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Interval fractional integro-differential equations without singular kernel by fixed point in partially ordered sets

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Abstract This work is devoted to the study of global solution for initial value problem of interval fractional integro-differential equations involving Caputo-Fabrizio fractional derivative without singular kernel admitting only the existence of a lower solution or an upper solution. Our method is based on fixed point in partially ordered sets. In this study, we guaranty the existence of special kind of interval H-difference that we will be faced it under weak conditions. The method is illustrated by an example.

Keywords. Interval fractional integro-differential equations, Caputo-Fabrizio fractional derivative, Method of upper or lower solutions, Fixed point in partially ordered sets.

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1. Introduction

The theory of fractional differential equations has attracted more attention in the last decades because of their applications in different fields of sciences (see e.g. [12, 16]). This theory is making use of the derivatives of fractional orders such as Riemann-Liouville, Caputo, Grünwal-Letnikov and etc. Recently Caputo and Fabrizio in [9] have presented the new definition of fractional derivative in where they have replaced a nonsingular kernel $\exp(-\frac{q}{1-q}(t-s))$ instead of the singular kernel $(t-s)^{-q}$. We believe that the large number of fractional derivatives does not constitute a disadvantage, since they can be used in different models which provide the best reflection of the behavior of the system. In many simultaneously occurring processes in modeling of the real world phenomena to obtain data, the field observations are needed. The modeling of a dynamical system based on the field observations becomes uncertain and vagueness or fuzziness, which is inherent in the systems behavior rather than being purely random or deterministic. The study of interval and fuzzy differential equations is an area of mathematics that has recently received a lot of attention (see e.g. [4, 11, 17, 18, 19, 20]). Recently, there are some papers dealing with the existence of solution for nonlinear set valued and fuzzy fractional differential equations whose methods are based on the monotone method, the method of upper and lower solutions and fixed point theorems [1, 2, 5, 6, 7, 10, 17]. Among of them, we can find results on existence of solution for fuzzy differential equations in presence of

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both lower and upper solutions relative to the problem considered. The contraction mapping theorem and the abstract monotone iterative technique are well known and are applicable to a variety of situations. Recently, there is a fixed point theorem to weaken the requirement on the contraction by considering metric spaces endowed with partial order. The existence of a unique fixed point is based on assuming that the operator considered is monotone in such a setting [14, 15, 21].

In this study, we consider the following interval fractional integro-differential equation involving Caputo-Fabrizio fractional derivative of order $0 < q < 1$

$$
\begin{aligned}
&{}^{CF}D^{q}u(t) = g(t) + \int_0^t f(t,s,u(s))ds, \qquad \forall\, t \in J, \\
&u(0) = u_0 \in \mathcal{K},
\end{aligned} \tag{1.1}
$$

where $J = [0, b]$, $g \in C(J, \mathcal{K})$, $f \in C(J \times J \times \mathcal{K}, \mathcal{K})$ and ${}^{CF}D^{q}u$ denotes interval Caputo-Fabrizio fractional derivative of order q.

Here, we consider just only a lower solution or an upper solution for the above interval initial value problem and use fixed point in partially ordered sets to prove the existence results. This approach allows us to weaken the assumptions on the function f where Problem (1.1) is under consideration. It is worth noting that the strategies of this work is based on overcoming some difficulties mentioned below. Firstly one of the main concerns related to interval and fuzzy differential equations is the existence of H-differences appeared in the problem being investigated. In this study, we guaranty the existence of such interval H-differences under weak conditions. Secondly applying such fixed point gives us a local solution. Here we divide the interval $[0, b]$ to subintervals and use the fixed point on the subintervals and then make the global solution on entire $[0, b]$ by using the obtained local solutions.

The paper is organized as follows. In Section 2, we introduce some basic knowledge for interval number and interval valued functions and state fixed point theorem in the partially ordered set. In Section 3, we state the main problem and define concept of Caputo-Fabrizio fractional derivative for interval space. Moreover, in this section we define three kind of solutions for our problem. We devote subsections 3.1-3.3 to the main theorems of existence and uniqueness of solutions for Problem (1.1).

2. Preliminaries

In this section, we gather together some definitions and results from the literature, which we will use throughout this paper.

$\mathcal{K}$ denotes the spaces of nonempty compact and convex sets of the real line $\mathbb{R}$. For $A \in \mathcal{K}$, we have $A = [a^-, a^+]$ where $a^- \leq a^+$. We denote the width of an interval A by $len(A) = a^+ - a^-$. Given two intervals $A, B \in \mathcal{K}$ and $k \in \mathbb{R}$, addition and scalar multiplication are defined by $A + B = [a^- + b^-, a^+ + b^+]$ and

$$
kA = \begin{cases} [ka^-, ka^+], & k \geq 0 \\ [ka^+, ka^-], & k < 0. \end{cases}
$$

Difference is defined as $A - B = A + (-1)B$. It is well known that addition is associative and commutative and with neutral element $\{0\}$. If $A, B \in \mathcal{K}$, and if there exists a unique interval $C \in \mathcal{K}$ such that $B + C = A$, then C is called the H-difference of A, B and is denoted by $A \ominus B$ (see e.g. [19]). For intervals $A, B \in \mathcal{K}$ the Hausdorff distance is defined as usual by

$$D(A,B) = max\{|a^- - b^-|, |a^+ - b^+|\}.$$

The following properties of distance D are well-known (see e.g. [22])
For all $A, B, C, E \in \mathcal{K}$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} &D(A+B, A+C) = D(B,C), \\ &D(\lambda A, \lambda B) = |\lambda| D(A,B), \quad \forall \lambda \in \mathbb{R}, \\ &D(A+B, C+E) \le D(A,C) + D(B,E), \end{aligned}$$

and $(\mathcal{K}, D)$ is a complete metric space.
We recall that if $F : [a,b] \to \mathcal{K}$ is an interval-valued function such that $F(t) = [f^-(t), f^+(t)]$, then $\lim_{t\to t_0} F(t)$ exists, if and only if $lim_{t\to t_0} f^-(t)$ and $lim_{t\to t_0} f^+(t)$ exist as finite numbers. In this case, we have

$$\lim_{t\to t_0} F(t) = [\lim_{t\to t_0} f^-(t), \lim_{t\to t_0} f^+(t)].$$

In particular, F is continuous if and only if f^- and f^+ are continuous.

Definition 2.1. (See e.g. [19]) Let $F : (a,b) \to \mathcal{K}$ and $x_0 \in (a,b)$. We say f is strongly generalized differentiable at x_0, if there exists an element $F'(x_0) \in \mathcal{K}$, such that for all $h > 0$ sufficiently small,
(i) there exist $F(x_0+h) \ominus F(x_0), F(x_0) \ominus F(x_0-h)$ and

$$\lim_{h\searrow 0} \frac{F(x_0+h) \ominus F(x_0)}{h} = \lim_{h\searrow 0} \frac{F(x_0) \ominus F(x_0-h)}{h} = F'(x_0),$$

or (ii) there exist $F(x_0) \ominus F(x_0+h), F(x_0-h) \ominus F(x_0)$ and

$$\lim_{h\searrow 0} \frac{F(x_0) \ominus F(x_0+h)}{-h} = \lim_{h\searrow 0} \frac{F(x_0-h) \ominus F(x_0)}{-h} = F'(x_0),$$

or (iii) there exist $F(x_0+h) \ominus F(x_0), F(x_0-h) \ominus F(x_0)$ and

$$\lim_{h\searrow 0} \frac{F(x_0+h) \ominus F(x_0)}{h} = \lim_{h\searrow 0} \frac{F(x_0-h) \ominus F(x_0)}{-h} = F'(x_0),$$

or (iv) there exist $F(x_0) \ominus F(x_0+h), F(x_0) \ominus F(x_0-h)$ and

$$\lim_{h\searrow 0} \frac{F(x_0) \ominus F(x_0+h)}{-h} = \lim_{h\searrow 0} \frac{F(x_0) \ominus F(x_0-h)}{h} = F'(x_0).$$

Remark 2.2. We say that a function is (i)- differentiable if it is differentiable as the case (i) of the definition above, etc.

Lemma 2.3. *(See [20].) Let $F : (a,b) \to \mathcal{K}$ be a strongly generalized differentiable interval function such that $F(t) = [f^-(t), f^+(t)]$.*

(1) *If F is (i)-differentiable, then f^-, f^+ are differentiable functions and*
$F'(t) = [(f^-)'(t), (f^+)'(t)]$.
(2) *If F is (ii)-differentiable, then f^-, f^+ are differentiable functions and*
$F'(t) = [(f^+)'(t), (f^-)'(t)]$.

Throughout this paper, we consider $J = [a, b]$ and we shall use the notation

$$C(J, \mathcal{K}) = \{\ F : J \to \mathcal{K}|\ F \text{ is continuous}\},$$

where the continuity is one-side at endpoints a, b. Also for $k = 1, 2$

$$C^1_{(i)}(J, \mathcal{K}) = \{F : J \to \mathcal{K}|\ F \ \text{ is } \ \text{(i)-differentiable and } F' \text{ is continuous}\},$$

$$C^1_{(ii)}(J, \mathcal{K}) = \{F : J \to \mathcal{K}|\ F \ \text{ is } \ \text{(ii)-differentiable and } \ F' \text{ is continuous}\}.$$

Define for $F, G \in C(J, \mathcal{K})$

$$H(F, G) = \sup_{t \in J} D(F(t), G(t)).$$

Remark 2.4. $(C(J, \mathcal{K}), H)$ is a metric space.

Let $F : [a, b] \to \mathcal{K}$ be an interval-valued function such that $F(t) = [f^-(t), f^+(t)]$ and f^- and f^+ are measurable and Lebesgue integrable on $[a, b]$. Then we define $\int_a^b F(t)dt$ by

$$\int_a^b F(t)dt = \left[\int_a^b f^-(t)dt, \int_a^b f^+(t)dt\right],$$

and we say that F is Lebesgue integrable on [a,b].

Lemma 2.5. *(See [22].) Let $F : [a, b] \to \mathcal{K}$ be a continuous interval-valued function. Then $G(x) = \int_a^x F(t)dt$ is (i)-differentiable and we have $G'(x) = F(x)$.*

Lemma 2.6. *(See [20].) Let $F : [a, b] \to \mathcal{K}$ be (i)-differentiable and C is an interval.Then $C + F$ is (i)-differentiable and $C \ominus f$ is (ii)-differentiable.*

Throughout this work, we use the following partial ordering(see e.g. [18]).

Definition 2.7. Suppose $x,\ y \in \mathcal{K}$. We say that $x \preceq y$ if and only if

$$x^- \leq y^-, \qquad \text{and} \qquad x^+ \leq y^+.$$

Let $h_1,\ h_2 \in C(J, \mathcal{K})$ be two interval functions, we say that $h_1 \preceq h_2$ if $h_1(t) \preceq h_2(t)$ for $t \in J$ $(j = 1, 2)$.

We recall some properties on the partial ordering $\preceq$ in space of interval functions, which are useful to our procedure.

Lemma 2.8. *(See [18].) Let $x, y, z, w \in \mathcal{K}$ and $c \in \mathbb{R}$, $c > 0$, $j = 1, 2$.*

- *$x = y$ if and only if $x \preceq y$ and $x \geq_j y$.*
- *If $x \preceq y$, then $x + z \preceq y + z$.*
- *If $x \preceq y$ and $z \preceq w$, then $x + z \preceq y + w$.*
- *If $x \preceq y$, then $cx \preceq cy$.*

Lemma 2.9. *(See [18].) Let $g,\ h \in C(J, \mathcal{K})$ and $g \preceq h$, then*

$$\int_a^t g(s)ds \preceq \int_a^t h(s)ds, \qquad \forall t \in J.$$

Definition 2.10. Let $(X, \leq)$ be a partially ordered set and $f : X \to X$. We say that f is monotone nondecreasing in x if for any $x, y \in X$,

$$x \preceq y \Rightarrow f(x) \preceq f(y)$$

and is monotone nonincreasing in y, if

$$x \preceq y \Rightarrow f(x) \succeq f(y).$$

Throughout this study we will use the following fixed point theorem in the partially ordered set.

Theorem 2.11. *(See [14, 15].) Let $(X, \leq)$ be a partially ordered set and suppose that d be a metric on X such that (X, d) is a complete metric space. Furthermore, let $T : X \to X$ be a monotone nondecreasing mapping such that*

$$\exists\ 0 \leq k < 1 \ni d(T(x), T(y)) \leq kd(x, y), \qquad \forall x \geq y.$$

Suppose that either T is continuous or X is such that if $\{x_n\} \to x$ is a nondecreasing (or respectively nonincreasing) sequence in X, then $x_n \leq x$ (or respectively $x_n \geq x$) for every $n \in \mathbb{N}$. If there exists $x_0 \in X$ comparable to $T(x_0)$, then T has a fixed point $\bar{x}$ and $\lim_{n\to\infty} T^n(x_0) = \bar{x}$.

The following lemma shows that a part of assumptions of Theorem 2.11 by considering $X = C(J, \mathcal{K})$ is satisfied.

Lemma 2.12. *If a nondecreasing (or nonincreasing) sequence $f_n \to f$ in $C(J, \mathcal{K})$, then $f_n \preceq f$ (or $f_n \succeq f$), $\forall n$ respectively.*

Proof. Since f_n is nondecreasing sequence in $C(J, \mathcal{K})$, $f_n(t)$ is nondecreasing sequence in $\mathcal{K}$ for $t \in J$. Also we have

$$f_1^-(t) \preceq ... \preceq f_n^-(t) \preceq ....$$

Hence $f_n^-(t)$ is a nondecreasing sequence that converges to $f^-(t)$ in $\mathbb{R}$. Therefore $f_n^-(t) \preceq f^-(t)$ for every n. Similarly we conclude $f_n^+(t) \preceq f^+(t)$ for every n. Thus $f_n \preceq f$ for every n. Also, the similar result can be conclude for nonincreasing function. □

The following lemma guaranties the existence of special kind of H-difference under some conditions that we will be faced it.

Lemma 2.13. *Let $x \in \mathcal{K}$ and $f : [a, b] \to \mathcal{K}$ be continuous with respect to t. If $x \in \mathcal{K} \setminus \mathbb{R}$ i.e. $x^- < x^+$ or if $x \in \mathbb{R}$ and $f(t) \in \mathbb{R}$ for all $t \in [a, b]$, then there exists $h > a$ such that the H-difference*

$$x \ominus \int_a^t f(s)ds,$$

exists for any $t \in [a, h]$.

Proof. The proof is given in [8] in fuzzy space. We give it for our special case in intervals space. In order to prove the existence of $x \ominus \int_a^t f(s)ds$, we have to prove that $[x^- - \int_a^t f^-(s)ds, x^+ - \int_a^t f^+(s)ds]$ is an interval. Therefore we have to check

$$\int_a^t f^+(s)ds - \int_a^t f^-(s)ds \leq x^+ - x^-.$$

The above condition is equivalent to

$$\int_a^t len(f(s))ds \leq len(x).$$

By continuity of f, there exists $M > 0$ such that $len(f(t)) \leq M$ for all $t \in [a, b]$. Now suppose $x \in \mathcal{K} \setminus \mathbb{R}$ and $t \in [a, a + \frac{len(x)}{M}]$, thus we have

$$\int_a^t len(f(s))ds \leq M(t-a) \leq len(x).$$

If $x, f(t) \in \mathbb{R}$ for all $t \in [a, b]$, then $len(x) = len(f(t)) = 0$ for all $t \in [a, b]$. $\square$

3. Interval Caputo-Fabrizio fractional integro-differential equations

To avoid of complications and for reduction of sentences, we investigate the following initial value problem for interval fractional integro-differential equation without singular kernel of order $0 < q < 1$

$$\begin{aligned} &{}^{CF}D^q u(t) = \int_0^t f(t, s, u(s))ds, \qquad \forall\, t \in J, \\ &u(0) = u_0 \in \mathcal{K}, \end{aligned} \tag{3.1}$$

where $J = [0, b]$, $f \in C(J \times J \times \mathcal{K}, \mathcal{K})$ and ${}^{CF}D^q u$ denotes interval Caputo-Fabrizio fractional derivative of order q introduced in below. We note that the method for Problem (1.1) is similar to Problem (3.1).

Definition 3.1. (Interval Caputo-Fabrizio fractional differential of order q)

(i) Let $u \in C^1_{(i)}(J, \mathcal{K})$ and $0 < q < 1$. We say that u is (i)-Caputo-Fabrizio fractional differentiable of order q at $t \in J$ if there exists an ${}^{CF}D^q u(t) \in \mathcal{K}$ such that

$${}^{CF}D^q u(t) = \frac{1}{1-q} \int_0^t \exp(-\frac{q}{1-q}(t-s))u'(s)ds.$$

(ii) Let $u \in C^1_{(ii)}(J, \mathcal{K})$ and $0 < q < 1$. We say that u is (ii)-Caputo-Fabrizio fractional differentiable of order q at $t \in J$ if there exists an ${}^{CF}D^q u(t) \in \mathbb{R}_{\mathcal{F}}$ such that

$${}^{CF}D^q u(t) = \frac{1}{1-q} \int_0^t \exp(-\frac{q}{1-q}(t-s))u'(s)ds.$$

The purpose of the current section is finding solutions $u \in C^1(J, \mathcal{K})$ of Problem (3.1), which are defined as below. We give the concept of a solution for Problem (3.1), in ways that the solutions can have different kinds of strongly generalized differentiability on subintervals of some partition into J.

Definition 3.2. We say that $u \in C^1_{(i)}(J, \mathcal{K})$ is (i)-solution of Problem(3.1), if u satisfies Problem (3.1).

Definition 3.3. We say that $u \in C^1_{(ii)}(J, \mathcal{K})$ is (ii)-solution of Problem (3.1), if u satisfies Problem (3.1).

Definition 3.4. We say that u is (ii,i)-solution of Problem (3.1), if there exists $c \in (0, b)$ such that u is (ii)-solution on $[0, c]$ and (i)-solution on $(c, b]$.

Remark 3.5. Authors in [3] have called (i)-solution and (ii)-solution as proper solutions and also (ii,i)-solution as mixed solution.

Lemma 3.6. *Problem (3.1) is equivalent to one of the following integral equations systems*

$$(F_1) \qquad u(t) = u_0 + (1-q)\int_0^t f(t,s,u(s))ds + q\int_0^t\int_0^s f(s,r,u(r))drds,$$

if u is (i)-Caputo-Fabrizio fractional differentiable of order q on J.

$$(F_2) \quad u(t) = u_0 \ominus (-1)\left((1-q)\int_0^t f(t,s,u(s))ds + q\int_0^t\int_0^s f(s,r,u(r))drds\right),$$

if u is (ii)-Caputo-Fabrizio fractional differentiable of order q on J.

$(F_3) \quad u(t) =$

$$\begin{cases} u_0 \ominus (-1)\cdot((1-q)\int_0^t f(t,s,u(s))ds + q\int_0^t\int_0^s f(s,r,u(r))drds), & t\in[0,c^*], \\ u(c^*) + \int_{c^*}^t f(t,s,u(s))ds + \int_{c^*}^t\int_0^s f(s,r,u(r))drds, & t\in[c^*,b], \end{cases}$$

if u is both (ii)-differentiable on $[0, c^*]$ and (i)-differentiable on $[c^*, b]$.

Proof. Let $u(t) = [u^-(t), u^+(t)]$ be the (i)-solution of Problem (3.1). u is (i)-differentiable, then by Lemma 2.3, $u'(t) = [{u^-}'(t), {u^+}'(t)]$. Therefore we have for all $t \in J$

$${}^{CF}D^q u^\pm(t) = \frac{1}{1-q}\int_0^t \exp(-\frac{q}{1-q}(t-s)){u^\pm}'(s)ds = \int_0^t (f(t,s,u(s)))^\pm ds,$$

and also $(u(0))^\pm = u_0^\pm$. By applying Laplace transform for the above crisp problem (see e.g. [13]), we have for all $t \in J$

$$u^\pm(t) = u_0^\pm + (1-q)\int_0^t (f(t,s,u(s)))^\pm ds + q\int_0^t\int_0^s (f(s,r,u(r)))^\pm drds.$$

Then we arrive at integral equation (F_1). To prove the inverse, by Lemmas 2.5 and 2.6, we conclude u in (F_1) belongs to $C^1_{(i)}(J, \mathcal{K})$ and clearly satisfies Problem (3.1).
Now let $u(t) = [u^-(t), u^+(t)]$ be the (ii)-solution of Problem (3.1). u is (ii)-differentiable, then by Lemma 2.3, $u'(t) = [{u^+}'(t), {u^-}'(t)]$. Therefore we have for all $t \in J$

$${}^{CF}D^q u^\pm(t) = \frac{1}{1-q}\int_0^t \exp(-\frac{q}{1-q}(t-s)){u^\mp}'(s)ds = \int_0^t (f(t,s,u(s)))^\pm ds,$$

and also $(u(0))^{\pm} = u_0^{\pm}$. Applying Laplace transform, we have for all $t \in J$

$$u^{\mp}(t) = u_0^{\mp} + (1-q)\int_0^t (f(t,s,u(s)))^{\pm} ds + q\int_0^t \int_0^s (f(s,r,u(r)))^{\pm} drds.$$

Then we arrive at the equation (F_2). Inversely, by Lemmas 2.5 and 2.6, u obtained by (F_2) is in $C^1_{(ii)}(J,\mathcal{K})$ and it is clear that u satisfies Problem (3.1). The rest of the proof can be conclude in the similar trend with Lemma 3.2 in [3]. $\square$

Remark 3.7. The equivalence between two equations means that any solution of an equation is a solution for the other one. Moreover, the continuous solution, obtained of integral equations (F_1) is corresponding to the (i)-solution, (F_2) is corresponding to the (ii)-solutions and (F_3) is corresponding to the (ii,i)-solution of Problem (3.1).

3.1. **Existence of (i)-solution.** Now we define the nonlinear mappings $\mathcal{A} : C(J,\mathcal{K}) \to C(J,\mathcal{K})$, related to (F_1), which plays a main role in our discussion, as following

$$[\mathcal{A}\phi](t) = u_0 + (1-q)\int_0^t f(t,s,\phi(s))ds + q\int_0^t \int_0^s f(s,r,\phi(r))drds, \tag{3.2}$$

Now we define upper and lower solution for Problem (3.1) as following:

Definition 3.8. Let $\underline{u}, \bar{u} \in C(J,\mathcal{K})$, we say that
(a) $\underline{u}$ is a lower solution for Problem (3.1) if

$$\underline{u}(t) \preceq [\mathcal{A}\underline{u}](t), \qquad t \in J,$$

(b) $\overline{u}$ is an upper solution for Problem (3.1) if

$$\overline{u}(t) \succeq [\mathcal{A}\overline{u}](t), \qquad t \in J.$$

In the following theorem, we state our main results. We apply fixed point Theorem 2.11 to prove the existence and uniqueness of global solution belonging to $C^1_{(i)}(J,\mathcal{K})$ for the interval initial value Problem (3.1) by the existence of just a lower solution or an upper solution.

Theorem 3.9. *Consider Problem (3.1) with f continuous and suppose f is nondecreasing in the last argument. Let exists a constant real number $l > 0$ such that*

$$D(f(t,s,x), f(t,s,y)) \le lD(x,y), \qquad \forall t,s \in J,$$

for $x \succeq y$. Then the existence of a lower solution $\underline{u}$ (or an upper solution $\overline{u}$) for Problem (3.1) provides the existence of a fixed point for $\mathcal{A}$ like u, and consequently (i)-solution to Problem (3.1) on $[0,b]$. Also, $\lim_{n\to\infty} \mathcal{A}^n(\underline{u}) = u$ (or $\lim_{n\to\infty} \mathcal{A}^n(\overline{u}) = u$). Moreover, if $w \in C(J,\mathcal{K})$ is another fixed point of $\mathcal{A}$ such that is comparable to $\underline{u}$, then $u = w$.

Proof. Since by Lemma 3.6, Problem (3.1) is equivalent to (F_1), we prove that the mapping $\mathcal{A}$ has a unique fixed point under assumption the existence a lower solution $\underline{u}$ for Problem (3.1). Because of similarity we omit the proof under assumption the existence of upper solution. Now we check that hypotheses in Theorem 2.11 are satisfied.

We consider $X = C(J, \mathcal{K})$ that is partially ordered set by the following order relation For $g, f \in C(J, \mathcal{K})$,

$$g \preceq f \Leftrightarrow g(t) \preceq f(t), \ \forall t \in J.$$

Since f is nondecreasing in its last arguments, the mapping $\mathcal{A}$, defined by (3.2), is nondecreasing on J. Obviously there exists $c > 0$ such that $\frac{b}{c} = N \in \mathbb{N}$ and $lc + l\frac{c^2}{2} < 1$. Firstly We consider the interval $[0, c]$. For $\phi \geq_j \psi$, we have

$$\begin{aligned} D([\mathcal{A}\phi](t), [\mathcal{A}\psi](t)) &\leq (1-q)\int_0^t D(f(s,\phi(s)), f(s,\psi(s)))ds \\ &+ q\int_0^t\int_0^s D(f(s,r,\phi(r)), f(s,r,\psi(r)))drds \\ &\leq l(1-q)cH(\phi,\psi) + lq\frac{c^2}{2}H(\phi,\psi). \end{aligned}$$

Then we have

$$H(\mathcal{A}\phi, \mathcal{A}\psi) \leq LH(\phi, \psi), \tag{3.3}$$

where $L = l(1-q)c + lq\frac{c^2}{2} < 1$. Applying Theorem 2.11, $\mathcal{A}$ has a fixed point $\mathbf{u} \in C([0,c], \mathcal{K})$ and $\lim_{n\to\infty} \mathcal{A}^n(\underline{u}) = \mathbf{u}$ (or $\lim_{n\to\infty} \mathcal{A}^n(\overline{u}) = \mathbf{u}$). Now suppose $w \in C([0,c], \mathcal{K})$ is another fixed point of $\mathcal{A}$ such that is comparable to $\underline{u}$. It means that $\underline{u} \preceq w$ or $w \preceq \underline{u}$. We claim that $H(\mathbf{u}, w) = 0$. Employing the nondecreasing property of the mapping $\mathcal{A}$, along with Lemma 2.12 and $\underline{u} \preceq \mathcal{A}$, we can infer $\underline{u} \preceq \mathbf{u}$. Then $\mathcal{A}^n\underline{u}$ is comparable to $\mathcal{A}^n\mathbf{u} = \mathbf{u}$ and $\mathcal{A}^n w = w$ for $n = 0, 1, 2, \ldots$. Utilizing (3.3) we have

$$\begin{aligned} H(\mathbf{u}, w) = H(\mathcal{A}^n\mathbf{u}, \mathcal{A}^n w) &\leq H(\mathcal{A}^n\mathbf{u}, \mathcal{A}^n\underline{u}) + H(\mathcal{A}^n w, \mathcal{A}^n\underline{u}) \\ &\leq L^n H(\mathbf{u}, \underline{u}) + L^n H(\underline{u}, w). \end{aligned}$$

Since $L < 1$, the right-hand side of above equation converges to zero as $n \to \infty$. Then $\mathcal{H}(\mathbf{u}, w) = 0$. It means that the fixed point is unique on $[0, c]$. Now by considering $\mathbf{u}$ as a fixed point for $\mathcal{A}$ on the interval $[0, c]$, we define another mapping on the interval $[c, 2c]$ as follows:

$$\begin{aligned} [\mathcal{T}\phi](t) &= \mathbf{u}(c) + q\int_c^t\int_0^c f(s,r,\mathbf{u}(r))drds + (1-q)\int_c^t f(t,s,\phi(s)ds \\ &+ q\int_c^t\int_c^s f(s,r,\phi(r))drds. \end{aligned}$$

Since f is nondecreasing in its last argument, the mapping $\mathcal{T} : C([c, 2c], \mathcal{K}) \to C([c, 2c], \mathcal{K})$ is nondecreasing. Now we will show that $\underline{u}(t) \preceq [\mathcal{T}\underline{u}](t)$ (or $\overline{u}(t) \succeq [\mathcal{T}\overline{u}](t)$) for $t \in [c, 2c]$. Due the fact that $\underline{u}$ is a lower solution of Problem (3.1) for

$t \in [0, b]$ and $\underline{u} \preceq \mathbf{u}$ for $t \in [0, c]$, we have

$$\begin{aligned}
\underline{u} &\preceq u_0 + (1-q)\int_0^t f(t,s,\underline{u}(s))ds + q\int_0^t\int_0^s f(s,r,\underline{u}(r))drds,\\
&= u_0 + (1-q)\int_0^c f(t,s,\underline{u}(s))ds + q\int_0^c\int_0^s f(s,r,\underline{u}(r))drds\\
&\quad +(1-q)\int_c^t f(t,s,\underline{u}(s))ds + q\int_c^t\int_0^s f(s,r,\underline{u}(r))drds,\\
&\preceq u_0 + (1-q)\int_0^c f(t,s,\mathbf{u}(s))ds + q\int_0^c\int_0^s f(s,r,\mathbf{u}(r))drds\\
&\quad +(1-q)\int_c^t f(t,s,\underline{u}(s))ds + q\int_c^t\int_0^s f(s,r,\underline{u}(r))drds,\\
&= \mathbf{u}(c) + (1-q)\int_c^t f(t,s,\underline{u}(s))ds + q\int_c^t\int_0^s f(s,r,\underline{u}(r))drds,\\
&= \mathbf{u}(c) + (1-q)\int_c^t f(t,s,\underline{u}(s))ds + q\int_c^t\int_0^c f(s,r,\underline{u}(r))drds\\
&\quad +q\int_c^t\int_c^s f(s,r,\underline{u}(r))drds,\\
&\preceq \mathbf{u}(c) + (1-q)\int_c^t f(t,s,\underline{u}(s))ds + q\int_c^t\int_0^c f(s,r,\mathbf{u}(r))drds\\
&\quad +q\int_c^t\int_c^s f(s,r,\underline{u}(r))drds,\\
&= [\mathcal{T}\underline{u}](t).
\end{aligned}$$

For $\phi \preceq \psi$, we conclude

$$\begin{aligned}
D([\mathcal{T}\phi](t),[\mathcal{T}\psi](t)) &\leq (1-q)\int_c^t D(f(t,s,\phi(s)),f(t,s,\psi(s)))ds\\
&+ q\int_c^t\int_c^s D(f(s,r,\phi(r)),f(s,r,\psi(r)))drds\\
&\leq l(1-q)cH(\phi,\psi) + lq\frac{c^2}{2}H(\phi,\psi).
\end{aligned}$$

Then we have

$$H(\mathcal{T}\phi,\mathcal{T}\psi) \leq LH(\phi,\psi),$$

where $L = l(1-q)c + lq\frac{c^2}{2}\} < 1$. All the conditions in Theorem 2.11 are satisfied, therefore the mapping $\mathcal{T}$ has a fixed point $\mathbf{v} \in C([c,2c],\mathcal{K})$ and $\lim_{n\to\infty}\mathcal{T}^n(\underline{u}) = \mathbf{v}$. If we suppose $w \in C([c,2c],\mathcal{K})$ is another fixed point of $\mathcal{T}$ such that is comparable to $\underline{u}$ on $[c,2c]$, then it is clear that $H(\mathbf{v},w) = 0$.
Obviously u as defined

$$u = \begin{cases} \mathbf{u}, & t \in [0,c],\\ \mathbf{v}, & t \in [c,2c],\end{cases}$$

is a fixed point of $\mathcal{A}$ defined by (3.2) on $[0,2c]$. By Lemma 2.6, u is (i)-differentiable on $[0,2c]$. In the same trend we can make a fixed point of $\mathcal{A}$ defined by (3.2) on $[0,Nc]=[0,b]$. Let $u\in C(J,\mathcal{K})$ is a fixed point of $\mathcal{A}$ where $J=[0,b]$. Therefore u is a solution of integral equation (F_1). By Remark 3.7, we can conclude u is a (i)-solution of Problem (3.1).
Now suppose $w\in C(J,\mathcal{K})$ is another fixed point of $\mathcal{A}$ such that is comparable to $\underline{u}$ on $J=[0,b]$. It is clear that $H(u,w)=0$.

□

3.2. **Existence of (ii)-solution.** Let $x_0\in\mathcal{K}$. We denote by $\bar{B}(x_0)=\{x\in\mathcal{K}: len(x)\leq len(x_0)\}$, a closed subset in $\mathcal{K}$. Now we are in a situation to define the nonlinear mappings $\mathcal{B}$ (related to (F_2)), which plays a main role in our discussion, as following

$$[\mathcal{B}\phi](t)=u_0\ominus(-1)((1-q)\int_0^t f(t,s,\phi(s))ds+q\int_0^t\int_0^s f(s,r,\phi(r))drds), \quad (3.4)$$

where $t\in J$. In general the $\mathcal{B}:C(J,\mathcal{K})\to C(J,\mathcal{K})$ is not well-defined. The following lemma guaranties the existence of H-differences involving in the mapping $\mathcal{B}$.

Lemma 3.10. *Let $u_0\in\mathcal{K}\setminus\mathbb{R}$ and $len(f(t,s,x))$ for all $x\in\bar{B}(u_0)$, $\forall t,s\in[0,b]$ are bounded. Then there exists $c^*>0$ such that the mapping $\mathcal{B}:C([0,c^*],\bar{B}(u_0))\to C([0,c^*],\bar{B}(u_0))$ is well-defined.*

Proof. Let $len(f(t,s,x))\leq M$ for all $x\in\bar{B}(u_0)$, $\forall t,s\in[0,b]$. By Lemma 2.13, for $t\in[0,\frac{2len(u_0)}{M(2(1-q)+bq)}]$, we can conclude

$$\begin{aligned}(1-q)\int_0^t len(f(t,s,\phi(s)))ds \quad &+\quad q\int_0^t\int_0^s len(f(s,r,\phi(r)))drds\\ &\leq\quad M((1-q)t+q\frac{t^2}{2})\leq len(u_0). \quad (3.5)\end{aligned}$$

Let consider $c^*=min\{(\frac{2len(u_0)}{M(2(1-q)+bq)},b\}$. Then from the relations (3.5), H-differences involving in the mapping $\mathcal{B}$ exist on $t\in[0,c^*]$. □

Definition 3.11. Let $\underline{u},\bar{u}\in C([0,c^*],\bar{B}(u_0))$, we say that
(a) $\underline{u}$ is a lower solution for Problem (3.1) if

$$\underline{u}(t)\preceq[\mathcal{B}\underline{u}](t), \qquad t\in[0,c^*],$$

(b) $\bar{u}$ is an upper solution for Problem (3.1) if

$$\bar{u}(t)\succeq[\mathcal{B}\bar{u}](t), \qquad t\in[0,c^*].$$

Remark 3.12. If $u_0\in\mathcal{K}\setminus\mathbb{R}$, then Definition 3.11 is well-defined.

Theorem 3.13. *Consider Problem (3.1) with f continuous and suppose f is nondecreasing in last argument. Let $u_0\in\mathcal{K}\setminus\mathbb{R}$ and $len(f(t,s,x))$ for all $x\in\bar{B}(u_0)$, $\forall t,s\in[0,b]$ is bounded with bound of M. Moreover, assume $c^*=min\{\frac{2len(u_0)}{M(2(1-q)+bq)},b\}$. Let exist $l>0$ such that*

$$D(f(t,s,x),f(t,s,y))\leq lD(x,y), \qquad \forall t\in[0,c^*],$$

for $x \geq_1 y$. Then the existence of a lower solution $\underline{u}$ (or an upper solution $\bar{u}$) for Problem (3.1) provides the existence of a fixed point for $\mathcal{B}$ like u, and consequently (ii)-solution to Problem (3.1) on $[0, c^*]$. Also, $\lim_{n\to\infty} \mathcal{B}^n(\underline{u}) = u$ (or $\lim_{n\to\infty} \mathcal{B}^n(\bar{u}) = u$). Moreover, if $w \in C([0, c^*], \bar{B}(u_0))$ is another fixed point of $\mathcal{B}$ such that is comparable to $\underline{u}$ in the partial ordering $\preceq$, then $u = w$.

Proof. Since by Lemma 3.6, Problem (3.1) is equivalent to (F_2), we prove that the mapping $\mathcal{B}$ has a unique fixed point under assumption the existence a lower solution $\underline{u}$ for Problem (3.1). Because of similarity we omit the proof under assumption the existence of upper solution. Now we check that hypotheses in Theorem 2.11 are satisfied.
We consider $X = C([0, c^*], \bar{B}(u_0))$ that is partially ordered set by the following order relation For $g, f \in C([0, c^*], \bar{B}(u_0))$,

$$g \preceq f \Leftrightarrow g(t) \preceq f(t), \ \forall t \in [0, c^*].$$

Obviously there exists $c > 0$ such that $\frac{c^*}{c} = N \in \mathbb{N}$ and $lc + l\frac{c^2}{2} < 1$. Firstly We consider the interval $[0, c]$. By Lemma 3.10, the mapping $\mathcal{B} : C([0, c], \bar{B}(u_0)) \to C([0, c], \bar{B}(u_0))$ defined by (3.4), is well-defined and since f is nondecreasing in its last arguments, the mapping $\mathcal{B}$ is nondecreasing on $[0, c]$. For $\phi \preceq \psi$, we have

$$\begin{aligned} D([\mathcal{B}\phi](t), [\mathcal{B}\psi](t)) &\leq (1-q)\int_0^t D(f(t, s, \phi(s)), f(t, s, \Psi(s)))ds \\ &+ q\int_0^t \int_0^s D(f(s, r, \phi(r)), f(s, r, \psi(r)))drds \\ &\leq l(1-q)cH(\phi, \psi) + lq\frac{c^2}{2}H(\phi, \psi). \end{aligned} \tag{3.6}$$

Then from (3.6), we have

$$H(\mathcal{B}\phi, \mathcal{B}\psi) \leq LH(\phi, \psi), \tag{3.7}$$

where $L = l(1-q)c + lq\frac{c^2}{2} < 1$. Applying Theorem 2.11, $\mathcal{B}$ has a fixed point $\mathbf{u} \in C([0, c], \bar{B}(u_0))$ and $\lim_{n\to\infty} \mathcal{B}^n(\underline{u}) = \mathbf{u}$ (or $\lim_{n\to\infty} \mathcal{B}^n(\overline{u}) = \mathbf{u}$). Now suppose $w \in C([0, c], \bar{B}(u_0))$ is another fixed point of $\mathcal{B}$ such that is comparable to $\underline{u}$ with respect to partial ordering $\preceq$. It means that $\underline{u} \preceq w$ or $w \preceq \underline{u}$. We claim that $H(\mathbf{u}, w) = 0$. Employing the nondecreasing property of the mapping $\mathcal{B}$, along with Lemma 2.12 and $\underline{u} \preceq \mathcal{B}\underline{u}$, we can infer $\underline{u} \preceq \mathbf{u}$. Then $\mathcal{B}^n\underline{u}$ is comparable to $\mathcal{B}^n\mathbf{u} = \mathbf{u}$ and $\mathcal{B}^n w = w$ for $n = 0, 1, 2, \ldots$. Utilizing (3.7) we have

$$\begin{aligned} H(\mathbf{u}, w) = H(\mathcal{B}^n\mathbf{u}, \mathcal{B}^n w) &\leq H(\mathcal{B}^n\mathbf{u}, \mathcal{B}^n\underline{u}) + H(\mathcal{B}^n w, \mathcal{B}^n\underline{u}) \\ &\leq L^n H(\mathbf{u}, \underline{u}) + L^n H(\underline{u}, w). \end{aligned}$$

Since $L < 1$, the right-hand side of above equation converges to zero as $n \to \infty$. Then $H(\mathbf{u}, w) = 0$. It means that the fixed point is unique on $[0, c]$
Now by considering $\mathbf{u}$ as a fixed point for $\mathcal{B}$ on the interval $[0, c]$, we define another

mapping on the interval $[c, 2c]$ as follows:

$$[\mathcal{T}\phi](t) = \mathbf{u}(c) \ominus (-1)(q\int_c^t \int_0^c f(s,r,\mathbf{u}(r))drds \quad + \quad (1-q)\int_c^t f(t,s,\phi(s))ds$$
$$+ \quad q\int_c^t \int_c^s f(s,r,\phi(r))drds).$$

The mapping $\mathcal{T} : C([c,2c],\bar{B}(u_0)) \to C([c,2c],\bar{B}(u_0))$ is well-defined, since for $t \in [c,2c]$ we have

$$\begin{aligned}
&q\int_c^t \int_0^c len(f(s,r,\mathbf{u}(r)))drds + (1-q)\int_c^t len(f(t,s,\phi(s)))ds\\
&+ q\int_c^t \int_c^s len(f(s,r,\phi(r)))drds + (1-q)\int_0^c (len(f(t,s,\mathbf{u}(s)))ds\\
&+ q\int_0^c \int_0^s len(f(s,r,\mathbf{u}(r)))drds\\
&\leq Mqc(t-c) + M(1-q)(t-c) + Mq\frac{(t-c)^2}{2} + M(1-q)c + Mq\frac{c^2}{2}\\
&= (1-q)Mt + Mq\frac{t^2}{2} \leq c^*(M(1-q) + Mq\frac{b}{2}) \leq len(u_0).
\end{aligned}$$

Since f is nondecreasing in last its argument, the mapping

$$\mathcal{T} : C([c,2c],\bar{B}(u_0)) \to C([c,2c],\bar{B}(u_0)),$$

is nondecreasing too.
Now we will show that $\underline{u}(t) \preceq [\mathcal{T}\underline{u}](t)$ (or $\overline{u}(t) \preceq [\mathcal{T}\overline{u}](t)$) for $t \in [c,2c]$. Due the fact that $\underline{u}$ is a lower solution of Problem (3.1) for $t \in [0,b]$ and $\underline{u} \preceq \mathbf{u}$ for $t \in [0,c]$, we

have

$$\underline{u} \preceq [\mathcal{B}\underline{u}](t) = u_0 \ominus (-1)((1-q)\int_0^t f(t,s,\underline{u}(s))ds$$
$$+ q\int_0^t \int_0^s f(s,r,\underline{u}(r))drds),$$
$$= u_0 \ominus (-1)(1-q)(\int_0^c f(t,s,\underline{u}(s))ds + \int_c^t f(t,s,\underline{u}(s))ds)$$
$$\ominus (-1)q(\int_0^c \int_0^s f(s,r,\underline{u}(r))drds + \int_c^t \int_0^s f(s,r,\underline{u}(r))drds)$$
$$\preceq u_0 \ominus (-1)((1-q)\int_0^c f(t,s,\mathbf{u}(s))ds + q\int_0^c \int_0^s f(s,r,\mathbf{u}(r))drds)$$
$$\ominus (-1)((1-q)\int_c^t f(t,s,\underline{u}(s)ds + q\int_c^t \int_0^s f(s,r,\underline{u}(r))drds)$$
$$= \mathbf{u}(c) \ominus (-1)((1-q)\int_c^t f(t,s,\underline{u}(s)ds + q\int_c^t \int_0^s f(s,r,\underline{u}(r))drds)$$
$$= \mathbf{u}(c) \ominus (-1)(q\int_c^t \int_0^c f(s,r,\underline{u}(r))drds + (1-q)\int_c^t f(t,s,\underline{u}(s))$$
$$+ q\int_c^t \int_c^s f(s,r,\underline{u}(r))drds)$$
$$\preceq \mathbf{u}(c) \ominus (-1)(q\int_c^t \int_a^c f(s,r,\mathbf{u}(r))drds + (1-q)\int_c^t f(,ts,\underline{u}(s)ds$$
$$+ q\int_c^t \int_c^s f(s,r,\underline{u}(r))drds)$$
$$= [\mathcal{T}\underline{u}](t).$$

For $\phi \succeq \psi$, we conclude

$$\begin{aligned} D([\mathcal{T}\phi](t),[\mathcal{T}\Psi](t)) &\leq (1-q)\int_c^t D(f(t,s,\phi(s)),f(t,s,\psi(s)))ds \\ &+ q\int_c^t \int_c^s D(f(s,r,\phi(r)),f(s,r,\psi(r)))drds \\ &\leq lc(1-q)H(\phi,\psi) + l\frac{c^2}{2}qH(\phi,\psi). \end{aligned}$$

Then we have

$$H(\mathcal{T}\phi,\mathcal{T}\psi) \leq LH(\phi,\psi), \tag{3.8}$$

where $L = lc(1-q) + l\frac{c^2}{2}q < 1$. All the conditions in Theorem 2.11 are satisfied, therefore the mapping $\mathcal{T}$ has a fixed point $\mathbf{v} \in C([c,2c],\bar{B}(u_0)) \times C([c,2c],\bar{B}(u_0))$ and $\lim_{n\to\infty} \mathcal{T}^n(\underline{u}) = \mathbf{v}$.
If we suppose $w \in C([c,2c],\bar{B}(u_0)) \times C([c,2c],\bar{B}(u_0))$ is another fixed point of $\mathcal{T}$ such

that is comparable to $\underline{u}$ on $[c, 2c]$, then it is clear that $H(\mathbf{v}, w) = 0$.
Obviously u as defined

$$u = \begin{cases} \mathbf{u}, & t \in [0, c], \\ \mathbf{v}, & t \in [c, 2c], \end{cases}$$

is a fixed point of $\mathcal{B}$ defined by (3.4) on $[0, 2c]$. By Lemma 2.6, u is (ii)-differentiable on $[0, 2c]$. In the same trend we can make a fixed point of $\mathcal{B}$ defined by (3.4) on $[0, Nc] = [0, c^*]$. Let $u =\in C([0, c^*], \bar{B}(u_0))$ is a fixed point of $\mathcal{B}$. Therefore u is a solution of integral equation (F_2). By Remark 3.7, we can conclude u is a (ii)-solution of Problem (3.1).
Now suppose $w \in C([0, c^*], \mathcal{K})$ is another fixed point of $\mathcal{B}$ such that is comparable to $\underline{u}$ on $[0, c^*]$. It is clear that $H(u, w) = 0$. □

Remark 3.14. As before mentioned, our purpose of this work is investigation on existence of global solution for Problem (3.1). On the other hand, the mapping $\mathcal{B}$ is well-defined just on $[0, c^*]$ and it is why Theorem 3.13 has proven the existence of local solution on subinterval $[0, c^*]$ of J. In order to overcome this difficulty, in the next subsection we show the existence of another kind solution for Problem (3.1) which is (ii)-differentiable on one part of J and (i)-differentiable on another part.

3.3. **Existence of (ii,i)-solution.** Now we define the nonlinear mappings $\mathcal{L}$, which plays a main role in our discussion, as following

$$[\mathcal{L}\phi](t) =$$
$$\begin{cases} u_0 \ominus (-1) \cdot ((1-q)\int_0^t f(t, s, \phi(s))ds + q\int_0^t \int_0^s f(s, r, \phi(r))drds), & t \in [0, c^*], \\ \phi(c^*) + \int_{c^*}^t f(t, s, \phi(s))ds + \int_{c^*}^t \int_0^s f(s, r, \phi(r))drds, & t \in [c^*, b]. \end{cases}$$

In general the $\mathcal{L} : C(J, \mathcal{K}) \to C(J, \mathcal{K})$ is not well-defined, but under the conditions of the next theorem, it will be well-defined.

Definition 3.15. Let $\underline{u}, \bar{u} \in C(J, \bar{B}(u_0)))$, we say that
(a) $\underline{u}$ is a lower solution for Problem (3.1) if

$$\underline{u}(t) \preceq [\mathcal{L}\underline{u}](t), \qquad t \in J,$$

(b) $\bar{U}$ is an upper solution for the problem (3.1) if

$$\bar{u}(t) \succeq [\mathcal{L}\bar{u}](t), \qquad t \in J.$$

Theorem 3.16. *Consider Problem (3.1) with f continuous and suppose f is nondecreasing in its last argument. Let $u_0 \in \mathcal{K} \setminus \mathbb{R}$ and $len(f(t, s, x))$ for all $x \in \bar{B}(u_0)$, $\forall t, s \in [0, c^*]$ is bounded with bounds of M, where $c^* = min\{\frac{2len(u_0)}{M(2(1-q)+qb)}, b\}$. Let exist $l > 0$ such that*

$$D(f(t, s, x), f(t, s, y)) \leq lD(x, y), \qquad \forall t \in [a, b],$$

for $x \geq_1 y$ and $x, y \in \bar{B}(u_0)$. Then the existence of a lower solution $\underline{u}$ (or an upper solution $\bar{u}$) for Problem (3.1) provides the existence of a fixed point for $\mathcal{L}$ like u, and consequently (ii,i)-solution to Problem (3.1) on $[0, b]$. Also, $\lim_{n\to\infty} \mathcal{L}^n(\underline{u}) = u$ (or $\lim_{n\to\infty} \mathcal{B}^n(\bar{u}) = u$). Moreover, if $w \in C([0, c^*], \bar{B}(u_0))$ is another fixed point of $\mathcal{B}$ such that is comparable to $\underline{u}$ in the partial ordering $\preceq$, then $u = w$.

Proof. By Theorem 3.13, the mapping $\mathcal{L}$ is well-defined and there exists a fixed point for the mapping $\mathcal{L}$ like $\mathbf{u} \in C([0,c^*], \bar{B}(u_0))$ and consequently (ii)-solution for Problem (3.1) on $[0,c^*]$. Now by considering $\mathbf{u}$ as a fixed point for $\mathcal{B}$ on the interval $[0,c^*]$, we define another mapping $\mathcal{T} : C([c^*,b],\mathcal{K}) \to C([c^*,b],\mathcal{K})$ as follows:

$$\begin{aligned}
[\mathcal{T}\phi](t) &= \mathbf{u}(c^*) + q\int_{c^*}^{t}\int_{0}^{c^*} f(s,r,\mathbf{u}(r))drds + (1-q)\int_{c^*}^{t}(f(t,s,\phi(s))ds \\
&+ q\int_{c^*}^{t}\int_{c^*}^{s} f(s,r,\phi(r))drds.
\end{aligned}$$

Now we will show that $\underline{u}(t) \preceq [\mathcal{T}\underline{u}](t)$ (or $\overline{u}(t) \succeq [\mathcal{T}\overline{u}](t)$) for $t \in [c^*,b]$. Due the fact that $\underline{u}$ is a lower solution of Problem (3.1) for $t \in [0,b]$ and $\underline{u} \preceq \mathbf{u}$ for $t \in [0,c^*]$, we have for $t \in [c^*,b]$

$$\begin{aligned}
\underline{u} &\preceq [\mathcal{L}\underline{u}](t) \\
&= \underline{u}(c^*) + (1-q)\int_{c^*}^{t} f(t,s,\underline{u}(s))ds + q\int_{c^*}^{t}\int_{0}^{s} f(s,r,\underline{u}(r))drds, \\
&\preceq \mathbf{u}(c^*) + (1-q)\int_{c^*}^{t} f(t,s,\underline{u}(s))ds + q\int_{c^*}^{t}\int_{o}^{s} f(s,r,\underline{u}(r))drds, \\
&= \mathbf{u}(c^*) + q\int_{c^*}^{t}\int_{a}^{c^*} f(s,r,\underline{u}(r))drds + (1-q)\int_{c^*}^{t} f(t,s,\underline{u}(s))ds \\
&+ q\int_{c^*}^{t}\int_{c^*}^{s} f(s,r,\underline{u}(r))drds, \\
&\preceq \mathbf{u}(c^*) + q\int_{c^*}^{t}\int_{0}^{c^*} f(s,r,\mathbf{u}(r))drds + (1-q)\int_{c^*}^{t} f(t,s,\underline{u}(s))ds \\
&+ q\int_{c^*}^{t}\int_{c^*}^{s} f(s,r,\underline{u}(r))drds, \\
&= [\mathcal{T}\underline{u}](t).
\end{aligned}$$

By Theorem 3.9, the mapping $\mathcal{T}$ has a fixed point $\mathbf{v} \in C([c^*,b],\mathcal{K})$. Obviously u as defined

$$u = \begin{cases} \mathbf{u}, & t \in [0,c^*], \\ \mathbf{v}, & t \in [c^*,b], \end{cases}$$

is a fixed point of $\mathcal{L}$ on $[0,b]$. $\square$

Example 3.17. *Consider the interval Caputo-Fabrizio fractional initial value problem*

$$\begin{aligned}
&{}^{CF}D^{\frac{1}{2}}u(t) = \int_0^t 2e^{(-(t-s))}u(s)ds, \qquad t \in [0,10], \\
&u(0) = [3,7]. \qquad (3.9)
\end{aligned}$$

$\underline{u} = [3,7]$ *is the lower solution of Problem (3.9) according to Definition 3.8. It is easy to see that all the conditions of Theorem 3.9 are fulfilled and as a conclusion, there* *exists a (i)-solution for this problem.* $u(t) = [3,7]e^t$ *is an exact solution on* $[0,10]$. *On the other hand,* $len(u_0) = len([3,7]) = 4$ *and*

$$len(f(t,s,x)) \leq 2e^{-(t-s)}len(u_0) \leq 8e^{10} \quad \forall x \in \bar{B}(u_0), \quad \forall t,s \in [0,10].$$

Also, $\underline{u} = [3,7]$ *is the lower solution of Problem (3.9) according to Definition 3.11 and* $c^* = \frac{e^{-10}}{6}$. *According to Theorem 3.13, there exists a (ii)-solution for this problem.* $u(t) = [-2,2]e^{-t} + [5,5]e^t$ *is an exact solution on* $[0, \frac{e^{-10}}{6}]$.

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