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New Solution for Fokker-Plank Equation of Special Stochastic Process with Lie Point Symmetries

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Abstract In this paper, Lie symmetry analysis is applied to find a new solution for Fokker Plank equation of Ornstein-Uhlenbeck process. This analysis classifies the solution format of the Fokker Plank equation by the Lie algebra symmetries of our considered stochastic process.

Keywords. Financial market, Ornstein-Uhlenbeck, Lie algebra symmetries, Fokker-Plank.

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1. Introduction

The Ornstein-Uhlenbeck process is an example of a Gaussian process that has a bounded variance and admits a stationary probability distribution. In contrast to the Wiener process, the difference between them is in their "drift" term. The Ornstein-Uhlenbeck process is also one of several approaches used to model interest rates, currency exchange rates, and commodity prices stochastically in a financial market.

In this paper we consider a financial market with price process $\{X_t\}_t$ for risky asset. Let the market be the free of arbitrage possibilities and be specified by the following stochastic differential equation

$$dX_t = -\alpha X_t dt + \sigma dW_t,$$

where $\{W_t\}_t$ is standard Brownian motion and the drift parameter α and the volatility σ are assumed to be constants.The above SDE is called Ornstein-Uhlenbeck process. Symmetry plays a very important role in various fields of nature. In fact Lie method is an effective method in solving a large number of equations which are not solved in the simple ways [10, 13, 14]. There are still many authors using this method to find the exact solutions of any given system of differential equation. There are many literatures on Lie point symmetry method and its applications in differential equations.

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In this paper in the first section Ornstein-Uhlenbeck process and its Fokker-Plank equation are observed. In the next section the Lie point symmetry of differential equation is presented. By the considered method, the solution format of the Fokker Plank equation is given in the last section.

2. Ornstein-Uhlenbeck process

Take

$$dX_t = -\alpha X_t dt + \sigma dW_t,$$

with $X_0 = x_0$, where $\{W_t\}_t$ is standard Brownian motion and α and σ are positive constants. There is a solution of the form $X_t = g(t)Y_t$ where $dY_t = h(t)dW_t$. According to Ito's formula, we have

$$dX_t = gdY_t + dgY_t + dgdY_t = ghdW_t + g'Y_t dt.$$

Comparing this with the original equation, we require

$$g'Y = -\alpha gY, \quad gh = \sigma.$$

So we have $g(t) = Ce^{-\alpha t}$. This gives

$$Y_t = Y_0 + \frac{\sigma}{C}\int_0^t e^{\alpha s} dW_s,$$

and hence

$$X_t = e^{-\alpha t}(CY_0 + \sigma \int_0^t e^{\alpha s} dW_s),$$

[7, 16].

Theorem 2.1. *Let $\{X_t\}_t$ be a solution to the stochastic differential equation*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

with infinitesimal generator $\mathcal{A}$ given by

$$\mathcal{A}f(s, y) = \mu(s, y)\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(s, y).$$

If the solution $\{X_u\}_{u\in[t,s]}$ has a transition density $p(s, y; t, x)$ then p will satisfy the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(s, y; t, x) = \mathcal{A}^* p(s, y; t, x), \quad (t, x) \in (0, T) \times \mathbf{R},$$

where $p(s, y; t, x) \longrightarrow \delta_y$ as $t \downarrow s$, and

$$\mathcal{A}^* f(t, x) = -\frac{\partial}{\partial x}[\mu(t, x)f(t, x)] + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(t, x),$$

[3, 12]

So obviously the Fokker-Plank equation for Ornstein-Uhlenbeck process is as follows

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2} + \alpha p + \alpha x \frac{\partial p}{\partial x}. \tag{2.1}$$

3. Lie Symmetries

Symmetry plays a very important role in various fields of nature. There are still many authors using this method to find the exact solutions of nonlinear differential equations [8, 13, 14]. It is also a powerful tool for finding exact solutions of nonlinear problems [14, 15]. Many eaxmples of applications to physical problems have been demonstrated in a huge number of papers and excellent books. The general procedure to obtain Lie symmetries of differential equations, and their applications to find analytic solutions of the equations are described in detail in several monographs on the subject (e.g. [8, 13, 14]) and in numerous papers in the literatures (e.g. [2, 6, 10, 11]).

Consider a system of DE (PDE or ODE) in the dependent variables $u^\alpha(1 \leq \alpha \leq m)$ and dependent variables $x^i(1 \leq i \leq n)$ of the form:

$$\Delta^s(x^i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, ...) = 0, \quad 1 \leq s \leq k, \tag{3.1}$$

where the subscripts denote partial derivatives (e.g. $u_i^\alpha = \partial u^\alpha / \partial x^i$). To determine continuous symmetries of (3.1), it is useful to consider infinitesimal Lie transformations of the following form.

$$\tilde{x}^i = x^i + \varepsilon \xi^i + O(\varepsilon^2), \quad \tilde{u}^\alpha = u^\alpha + \varepsilon \eta^\alpha + O(\varepsilon^2), \tag{3.2}$$

that leave the equation system invariant to $O(\varepsilon^2)$. Lie point symmetries correspond to the case where the infinitesimal generators $\xi^i = \xi^i(x^i, u^\alpha)$ and $\eta^\alpha = \eta^\alpha(x^i, u^\alpha)$ depend only on the x^i and the u^α and not on the derivatives or integrals of the u^α. Generalized Lie symmetries are obtained in the case when the transformations (3.2) also depend on the derivatives or integrals of the u^α.

The infinitesimal transformations for the first and second derivatives to $O(\varepsilon^2)$ are given by the prolongation formula

$$\tilde{u}_i^\alpha = u_i^\alpha + \varepsilon \zeta_i^\alpha, \quad \tilde{u}_{ij}^\alpha = u_{ij}^\alpha + \varepsilon \zeta_{ij}^\alpha, \tag{3.3}$$

where

$$\zeta_i^\alpha = D_i \hat{\eta}^\alpha + \xi^s u_{si}^\alpha, \quad \zeta_{ij}^\alpha = D_i D_j \hat{\eta}^\alpha + \xi^s u_{sij}^\alpha. \tag{3.4}$$

Here

$$\hat{\eta}^\alpha = \eta^\alpha - \xi^s u_s^\alpha, \tag{3.5}$$

corresponds to the canonical Lie transformation for which $\tilde{x}^i = x^i$ and $\tilde{u}^\alpha = u^\alpha + \varepsilon \hat{\eta}^\alpha$. The symbol D_i in (3.4) denotes the total derivative operator with respect to x^i. Similar formula to (3.4) apply for the transformation of the higher order derivatives.

The condition for invariance of the DE system (3.1) to $O(\varepsilon^2)$ under the Lie transformation (3.2) can be expressed in the following form.

$$\mathcal{L}_{\mathbf{v}} \Delta^s \equiv \tilde{\mathbf{v}}(\Delta^s) = 0 \quad \text{whenever} \quad \Delta^s = 0, \quad 1 \leq s \leq k, \tag{3.6}$$

where

$$\tilde{\mathbf{v}} = \mathbf{v} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \tag{3.7}$$

is the prolongation of the vector field

$$\mathbf{v} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{3.8}$$

associated with the infinitesimal transformation (3.2). The symbol $\mathcal{L}_{\mathbf{v}}\Delta^s$ in (3.6) denotes the Lie derivative of Δ^s with respect to the vector field $\mathbf{v}$ (i.e. $\mathcal{L}_{\mathbf{v}}\Delta^s = \frac{\mathrm{d}\Delta^s}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$).

The Lie symmetries of the Fokker Plank equation (2.1) with variables x, t and p can be found by solving the Lie determining equation (3.6) for the infinitesimal generators of the Lie group. In the sequel we first write down the Lie determining equations that correspond to the point Lie group. The point Lie algebra system is briefly described, and the symmetries are used to obtain some results for the solutions of system (3.6).

The infinitesimal Lie transformations for the system (3.1) are of the following form:

$$\tilde{t} = t + \varepsilon\xi^t, \qquad \tilde{x} = x + \varepsilon\xi^x, \qquad \tilde{p} = p + \varepsilon\eta. \tag{3.9}$$

The corresponding canonical symmetry generator $\hat{\eta}$ is given by the formula analogous to (3.5). Thus

$$\hat{\eta} = \eta - \xi^t\eta_t - \xi^x\eta_x, \tag{3.10}$$

relates the canonical symmetry generator $\hat{\eta}$ to η.

The Lie determining equations (3.6) for the infinitesimal generators of the equation (2.1) can be written in the following form:

$$\begin{aligned}
&\xi^1_x = \xi^2_x = \xi^2_p = \eta_{pp} = 0, \qquad (3.11)\\
&\xi^1_x = \frac{1}{2}\xi^2_t, \qquad \xi^1_{tt} = \frac{1}{2}\alpha^2(2\xi^1 + 3x\xi^2_t), \qquad \xi^2_{ttt} = 4\alpha^2\xi^2_t,\\
&\eta_{xx} = \frac{1}{\sigma^2}\left(2\eta_t + 2\alpha p\eta - 2\alpha\eta - 2\alpha x\eta_x - 2\alpha p\xi^2_t\right),\\
&\eta_{xx} = \frac{-2p(\alpha-\sigma^2)\eta_p + 2x(\alpha - 2\sigma^2)\eta_x + 2p(\alpha-\sigma^2)\xi^1_t + 2\eta_t + 2(\alpha-\sigma^2)\eta_p}{\sigma^2 x^2},\\
&\eta_{xp} = \frac{2x\left(\alpha - \frac{3}{2}\sigma^2\right)\xi^1_t - 4\xi^2_t}{4\sigma^2 x^2},
\end{aligned}$$

for the vector field

$$\mathbf{v} = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial p}. \tag{3.12}$$

Thus, the general vector field $\mathbf{v}$ in the point Lie algebra corresponding to the transformations (3.2) can be written in the following form:

$$\mathbf{v} = \sum_{i=1}^{6} a_i\mathbf{v}_i, \tag{3.13}$$

TABLE 1. Commutators Table of $\mathcal{G}$.

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	0	0	$-\alpha\mathbf{v}_3$	$\alpha\mathbf{v}_4$	$-2\alpha\mathbf{v}_5$	$2\alpha\mathbf{v}_6$
$\mathbf{v}_2$	0	0	0	0	0	0
$\mathbf{v}_3$	$\alpha\mathbf{v}_3$	0	0	$-\frac{2\alpha}{\sigma^2}\mathbf{v}_2$	0	$2\alpha\mathbf{v}_4$
$\mathbf{v}_4$	$-\alpha\mathbf{v}_4$	0	$\frac{2\alpha}{\sigma^2}\mathbf{v}_2$	0	$-2\alpha\mathbf{v}_3$	0
$\mathbf{v}_5$	$2\alpha\mathbf{v}_5$	0	0	$2\alpha\mathbf{v}_3$	0	$2\alpha\mathbf{v}_1 - 2\alpha^2\mathbf{v}_2$
$\mathbf{v}_6$	$-2\alpha\mathbf{v}_6$	0	$-2\alpha\mathbf{v}_4$	0	$-2\alpha\mathbf{v}_1 + 2\alpha^2\mathbf{v}_2$	0

where the basis vector fields $\{\mathbf{v}_i : 1 \leq i \leq 6\}$ are

$$\begin{aligned}
&\mathbf{v}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_2 = p\frac{\partial}{\partial p}, \qquad \mathbf{v}_3 = e^{-\alpha t}\frac{\partial}{\partial x},\\
&\mathbf{v}_4 = e^{\alpha t}\frac{\partial}{\partial x} - \frac{2\alpha}{\sigma^2}e^{\alpha t}xp\frac{\partial}{\partial p},\\
&\mathbf{v}_5 = -\alpha x e^{-2\alpha t}\frac{\partial}{\partial x} + e^{-2\alpha t}\frac{\partial}{\partial t} - \alpha p e^{-2\alpha t}\frac{\partial}{\partial p},\\
&\mathbf{v}_6 = \alpha x e^{2\alpha t}\frac{\partial}{\partial x} + e^{2\alpha t}\frac{\partial}{\partial t} - \frac{2\alpha^2}{\sigma^2}x^2 p e^{\alpha^2 t^2}\frac{\partial}{\partial p}.
\end{aligned}$$

The commutator table of the Lie algebra $\mathcal{G}$ spanned by the vector fields $\mathbf{v}_i$'s are given in Table (1). thus these vector fields make a Lie algebra with respect to Lie bracket.

3.1. **Classification of the Solutions.**

The one-parameter groups g_i generated by the $\mathbf{v}_i$ are given in the following list. The entires give the transformed point $\exp(\varepsilon\mathbf{v}_i)(x,t,p) = (\tilde{x}, \tilde{t}, \tilde{p})$:

$$\begin{aligned}
g_1 &:= \exp(\varepsilon\mathbf{v}_1)(x,t,p) = (x, t+\varepsilon, p),\\
g_2 &:= \exp(\varepsilon\mathbf{v}_2)(x,t,p) = (x, t, e^{\varepsilon}p),\\
g_3 &:= \exp(\varepsilon\mathbf{v}_3)(x,t,p) = (x + e^{\alpha t}\varepsilon + x, t, p),\\
g_4 &:= \exp(\varepsilon\mathbf{v}_4)(x,t,p) = \left(e^{-\alpha t}\varepsilon + x, t, \frac{-2\alpha}{\sigma^2}xp\varepsilon e^{\alpha t}\right),\\
g_5 &:= \exp(\varepsilon\mathbf{v}_5)(x,t,p) = \left(x - e^{-\alpha t}\alpha x\varepsilon, t + e^{-2\alpha t}\varepsilon, p - e^{-2\alpha t}\alpha p\varepsilon\right),\\
g_6 &:= \exp(\varepsilon\mathbf{v}_6)(x,t,p) = \left(x + \alpha x e^{2\alpha t}\varepsilon, t + e^{2\alpha t}\varepsilon, -\frac{2\alpha^2}{\sigma^2}x^2 p e^{2\alpha t}\varepsilon\right).
\end{aligned}$$

Since each group element g_i is a symmetry, then, if $p = f(x,t)$ is a solution of the equation (2.1), so we have the following functions

$$p_1 = f(x, t-\varepsilon), \tag{3.14}$$
$$p_2 = e^{\varepsilon} f(x,t), \tag{3.15}$$
$$p_3 = f(x - e^{\alpha t} t\varepsilon, t), \tag{3.16}$$
$$p_4 = -\frac{2\alpha}{\sigma^2} e^{\alpha t}\varepsilon\left(x - e^{\alpha t} t\varepsilon\right) f\left(x - e^{\alpha t} t\varepsilon, t\right), \tag{3.17}$$
$$\begin{aligned} p_5 = &\left(1 - \alpha \exp\left\{-\alpha\left(t + e^{-2\alpha t}\varepsilon\right)\right\}\right) \\ &\times f\left(\frac{x}{1 - \alpha \exp\left\{-\alpha\left(t + e^{-2\alpha t}\varepsilon\right)\right\}}, t + e^{-2\alpha t}\varepsilon\right), \end{aligned} \tag{3.18}$$
$$\begin{aligned} p_6 = &\left(\frac{x}{1 + \alpha \exp\left\{2\alpha\left(t + e^{2\alpha t}\varepsilon\right)\right\}}\right)^2 \varepsilon \\ &\times f\left(\frac{x}{1 + \alpha \exp\left\{2\alpha\left(t + e^{2\alpha t}\varepsilon\right)\right\}}, t + e^{2\alpha t}\right) \\ &- \frac{2\alpha^2}{\sigma^2} \exp\left\{2\alpha(t + e^{2\alpha t})\right\}. \end{aligned} \tag{3.19}$$

4. Invariant Functions

Given a group of point transformations G acting on the space of variables of the system called E, the charachteristic of all $G-$invariant functions $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is of great importance.

Definition 4.1. A function $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is said to be *invariant* under the group transformation G if its graph $\{(\mathbf{x}, \mathbf{f}(\mathbf{x}))\}$ is a (locally) $G-$invariant subset.

For example, the graph of any invariant function for the rotation group SO(2) must be an arc of a circle centered at the origin, so $u = \pm\sqrt{c^2 - x^2}$.

The fundamental feature of Lie groups is the ability to work infinitesimally, thereby effectively linearizing complicated invariance criteria.

Theorem 4.2. *Let G be a connected Lie group of transformations acting on total space E. A function $I : E \to \mathbf{R}$ is invariant under G if and only if for all $(\mathbf{x}, \mathbf{u}) \in E$ and every infinitesimal generator $\mathbf{v} =\in \mathcal{G}$ of G,*

$$\boldsymbol{v}[I(\mathbf{x}, \mathbf{u})] = 0. \tag{4.1}$$

Thus, according to theorem 4.2, the invariant $v = I(\mathbf{x}, \mathbf{u})$ of a one-dimensional group with infinitesimal generator $\mathbf{v}$, obtained from (4.1), satisfy the first order, linear, homogeneous partial differential equation

$$\xi^i(\mathbf{x}, \mathbf{u})\frac{\partial v}{\partial x^i} + \eta^{\alpha}(\mathbf{x}, \mathbf{u})\frac{\partial v}{\partial u^{\alpha}} = 0. \tag{4.2}$$

The solutions of (4.2) are effecyively found by the method of charachteristics. We replace the partial differential equation by the charachteristic system of ordinary differential equations

$$\frac{dx^1}{\xi^1(\mathbf{x},\mathbf{u})} = \cdots = \frac{dx^p}{\xi^p(\mathbf{x},\mathbf{u})} = \frac{du^1}{d\eta^1(\mathbf{x},\mathbf{u})} = \cdots = \frac{du^\alpha}{\eta^\alpha(\mathbf{x},\mathbf{u})}. \tag{4.3}$$

The general soluition to (4.3) can be written in the form $I_1(\mathbf{x},\mathbf{u}) = c_1, ..., I_{p+q-1}(\mathbf{x},\mathbf{u}) = c_{p+q-1}$, where c_i are constants of integrations.

Lemma 4.3. *The resulting functions $I_1, ..., I_{p+q-1}$ form a complete set of functionally independent invariants of the one-dimensional Lie algebra spanned by differential operator $\mathbf{v}$.*

For example a one-dimensional Lie algebra spanned by the differential operator $\mathbf{v} = -y\partial_x + x\partial_y + (1+z^2)\partial_z$ are obtained by solving the charachteristic system

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1+z^2},$$

thus, there are two functionally independent invariant functions $r = \sqrt{x^2+y^2}$ and $w = (xz-y)/(yz+x)$. A fundamental theorem obtained from differential geometry charachterizes the number of functionally independent invariants of a group action.

Theorem 4.4. *Let G be a transformation group acting semi-regularly (all the orbits have same dimension) on total space E with s-dimensional orbits. Let $I_1(\mathbf{x},\mathbf{u})$,..., $I_{p-s}(\mathbf{x},\mathbf{u}), J_1(\mathbf{x},\mathbf{u}), ..., J_q(\mathbf{x},\mathbf{u})$, be a complete set of functionally independent invariants for G. Then any G-invariant function $\mathbf{u} = \mathbf{f}(\mathbf{x})$, can locally be written in the implicit form*

$$\mathbf{w} = \mathbf{h}(\mathbf{y}), \qquad \textit{where} \qquad \mathbf{y} = \mathbf{I}(\mathbf{x},\mathbf{u}), \qquad \mathbf{w} = \mathbf{J}(\mathbf{x},\mathbf{u}). \tag{4.4}$$

Remark 4.5. A "similarity solution" or "group-invariant solution" which is a main subject of next section, of a system of partial differential equations is just an invariant function for a group of scalling transformations. For example, consider the one-dimensional group $\mathbf{R}^+$ acting on $\mathbf{R}^3$ with the transformation $(x, y, u) \mapsto (\lambda x, \lambda^\alpha y, \lambda^\beta u)$. The independents invariants are provided by the rotios $y = y/x^\alpha, w = u/x^\beta$, so any scale-invariant function can be written as $w = h(y)$, or explicitely $u = x^\beta h(y/x^\alpha)$.

As usual, the most convenient charachterization of the invariant functions is based on an infinitesimal conditions. Since the graph of a function is defined by the vanishing of its components $u^\alpha - f^\alpha(\mathbf{x})$, the general invariance formulation (3.6) imposes the infinitesimal invariance conditions

$$0 = \mathbf{v}(u^\alpha - f^\alpha(\mathbf{x})) = \eta^\alpha(\mathbf{x},\mathbf{u}) - \xi^i(\mathbf{x},\mathbf{u})\frac{\partial f^\alpha}{\partial x^i},$$

which must hold whenever $\mathbf{u} = \mathbf{f}(\mathbf{x})$, for every infinitesimal generator $\mathbf{v} \in \mathcal{G}$. These first order partial differential equations are known in the literature as the invariant surface conditions associated with the given transformation group.

5. Group-Invariant Solutions

When we confronted with a complicated system of partial differential equations in some physically important problem, the discovery of any explicit solutions whatsoever is of great interest. Explicit solutiona can be used as models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant bahaviour of more general types of solutions. The method used to find group-invariant solutions, generalizing the well-known techniques for finding similarity solutions, provide a systematic computational method for determining large classes of special solutions. These group-invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations; the more symmetrical the solution, the easier it is to construct. The fandamental theorem on group-invariant solutions roughly states that the solutions which are invariant under a given $r-$dimensional symmetry group of the system can all the find by solving r fewer independent variables than the original system. In particular, if the number of parameters is one less that the number of independent variables in the physical system: $r = p - 1$, then all the corresponding group-invariamt solutions can be found by solving a system of **ordinary differential equations.** In this way, one reduces an intractable set of partial differential equations to a simpler set of ordinary differential equations which one might stand a chance of solving explicitly. In practical applications, these group-invariant solutions can, in most instances, be effectively found and, often, are the only explicit solutions which are known.

5.1. **Construction of Group-Invariant Solutions.** Consider a system of partial differential equations $\Delta = 0$ with $p-$independent and $q-$dependent variables. Let G be a group of transformations acting on E. A solution $\mathbf{u} = \mathbf{f}(\mathbf{x})$ of the system is said to be *G-invariant* if it is left unchanged by all the group transformations in G, meaning that for each $g \in G$, the function $\mathbf{f}$ and $g \cdot \mathbf{f}$ agree on their common domains of definition.

If G is a symmetry group of a system of partial differential equations $\Delta = 0$, then, we can find all the $G-$invariant solution to Δ by solving a reduced system of differential equations, denoted by Δ/G, which will involve fewer independent variables than the original system Δ. To see how this reduction effected, we begin by making the simplifying assumption that G acts *projectably* on M. This means that the transformations in G all takes the form $(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = g \cdot (\mathbf{x}, \mathbf{u}) = (\Xi^g(\mathbf{x}), \Phi_g(\mathbf{x}, \mathbf{u}))$ for $g \in G$, i.e., the changes in the independent variable $\mathbf{x}$ do not depend on the dependent variables $\mathbf{u}$. There is then a projected group action $\tilde{\mathbf{x}} = g \cdot \mathbf{x} = \Xi_g(\mathbf{x})$ on an open sebset $\Omega \subset X$. We make the regularity assumption that both the action of G on E and the projected action of G on Ω is *regular*, i.e., all the orbit dimension of the action are same as s, where s is strictly less than p. (The case $s = p$ is fairly trivial, while if $s > p$, no $G-$invariant functions exist, Usually s will be the same as the dimension of G itself, but this need not be the case.) Under these assumption there exist $p - s$ functionally independent invariants $y^1 = \eta^1(\mathbf{x}), ..., y^{p-s} = \eta^{p-s}(\mathbf{x})$ of the projected group action on $\Omega \subset X$. Each of this functions is also an invariant of the full group action on E , and furthermore, we can find q additional invariant of the action of G on E, of the form $\mathbf{v}^1 = \zeta^1(\mathbf{x}, \mathbf{u}), ..., \mathbf{v}^q = \zeta^q(\mathbf{x}, \mathbf{u})$, which, together with the $\eta's$ provide a complete set of $p+q-s$ functionally independent invariants for G on E, We write this complete collection on invariants as

$$\mathbf{y} = \eta(\mathbf{x}), \qquad \mathbf{v} = \zeta(\mathbf{x}, \mathbf{u}). \tag{5.1}$$

In the construction of the reduced system of differential equations for the $G-$ invariant solutions to Δ, then $\mathbf{y}'s$ will play the role of the new independent variables, and the $\mathbf{v}'s$ the role of the new dependent variables. Note in particular that there are s few independent variables $y^1, ..., y^{p-s}$ which will appear in this reduced system, where s is the dimension of the orbits of G.

There in a one-to-one correspondence between $G-$invariant function $\mathbf{u} = \mathbf{f}(\mathbf{x})$ on E and arbitrary functions $\mathbf{v} = \mathbf{h}(\mathbf{y})$ involving the new variables. To explain this correspondence, we begin by invoking the inplicit function theorem to solve the system $\mathbf{y} = \eta(\mathbf{x})$ for $p-s$ of the independent variables, say $\tilde{\mathbf{x}} = (x^{i_1}, ..., x^{i_p-s})$, in the terms of the new variables $y^1, ..., y^{p-s}$ and the nremaining s old independent variables, denoted as $\hat{\mathbf{x}} = (x^{j_1}, ..., x^{j_s})$. Thus we have the solutions

$$\tilde{\mathbf{x}} = \rho(\hat{\mathbf{x}}, \mathbf{y}), \tag{5.2}$$

for some well-defined function ρ. The first $p-s$ of the old independent variables $\tilde{\mathbf{x}}$ are known as *principle variables*, and the remaining s of these variables $\hat{\mathbf{x}}$ are the *parametric variables*, as they will, enter parametrically into all the subsequent fromulae. The preciese manner in which oner splits the variables $\mathbf{x}$ into principle and parametric variables is restricted only by the requirement that the $(p-s)\times(p-s)$ submatris $(\partial\eta^j/\partial\tilde{x}^i)$ of the full Jacobian matrix $\partial\eta/\partial\mathbf{x}$ is invertible, so that the implicit function theorem is applicable; otherwise, the choice is entirely arbitrary. We need to make a further transversality assumption on the action of G on E, that allows us to solve the other system of invariants $\mathbf{v} = \zeta(\mathbf{x}, \mathbf{u})$ for all the derpendent variables $u^1, ..., u^q$ interms of $x^1, ..., x^p$, and $v^1, ..., v^q$, and hence in terms of new variables $\mathbf{y}, \mathbf{v}$ and parametric variables $\hat{\mathbf{x}}$:

$$\mathbf{u} = \tilde{\mu}(\mathbf{x}, \mathbf{v}) = \tilde{\mu}(\hat{\mathbf{x}}, \rho(\hat{\mathbf{x}}, \mathbf{y}), \mathbf{v}) = \mu(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}), \tag{5.3}$$

near any point $(\mathbf{x}_0, \mathbf{u}_0) \in E$.

If $\mathbf{v} = \mathbf{h}(\mathbf{y})$ is any smooth function, then (5.3) coupled with (5.1) produces a corresponding $G-$invariant function on E, of the form

$$\mathbf{u} = \mathbf{f}(\mathbf{x}) = \mu(\hat{\mathbf{x}}, \eta(\mathbf{x}), \mathbf{h}(\eta(\mathbf{x}))). \tag{5.4}$$

Conversely, if $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is any $G-$invariant function on E, then it is not too difficult to see that there necessarily exist a function $\mathbf{v} = \mathbf{h}(\mathbf{y})$ such that $\mathbf{f}$ and the corresponding function (5.4) locally agree. Thus, we have seen how $G-$invariance of functions sereves to decrease the number of variables upon which they depend.

We are now interested in finding all the $G-$invariant solutions to some system of partial differential equations (5.4). In other words, we want to know when a function of the form (5.3) corresponding to a function $\mathbf{v} = \mathbf{h}(\mathbf{y})$ is a solution to Δ. This will impose certain constrains on the function $\mathbf{h}$; these are found by computing the formulae for the derivatives of $\mathbf{v} = \mathbf{h}(\mathbf{y})$ with respect to $\mathbf{y}$, and then substituting these into the system of differential equations Δ.Thus we need to know how the derivatives of the functions $\mathbf{v} = \mathbf{h}(\mathbf{y})$ are related to the derivatives of the corresponding

$G-$invariant function $\mathbf{u} = \mathbf{f}(\mathbf{x})$. However, this is an easy application of the chain rule. Differentiating (5.4) with respect to $\mathbf{x}$ leads to asystem equation of the form

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left[\mu(\hat{\mathbf{x}}, \mathbf{y},)\right] = \frac{\partial \mu}{\partial \hat{\mathbf{x}}} + \frac{\partial \mu}{\partial \mathbf{y}}\frac{\partial \eta}{\partial \mathbf{x}} + \frac{\partial \mu}{\partial \mathbf{v}}\frac{\partial \mathbf{v}}{\partial \mathbf{y}}\frac{\partial \eta}{\partial \mathbf{x}},$$

since $\mathbf{y} = \eta(\mathbf{x})$. Here, $\partial \mathbf{u}/\partial \mathbf{x}$, etc., denoted Jacobian matrices of first order derivatives of indicated variables. Moreover, using (5.2), we can rewrite $\partial \eta/\partial \mathbf{x}$ in terms of $\mathbf{y}$ anf parametric variables $\hat{\mathbf{x}}$, Thus we obtain an equation of the form

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mu_1\left(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}, \frac{\partial \mathbf{v}}{\partial \mathbf{y}}\right),$$

expressing the first order derivatives of any $G-$invariant function $\mathbf{u}$ with respect to $\mathbf{x}$ in teroms of $\mathbf{y}, \mathbf{v}$, the first order derivatives of $\mathbf{v}$ with respect to $\mathbf{y}$ together with parametric variables $\hat{\mathbf{x}}$. Continuing to differentiate using the chain rule, and substituting to (5.3) whenever necessary, we are led to general formulae

$$\mathbf{u}^{(n)} = \mu^{(n)}(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}^{(n)}),$$

for all the derivatives of such a $\mathbf{u}$ up to order n with respect to $\mathbf{x}$ in terms of $\mathbf{y}, \mathbf{v}$, the derivatives of $\mathbf{v}$ with respect to $\mathbf{y}$ up to order n, and the ubiquitous parametric variable $\hat{\mathbf{x}}$.

Once the relevant formulae relating derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$ to those of $\mathbf{v}$ with respect to $\mathbf{y}$ have been determined, the reduced system of differential equations for the $G-$invariant solutions to the system Δ is determined by substituting these expressions into the system whenever they occur. In general, this leads to system of differential equations of the form

$$\tilde{\Delta}_\nu\left(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}^{(n)}\right) = 0, \qquad ,\nu = 1, .., \ell,$$

still involving parametric variables $\hat{\mathbf{x}}$. If G is a symmetry group for Δ, this resulting system will in fact always be equivalent to a system of equations denoted by

$$(\Delta/G)_\nu(\mathbf{y}, \mathbf{v}^{(n)}) = 0, \qquad \nu = 1, ..., \ell,$$

which are independent of the parametric variables, and thus constitute a genuine system of differential equations for $\mathbf{v}$ as a function of $\mathbf{y}$. This is the reduced system Δ/G for the $G-$invariant solutions to the system Δ. Every solution $\mathbf{v} = \mathbf{h}(\mathbf{y})$ of Δ/G will corresponds, via (5.3), to a $G-$invariant solution to Δ, and moreover every $G-$invariant solution can be constructed in this manner.

5.2. **Group-Invariant Solution of The Equation (2.1).** In this part we will find all group-invariant solution correspond to Lie symmetries of the equation (2.1).

Consider the symmetry $\mathbf{v}_1$ which is translation on the parameter t. Theorem 4.2 shows that the invariant of this inifinetimal is $p = v(y)$ where $y = x$. Substituiting these new variables to (2.1), we obtain the reduced equation

$$\frac{1}{2}\sigma^2\frac{d^2v}{dy^2} + \alpha r\frac{dv}{dy} + \alpha v = 0. \tag{5.5}$$

TABLE 2. Group-Invarint Solutions

Symmetry	Invariants	Group − Invarint Solution
$\mathbf{v}_1$	$x=y, p=v$	$p=e^{-\frac{\alpha}{\sigma^2}x^2}\left(\operatorname{erf}\left(\frac{\sqrt{-\alpha}}{\sigma}x\right)C_1+C_2\right)$
$\mathbf{v}_2$		Any solution is a group − invariant solution.
$\mathbf{v}_3$	$t=y, p=v$	$p=C_1e^{\alpha t}$
$\mathbf{v}_4$	$x=y, p=ve^{\frac{\alpha}{\sigma^2}y^2}$	$p=C_1e^{-\frac{\alpha}{\sigma^2}x^2}$
$\mathbf{v}_5$	$t=y, p=vx$	$p=e^{\alpha t}\left(C_1+C_2xe^{\alpha t}\right)$
$\mathbf{v}_6$	$x=e^{\alpha(t-y)}, p=ve^{-\frac{\alpha^3}{\sigma^2}}(t-y)^2$	$p=e^{-\frac{\alpha}{\sigma^2}}\left(C_1+C_2xe^{-\alpha t}\right)$

The solution of the equation (5.5) is

$$v=e^{-\frac{\alpha}{\sigma^2}y^2}\left(\operatorname{erf}\left(\frac{\sqrt{-\alpha}}{\sigma}y\right)C_1+C_2\right), \tag{5.6}$$

where erf is the error function. Replacing the variables in the (5.6) leads us to an group-invariant solution

$$p=e^{-\frac{\alpha}{\sigma^2}x^2}\left(\operatorname{erf}\left(\frac{\sqrt{-\alpha}}{\sigma}x\right)C_1+C_2\right). \tag{5.7}$$

A similar procedure gives another group-invariant solution for the equation (2.1). The results come in the Table 2.

5.3. **Some New Solutions for the Equation (2.1).** In this part we use the general form of the solutions (15-20) arises from one-parameter groups to find new solutions fo the equation (2.1). It is noteworthy we only work on the solution (5.7) an its obtained solutions. The same method would be applied for all solutions specially group-invariant solution.

1. For the case (15), if (5.7) be a considered solution then no new solution is obtained.
2. For the case (16), if (5.7) be a considered solution then

$$p=e^{-\frac{\alpha}{\sigma^2}x^2+\varepsilon}\left(\operatorname{erf}\left(\frac{\sqrt{-\alpha}}{\sigma}x\right)C_1+C_2\right),$$

is a new solution.

3. For the case (17), if (5.7) be a considered solution then

$$p=\exp\left\{-\frac{\alpha}{\sigma^2}\left(x-e^{-\alpha t\varepsilon}\right)^2\right\}\left(\operatorname{erf}\left(\frac{\sqrt{-\alpha}}{\sigma}x\right)C_1+C_2\right),$$

is a new solution.

4. For the case (18), if (5.7) be a considered solution then

$$p=-\frac{2\alpha\varepsilon}{\sigma^2}e^{\alpha t}\left(x-e^{\alpha t}\varepsilon t\right)\exp\left\{-\frac{\alpha}{\sigma^2}\left(x-e^{\alpha t\varepsilon}\right)^2\right\}$$
$$\times\left(\operatorname{erf}\left(\frac{\sqrt{-\alpha}}{\sigma}x\right)C_1+C_2\right),$$

is a new solution.

5. For the case (19), if (5.7) be a considered solution then

$$\begin{aligned} p = &\left(1 - \alpha \exp\left\{-\alpha(t + e^{-2\alpha t}\varepsilon)\right\}\right) \\ &\times \exp\left\{-\frac{\alpha}{\sigma^2}\left(\frac{x^2}{(1 - \alpha\exp\{-\alpha(t + e^{-2\alpha t}\varepsilon)\})^2}\right)\right\} \\ &\times \left(\operatorname{erf}\left(\frac{\sqrt{-\alpha}}{\sigma}\frac{x}{1 - \alpha\exp\{-\alpha(t + e^{-2\alpha t}\varepsilon)\}}\right) C_1 + C_2\right), \end{aligned}$$

is a new solution.

6. For the case (20), if (5.7) be a considered solution then

$$\begin{aligned} p = &\left(\frac{x}{1 + \alpha\exp\left\{2\alpha\left(t + e^{2\alpha t}\varepsilon\right)\right\}}\right)^2 \varepsilon \\ &\times \exp\left\{-\frac{\alpha}{\sigma}\left(\frac{x^2}{(1 + \alpha\exp\{2\alpha(t + e^{2\alpha t}\varepsilon)\})}\right)\right\} \\ &\times \left(\operatorname{erf}\left(\frac{\sqrt{-\alpha}}{\sigma}\frac{x}{(1 + \alpha\exp\{2\alpha(t + e^{2\alpha t}\varepsilon)\})}\right) C_1 + C_2\right) \\ &-\frac{2\alpha^2}{\sigma^2}\exp\{2\alpha(t + e^{2\alpha t}\varepsilon)\}, \end{aligned}$$

is a new solution.

Conslusion

As we know partial differential equations plays a vast roll in sciences such as financial mathematics. In this paper we used a geometrical method called Lie theory of differential equations to find exact solutions for a kind of Fokker-Plank equation. Doing as well as the last section we can find another new solutions. Some packages like Maple and Mathematica is usefull for our mail goal.

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