functions method [1]	
m=8	3.4e-3
m=16	9.9e-4
Triangular functions methods [14]	
m=8	9.9e-3
m=16	2.5e-3
Present method	
N=4	6.6e-4
N=5	5.8e-4

with the rationalized hat functions method [1] and triangular functions method [14] in Table 2.

For second example, in Eq. (1.1), let $\lambda_1 = 1$ and $\lambda_2 = 1$, then we get the Fredholm-Volterra integral equation.

Example 7.2. Consider the following linear Fredholm-Volterra integral equation

$$\begin{aligned} f(x) = & -x^4 - x^3 + 12x^2 - x - 5 + \int_0^x (x-y)f(y)dy \\ & + \int_0^1 (x+y)f(y)dy, \qquad x \in [0,1]. \end{aligned} \tag{7.3}$$

The exact solution of this equation is $f(x) = 12x^2 + 6x$.
Table 3 shows the comparison between the exact solution and approximate solution obtained by proposed method for different values of N. Also, in this table we compare proposed method with modification of hat functions method (MHFs) [16] for different values of m. Moreover, L^2-norm of error obtained by the present method are compared

FIGURE 1. Absolute value of errors for Example 7.1 with $N = 4, 5$.

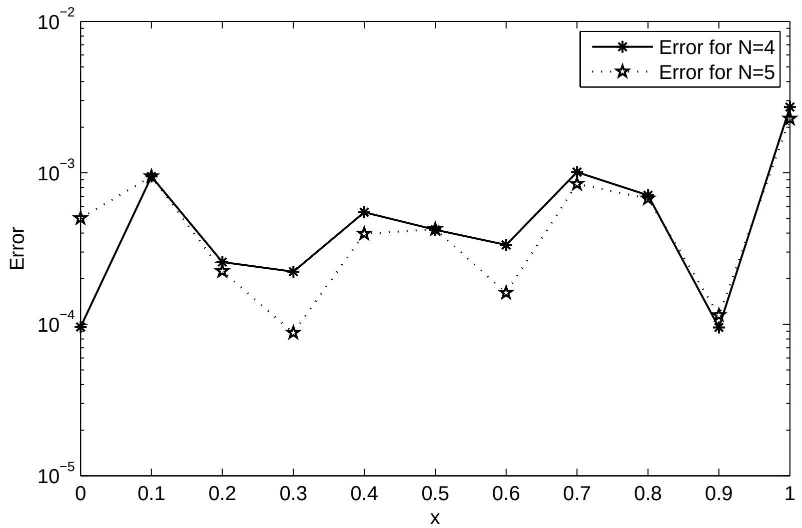


TABLE 3. Numerical results of Example 7.2.

x_i	Exact solution	MHFs method		Present method	
		m=8	m=16	N=4	N=5
0.0	0.000	0.001326	-0.000083	0.0000135	-0.0007507
0.1	0.720	0.7191535	0.7199322	0.7200158	0.7190989
0.2	1.680	1.6786887	1.6799324	1.6800184	1.6789492
0.3	2.880	2.8784452	2.8799170	2.8800212	2.8787923
0.4	4.320	4.3184184	4.3198861	4.3200241	4.3186265
0.5	6.000	5.9974416	5.9998395	6.0000268	5.9984601
0.6	7.920	7.9178431	7.9198500	7.9200290	7.9183138
0.7	10.08	10.077287	10.079844	10.0800303	10.0782247
0.8	12.48	12.476939	12.479823	12.4800301	12.4782502
0.9	15.12	15.116792	15.119784	15.1200275	15.1184717
1.0	18.00	17.995680	17.999729	18.0000219	17.9989970

with the rationalized hat functions method [1] and triangular functions method [14] in Table 4.

For third example, in Eq. (1.1), let $\lambda_1 = 0$, then we get the Volterra integral equation.

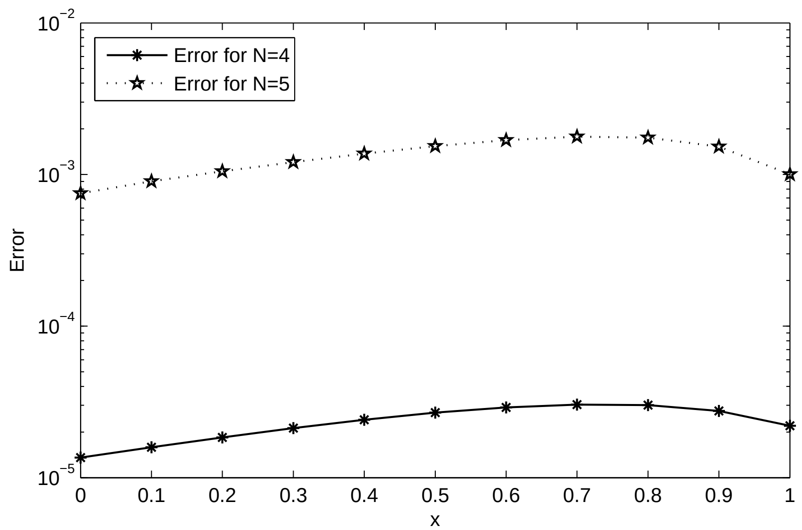
Example 7.3. Consider the following nonlinear Volterra integral equation

$$f(x) = -\frac{e^{-2x}}{2} + \frac{3}{2} - \int_0^x (f^2(y) + f(y))dy, \qquad x \in [0, 1]. \tag{7.4}$$

TABLE 4. Comparison of the errors E_2 of Example 7.2.

Methods	E_2
Hat functions method [1]	
m=8	1.4e-1
m=16	3.6e-2
Triangular functions methods [14]	
m=8	5.3e-2
m=16	1.8e-2
Present method	
N=4	2.4e-5
N=5	1.4e-3

FIGURE 2. Absolute value of errors for Example 7.2 with $N = 4, 5$.



The exact solution of this equation is $f(x) = e^{-x}$.
Table 5 shows the comparison between the exact solution and approximate solution obtained by proposed method for different values of N. Also, in this table we compare proposed method with modification of hat functions method (MHFs) [16] for different values of m. Moreover, L^2-norm of error obtained by the present method are compared with the rationalized hat functions method [1] and triangular functions method [14] in Table 6.

Example 7.4. Consider the following nonlinear Volterra integral equation

$$f(x) = -100e^x + \frac{1}{1+x} + 100 \int_0^x e^y \frac{1}{f(y)} dy, \qquad x \in [0, 1]. \tag{7.5}$$

TABLE 5. Numerical results of Example 7.3.

x_i	Exact solution	MHFs method		Present method	
		m=8	m=16	N=4	N=5
0.0	1.0000000	1.0000000	1.0000000	1.0002219	1.0000168
0.1	0.9048374	0.9049781	0.9047612	0.9048110	0.9050053
0.2	0.8187308	0.8181718	0.8186955	0.8187808	0.8188386
0.3	0.7408182	0.7411201	0.7408354	0.7411202	0.7412613
0.4	0.6703200	0.6701145	0.6703539	0.6712819	0.6725037
0.5	0.6065307	0.6065412	0.6065300	0.6091827	00.614477
0.6	0.5488116	0.5488997	0.5487812	0.5552029	0.5719678
0.7	0.4965853	0.4963738	0.4965719	0.5101869	0.5538320
0.8	0.4493290	0.4494735	0.4493376	0.4754428	0.5741903
0.9	0.4065697	0.4064958	0.4065853	0.4527426	0.6536217
1.0	0.3678794	0.3678941	0.3678801	0.4443221	0.8203590

TABLE 6. Comparison of the errors E_2 of Example 7.3.

Methods	E_2
Hat functions method [1]	
m=8	4.6e-3
m=16	1.3e-3
Triangular functions methods [14]	
m=8	9.4e-4
m=16	2.3e-4
Present method	
N=4	2.3e-3
N=5	3.6e-4

The exact solution of this equation is not available.
Table 7 shows the values of approximate solution obtained by proposed method for different values of N.

8. Conclusion

In this work, we suggest a numerical method to solve nonlinear Fredholm-Volterra integral equations based on Bell polynomials. By using this method nonlinear Fredholm-Volterra integral equations convert to a nonlinear system of algebraic equations which can be solved by an appropriate numerical method such as Newton's method. Furthermore, we established the proposed method is convergent. Some examples are included to show accuracy and efficiency of the proposed method. The comparison of the results achieved by the present method with the exact solution and the other methods reveals that the method is very effective.

FIGURE 3. Absolute value of errors for Example 7.3 with $N = 4, 5$.

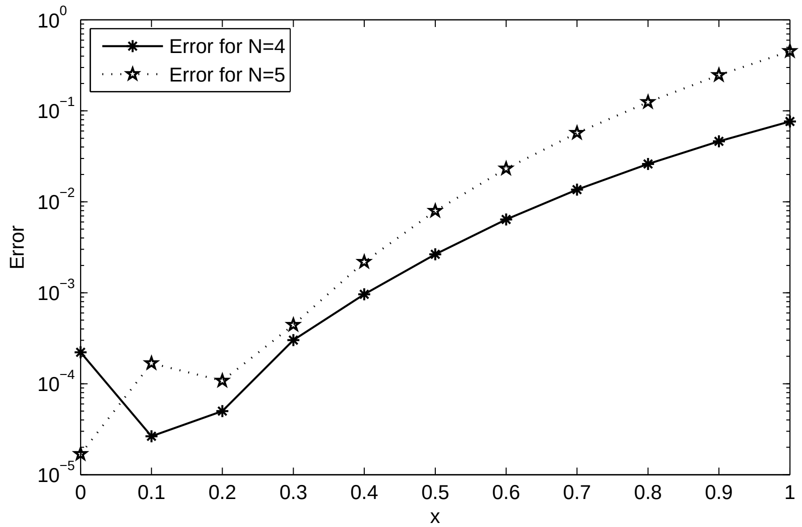


TABLE 7. Numerical results of Example 7.4.

x_i	N=4	N=5	N=6
0.1	-109.5048299993	-109.5072406720	-109.5074585361
0.2	-121.1045297108	-121.1054499714	-121.1054869563
0.3	-133.9162866472	-133.9146524248	-133.9146974057
0.4	-148.0678315943	-148.0659705330	-148.0661621869
0.5	-163.7032141390	-163.7031988128	-163.7032986978
0.6	-180.9828026704	-180.9846332075	-180.9839536510
0.7	-200.0832843804	-200.0849004230	-200.0821913311
0.8	-221.1976652651	-221.1967871903	-221.1897872222
0.9	-244.5352701253	-244.5330694549	-244.5174283539
0.91	-246.999853847	-246.9978724886	-246.9809939751
0.93	-252.003387037	-252.0021806956	-251.9825596955
0.95	-257.107657720	-257.1077673182	-257.0850062569
0.97	-262.314588540	-262.3166769574	-262.2903243749
0.99	-267.626128252	-267.6309935134	-267.6005378639

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