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An efficient approximate method for solution of the heat equation using Laguerre-Gaussians radial functions

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Abstract In the present paper, a numerical method is considered for solving one-dimensional heat equation subject to both Neumann and Dirichlet initial boundary conditions. This method is a combination of collocation method and radial basis functions (RBFs). The operational matrix of derivative for Laguerre-Gaussians (LG) radial basis functions is used to reduce the problem to a set of algebraic equations. The results of numerical experiments are presented to confirm the validity and applicability of the presented scheme.

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1. Introduction

Partial differential equations have a wide range of applications in chemistry and physics. Theory and numerical schemes for solving initial boundary value problems have attracted the attention of researchers. Yousefi proposed Bernstein Tau technique to solve the one-dimensional parabolic equation in [22] and Tohidi presented the solution of this problem by the Legendre collocation method in [17]. Numerical solutions of parabolic equation with an initial-boundary value problem that combines Neumann and Dirichlet conditions were investigated in [1,2,3,4,5,21]. The general form of equation is given as:

$$u_t(x,t) = u_{xx}(x,t) + Q(x,t), \quad 0 < x < L, 0 < t \leq T, \tag{1.1}$$

with the initial condition:

$$u(x,0) = f(x), \quad 0 \leq x \leq L, \tag{1.2}$$

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and the boundary conditions

$$u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \quad 0 < t \leq T, \tag{1.3}$$

$$u_x(0,t) = g_2(t), \quad u_x(L,t) = g_3(t), \quad 0 < t \leq T, \tag{1.4}$$

where $Q(x,t)$, $f(x)$, $g_0(t)$, $g_1(t)$, $g_2(t)$ and $g_3(t)$ are suitably given functions. He in [9,10] and Cheniguel in [5] proposed the homotopy perturbation method (HPM) to solve initial boundary value problems. Mohebbi presented a class of new finite difference schemes to solve the one-dimensional heat and advection-diffusion equations in [13]. Sun in [16] proposed a class of new finite difference methods, CBVM, to solve the one dimension heat equations.

The organization of this article is as follow. We describe radial basis functions and their properties in Section 2. In Section 3, the use of this basis is discussed for solving one-dimensional heat equation. In Section 4, we give some computational results of numerical experiments with RBFs method to support our theoretical discussion. The conclusion is presented in Section 5.

2. Radial basis function approximation

Polynomials (e.g., Legendre and Chebyshev) are very efficient tools for interpolating a set of points in one-dimensional domains but in irregular domains and higher-dimensional the use of these functions is not effective. The main benefit of radial basis functions is that this method is independent of the dimension of the problem and needs neither domain nor surface discretization. The method is meshless and is not difficult.

2.1. **Definition of RBF.** Let $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$, $\|.\|_2$ denotes the Euclidean norm and $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $\varphi(0) \geq 0$. A radial basis function on $\mathbb{R}^d$ is a function of the form:

$$\phi_i(\mathbf{x}) = \varphi(\|\mathbf{x} - \mathbf{x}_i\|_2),$$

which depends only on the distance between $\mathbf{x} \in \mathbb{R}^d$ and a fixed point $\mathbf{x}_i \in \mathbb{R}^d$. So that the radial basis function ϕ_i is radially symmetric about the center $\mathbf{x}_i$.

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N \in \Omega \subset \mathbb{R}^d$ be a given set of scattered data. A radial basis function interpolation problem may be described as:

$$S_f(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i \phi_i(\mathbf{x}),$$

for given data $f_i = f(\mathbf{x}_i)$, $i = 1, 2, \cdots, N$, where λ_i are chosen in order to $S_f(\mathbf{x}_j) = f_j$, $j = 1, 2, \cdots, N$, that the interpolation conditions provide the linear system:

$$A\lambda = f,$$

where for $i, j \in \{1, 2, \cdots, N\}$, $A_{ij} = \phi_i(\mathbf{x}_j)$, $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_N]^T$ and $f = [f_1, f_2, \cdots, f_N]^T$. Let r be the Euclidean distance between a fixed point $\mathbf{x}_i \in \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$, i.e. $\|\mathbf{x} - \mathbf{x}_i\|_2$. Some well-known RBFs are listed in Table 1. The kind of RBFs, we will be mostly interested in, are the Gaussians $\varphi(r) = e^{-\varepsilon^2 r^2}$. Other families of radial

TABLE 1. Some well-known functions that generate RBFs.

Name of Radial Basis Function	Definition
Multiquadric(MQ)	$\varphi(r)=\sqrt{\varepsilon^2+r^2}$
Inverse Quadratic(IQ)	$\varphi(r)=\frac{1}{(\varepsilon^2+r^2)}$
Inverse Multiquadric(IMQ)	$\varphi(r)=\frac{1}{\sqrt{\varepsilon^2+r^2}}$
Gaussian(GA)	$\varphi(r)=e^{-\varepsilon^2 r^2}$
Thin Plate Splines(TPS)	$\varphi(r)=r^2 log(r)$

basis functions are the Matérn functions, also known as Sobolev splines. Examples are listed in Table 2. Another RBFs are the Laguerre-Gaussians. The definition of

TABLE 2. Matérn functions.

name	Definition
basic	$\varphi(r)=e^{-\varepsilon r}$
linear	$\varphi(r)=(1+\varepsilon r)e^{-\varepsilon r}$
quadratic	$\varphi(r)=(1+\varepsilon r+\frac{(\varepsilon r)^2}{3})e^{-\varepsilon r}$
cubic	$\varphi(r)=(15+15\varepsilon r+6(\varepsilon r)^2+(\varepsilon r)^3)e^{-\varepsilon r}$

Laguerre-Gaussians functions family comes from the generalized Laguerre polynomials of degree n and order $s/2$ [14]. Specific examples are listed in Table 3.
The shape parameter ε, which appears in tables affects both the accuracy of the estimate and the conditioning of the interpolation matrix [15]. Almost, for fixed values of the shape parameter ε, the condition number increases with N. For a fixed number N, smaller shape parameters produce more accurate approximations, but they are also associated with a poorly conditioned A. However, many researchers have attempted to develop algorithms for choosing optimal values of the shape parameter but the optimal choice of the shape parameter is still an open question and it is most often selected by brute force. Franke [7] suggested $\varepsilon^2=1.25D/\sqrt{N}$ in MQ basis, where D is the diameter of the smallest circle containing all data points and N is the number of data points. Hardy [8] recommended the use of $\varepsilon^2=0.815d$ where $d=(1/N)\sum_{i=1}^{N} d_i$ and d_i is the distance from the data point x_i to its nearest neighbor. Recently, Fornberg developed a Contour-Padé algorithm that is capable of stably computing the RBF approximation for all $\varepsilon>0$ [6]. Micchelli [12] and Wendland [18] showed that the interpolation matrix for the RBFs is invertible for distinct interpolation points.
We have the following theorem about the convergence of RBFs interpolation.

Theorem 2.1. *Assume $\{\mathbf{x}_i\}_{i=1}^{N}$ are N nodes in $\Omega\subset\mathbb{R}^d$ which is convex, let:*

$$h=\max_{\mathbf{x}\in\Omega}\min_{1\le i\le N}\|\mathbf{x}-\mathbf{x}_i\|_2,$$

when $\hat{\phi}(\eta)<c(1+\mid\eta\mid)^{-2l+d}$, for any y satisfing $\int(\hat{y}(\eta))^2/\hat{\phi}(\eta)d\eta<\infty$, we have:

$$\|y_N^{(\alpha)}-y^{(\alpha)}\|<ch^{l-\alpha},$$

TABLE 3. Laguerre-Gaussians radial functions.

s	$n=1$	$n=2$
1	$\varphi(r) = (\frac{3}{2} - (\varepsilon r)^2)e^{(-\varepsilon r)^2}$	$\varphi(r) = (\frac{15}{8} - \frac{5}{2}(\varepsilon r)^2 + \frac{1}{2}(\varepsilon r)^4)e^{(-\varepsilon r)^2}$
2	$\varphi(r) = (2 - (\varepsilon r)^2)e^{(-\varepsilon r)^2}$	$\varphi(r) = (3 - 3(\varepsilon r)^2 + \frac{1}{2}(\varepsilon r)^4)e^{(-\varepsilon r)^2}$
3	$\varphi(r) = (\frac{5}{2} - (\varepsilon r)^2)e^{(-\varepsilon r)^2}$	$\varphi(r) = (\frac{35}{8} - \frac{7}{2}(\varepsilon r)^2 + \frac{1}{2}(\varepsilon r)^4)e^{(-\varepsilon r)^2}$

where ϕ is RBFs and the constant c depends on the RBFs, $\hat{\phi}$ and $\hat{y}$ are supposed to be the Fourier transforms of ϕ and y respectively, $y^{(\alpha)}$ denotes the αth derivative of y, y_N is the RBFs approximation of y, d is space dimension, l and α are nonnegative integers.

Proof. A complete proof is given by authors [19,20].

2.2. **Function approximation.** Let $X = L^2(\Omega)$ where $\Omega = [0, L] \times [0, T]$ and

$$\{\psi_{11}(x,t), ..., \psi_{1M}(x,t), \psi_{21}(x,t), ..., \psi_{2M}(x,t), ..., \psi_{N1}(x,t), ..., \psi_{NM}(x,t)\} \subset X$$

be the set of LG-RBFs where $\psi_{ij}(x,t) = (2-\varepsilon^2((x-x_i)^2+(t-t_j)^2))e^{-\varepsilon^2((x-x_i)^2+(t-t_j)^2)}$ and

$$Y = span\{\psi_{11}(x,t), ..., \psi_{1M}(x,t), \psi_{21}(x,t), ..., \psi_{2M}(x,t), ..., \psi_{N1}(x,t), ..., \psi_{NM}(x,t)\},$$

suppose that y be an arbitrary element in X. Since Y is a finite dimensional vector space, y has the unique best approximation out of Y as $y_{NM} \in Y$, that is [11]:

$$\forall g \in Y, \|y - y_{NM}\|_2 \leq \|y - g\|_2.$$

Since $y_{NM} \in Y$, there exist unique coefficients $c_{11}, ..., c_{1M}, c_{21}, ..., c_{2M}, ..., c_{N1}, ..., c_{NM}$ such that:

$$y \simeq y_{NM} = \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}\psi_{ij}(x,t) = C^T\Psi_{NM}(x,t) = \Psi_{NM}^T(x,t)C,$$

where C and $\Psi_{NM}(x,t)$ are vectors with the form:

$$C = [c_{11}, ..., c_{1M}, c_{21}, ..., c_{2M}, ..., c_{N1}, ..., c_{NM}]^T, \tag{2.1}$$

$$\Psi_{NM}(x,t) = [\psi_{11}(x,t), ..., \psi_{1M}(x,t), \psi_{21}(x,t)..., \psi_{NM}(x,t)]^T. \tag{2.2}$$

3. THE OPERATIONAL MATRIX OF DERIVATIVE

Let $x_i = L\frac{i}{N}, i = 1, 2, ..., N$, and $t_j = T\frac{j}{M}, j = 1, 2, ..., M$. The unknown function $u(x,t)$ in (1.1)-(1.4) can be approximated as:

$$u(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}\psi_{ij}(x,t) = C^T\Psi_{NM}(x,t). \tag{3.1}$$

The differentiation with respect to x of vectors Ψ_{NM} in (2.2) can be expressed as:

$$\frac{\partial}{\partial x}\Psi_{NM}(x,t) = D_N(x)\Psi_{NM}(x,t) + D_N(x)\Phi_{NM}(x,t), \tag{3.2}$$

where $D_N(x)$ is the operational matrix of derivative with respect to x and

$$\Phi_{NM}(x,t) = [\phi_{11}(x,t), ..., \phi_{1M}(x,t), \phi_{21}(x,t), ..., \phi_{NM}(x,t)]^T, \tag{3.3}$$

where $\phi_{ij}(x,t)$ is the GA-RBFs, i.e. $\phi_{ij}(x,t) = e^{-\varepsilon^2((x-x_i)^2+(t-t_j)^2)}$. The matrix $D_N(x)$ can be obtained as:

$$\frac{\partial}{\partial x}\Psi_{NM}(x,t) = \begin{bmatrix} -2\varepsilon^2(x-x_1)(\psi_{11}(x,t)+\phi_{11}(x,t)) \\ \vdots \\ -2\varepsilon^2(x-x_1)(\psi_{1M}(x,t)+\phi_{1M}(x,t)) \\ -2\varepsilon^2(x-x_2)(\psi_{21}(x,t)+\phi_{21}(x,t)) \\ \vdots \\ -2\varepsilon^2(x-x_2)(\psi_{2M}(x,t)+\phi_{2M}(x,t)) \\ \vdots \\ -2\varepsilon^2(x-x_N)(\psi_{N1}(x,t)+\phi_{N1}(x,t)) \\ \vdots \\ -2\varepsilon^2(x-x_N)(\psi_{NM}(x,t)+\phi_{NM}(x,t)) \end{bmatrix}. \tag{3.4}$$

By comparing (3.2) and (3.4), we can write:

$$\frac{\partial}{\partial x}\Psi_{NM}(x,t) = \begin{bmatrix} M_1(x) & 0 & ... & 0 \\ 0 & M_2(x) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & M_N(x) \end{bmatrix} \begin{bmatrix} \psi_{11}(x,t)+\phi_{11}(x,t) \\ \vdots \\ \psi_{1M}(x,t)+\phi_{1M}(x,t) \\ \psi_{21}(x,t)+\phi_{21}(x,t) \\ \vdots \\ \psi_{2M}(x,t)+\phi_{2M}(x,t) \\ \vdots \\ \psi_{N1}(x,t)+\phi_{N1}(x,t) \\ \vdots \\ \psi_{NM}(x,t)+\phi_{NM}(x,t) \end{bmatrix}, \tag{3.5}$$

where $M_i(x) = -2\varepsilon^2(x-x_i)I_M, i = 1, 2, ..., N$ and I_M is the $M \times M$ identity matrix. Thus we have:

$$D_N(x) = \begin{bmatrix} M_1(x) & 0 & ... & 0 \\ 0 & M_2(x) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & M_N(x) \end{bmatrix}. \tag{3.6}$$

Similarly, the differentiation of vectors Ψ_{NM} with respect to t in (2.2) can be expressed as:

$$\frac{\partial}{\partial t}\Psi_{NM}(x,t) = D_M(t)\Psi_{NM}(x,t) + D_M(t)\Phi_{NM}(x,t), \tag{3.7}$$

where $D_M(t) = diag(N_1(t), N_2(t), ..., N_M(t))$ and $N_i(t) = -2\varepsilon^2(t-t_i)I_N, i = 1, 2, ..., M$. Also:

$\frac{\partial^2}{\partial x^2}\Psi_{NM}(x,t) =$

$$\begin{bmatrix} (-2\varepsilon^2 + (-2\varepsilon^2(x-x_1))^2)\psi_{11}(x,t) + (-2\varepsilon^2 + 2(-2\varepsilon^2(x-x_1))^2)\phi_{11}(x,t) \\ \vdots \\ (-2\varepsilon^2 + (-2\varepsilon^2(x-x_1))^2)\psi_{1M}(x,t) + (-2\varepsilon^2 + 2(-2\varepsilon^2(x-x_1))^2)\phi_{1M}(x,t) \\ (-2\varepsilon^2 + (-2\varepsilon^2(x-x_2))^2)\psi_{21}(x,t) + (-2\varepsilon^2 + 2(-2\varepsilon^2(x-x_2))^2)\phi_{21}(x,t) \\ \vdots \\ (-2\varepsilon^2 + (-2\varepsilon^2(x-x_2))^2)\psi_{2M}(x,t) + (-2\varepsilon^2 + 2(-2\varepsilon^2(x-x_2))^2)\phi_{2M}(x,t) \\ \vdots \\ (-2\varepsilon^2 + (-2\varepsilon^2(x-x_N))^2)\psi_{N1}(x,t) + (-2\varepsilon^2 + 2(-2\varepsilon^2(x-x_N))^2)\phi_{N1}(x,t) \\ \vdots \\ (-2\varepsilon^2 + (-2\varepsilon^2(x-x_N))^2)\psi_{NM}(x,t) + (-2\varepsilon^2 + 2(-2\varepsilon^2(x-x_N))^2)\phi_{NM}(x,t) \end{bmatrix}. \tag{3.8}$$

So we can write:

$$\frac{\partial^2}{\partial x^2}\Psi_{NM}(x,t) = (R + D_N^2(x))\Psi_{NM}(x,t) + (R + 2D_N^2(x))\Phi_{NM}(x,t), \tag{3.9}$$

where

$$R = \begin{bmatrix} -2\varepsilon^2 & 0 & ... & 0 \\ 0 & -2\varepsilon^2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & -2\varepsilon^2 \end{bmatrix}. \tag{3.10}$$

Using Eqs. (3.7) and (3.9) in Eq. (1.1), we obtain:

$$C^T(D_M(t) - D_N^2(x) - R)\Psi_{NM}(x,t) + C^T(D_M(t) - 2D_N^2(x) - R)\Phi_{NM}(x,t) - Q(x,t) = 0, \tag{3.11}$$

and using Eqs. (3.1) and (3.2) in (1.2)-(1.4) yields:

$$C^T\Psi_{NM}(x,0) - f(x) = 0, \tag{3.12}$$

$$C^T\Psi_{NM}(0,t) - g_0(t) = 0, \tag{3.13}$$

$$C^T\Psi_{NM}(L,t) - g_1(t) = 0, \tag{3.14}$$

$$C^T D_N(0)\Psi_{NM}(0,t) + C^T D_N(0)\Phi_{NM}(0,t) - g_2(t) = 0, \tag{3.15}$$

$$C^T D_N(L)\Psi_{NM}(L,t) + C^T D_N(L)\Phi_{NM}(L,t) - g_3(t) = 0. \tag{3.16}$$

We collocate (3.11) in $(N-2)\times(M-1)$ interior points $\{(x_l,t_s) \mid l = 2,...,N-1, s = 2,...,M\}$, so we have:

$$C^T(D_M(t_s) - D_N^2(x_l) - R)\Psi_{NM}(x_l,t_s) + C^T(D_M(t_s) - 2D_N^2(x_l) - R)\Phi_{NM}(x_l,t_s) - Q(x_l,t_s) = 0. \tag{3.17}$$

Now, collocating (3.12) in N points $x_l, l = 1,2,...,N$, leads to:

$$C^T\Psi_{NM}(x_l,0) - f(x_l) = 0. \tag{3.18}$$

By collocating (3.13) and (3.14) or (3.15) and (3.16) in (M-2) points $t_s, s = 2,3,...,M$, we have:

$$C^T\Psi_{NM}(0,t_s) - g_0(t_s) = 0, \tag{3.19}$$

$$C^T\Psi_{NM}(L,t_s) - g_1(t_s) = 0, \tag{3.20}$$

$$C^T D_N(0)\Psi_{NM}(0,t_s) + C^T D_N(0)\Phi_{NM}(0,t_s) - g_2(t_s) = 0, \tag{3.21}$$

$$C^T D_N(L)\Psi_{NM}(L,t_s) + C^T D_N(L)\Phi_{NM}(L,t_s) - g_3(t_s) = 0, \tag{3.22}$$

respectively.
Eqs. (3.17)-(3.20) or Eqs. (3.17), (3.18), (3,21) and (3.22) give an $N\times M$ system of linear equations, which can be solved for $c_{ij}, i = 1,...,N, j = 1,...,M$.

4. Numerical examples

In this section we give some computational results of numerical experiments with the method based on the preceding sections, to support our theoretical discussion. In the process of computation, all the symbolic and numerical computations were performed using Maple and shape parameters were chosen by trial and error. The readers can see the efficiency of the proposed method from the provided figures and tables in the following examples.

Example 1. Consider Eqs. (1.1)-(1.4) with $L = 1$, $T = 1$ and [17,22]

$$Q(x,t) = (\pi^2+1)e^t\sin(\pi x), \tag{4.1}$$

$$f(x) = \sin(\pi x),\ g_0(t) = 0,\ g_1(t) = 0, \tag{4.2}$$

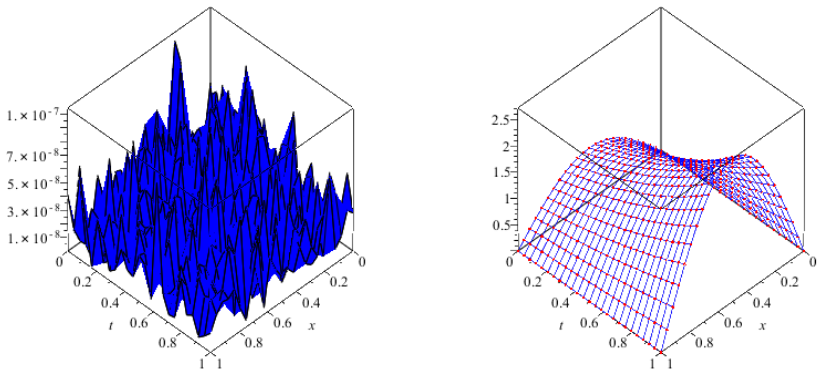
with the exact solution

$$u(x,t) = e^t\sin(\pi x). \tag{4.3}$$

TABLE 4. Absolute values of error for u from Example 1.

(x,t)	Method [17]	Method [22]	Present method	
			M=N=10, ε=0.6	M=N=16, ε=0.8
(0.1,0.1)	2.3106E–06	7.8747E–05	8.1463E–07	6.5626E–08
(0.2,0.2)	2.5218E–06	2.0530E–04	5.53440E–07	2.4356E–08
(0.3,0.3)	2.8753E–06	3.1140E–04	5.6466E–07	8.6643E–09
(0.4,0.4)	3.2063E–06	3.7870E–04	2.9014E–07	8.0138E–08
(0.5,0.5)	3.5608E–06	4.0294E–05	2.6930E–07	5.6300E–08
(0.6,0.6)	3.9745E–06	3.8488E–04	1.0142E–07	2.8247E–09
(0.7,0.7)	4.4678E–06	3.2818E–04	1.9319E–07	4.6188E–08
(0.8,0.8)	5.0163E–06	2.3840E–04	2.3859E–08	9.4859E–08
(0.9,0.9)	6.0805E–06	1.2546E–04	4.0823E–07	6.7650E–09
(1,1)	7.6345E–30	1.3177E–07	1.16E–07	2.01E–08

FIGURE 1. (a) Absolute errors of the solution, (b) Analytical (line) and estimated (point) solutions with $dx = dt = 0.1$ and $\varepsilon = 0.6$ for Example 1.



In Table 4 we give the absolute errors for Laguerre-Gaussians radial basis functions with $dx = dt = 0.1$ with shape parameter $\varepsilon = 0.6$ and $dx = dt = 0.0625$ with shape parameter $\varepsilon = 0.8$. The absolute errors of our method are compared with the Bernstein Tau method [22] and Legendre collocation method [17]. The absolute errors of estimated solution, and the exact and estimated solutions are given in Figure 1.

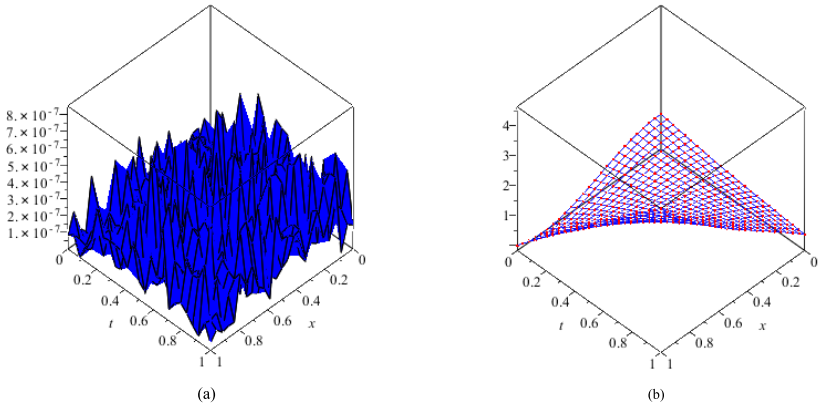
Example 2. Consider Eqs. (1.1)-(1.4) with $L = 1$, $T = 1$ and [5]

$$Q(x,t) = (\pi^2 - 1)e^{-t}\cos(\pi x) + 4x - 2, \tag{4.4}$$

TABLE 5. Absolute values of error for u from Example 2.

x	Method [5]		Present method with	
	u_{ex}	u_{hpm}	M=N=12, ε=0.9	M=N=20, ε=0.9
0.1	0.9589	0.9589	1.3085E–06	2.6748E–08
0.2	0.8490	0.8490	7.5991E–07	4.6612E–08
0.3	0.6802	0.6802	1.2073E–06	2.1270E–07
0.4	0.4742	0.4742	2.5052E–06	7.4759E–08
0.5	0.2580	0.2580	8.950E–07	2.6E–07
0.6	0.0618	0.0618	8.3476E–07	2.9476E–07
0.7	-0.0842	-0.0842	1.1426E–06	3.2698E–08
0.8	-0.1530	-0.1530	6.2009E–07	2.5991E–07
0.9	-0.1229	-0.1229	4.1145E–07	1.8548E–08
1	0.0200	0.0200	1.0656E–08	2.9656E–08

FIGURE 2. (a) Absolute errors of the solution, (b) Analytical (line) and estimated (point) solutions with $dx = dt = 0.0833$ and $\varepsilon = 0.9$ for Example 2.



(a) (b)

$$f(x) = \cos(\pi x) + x^2,\ g_0(t) = e^{-t},\ g_1(t) = -e^{-t} + 4t + 1, \tag{4.5}$$

with the exact solution

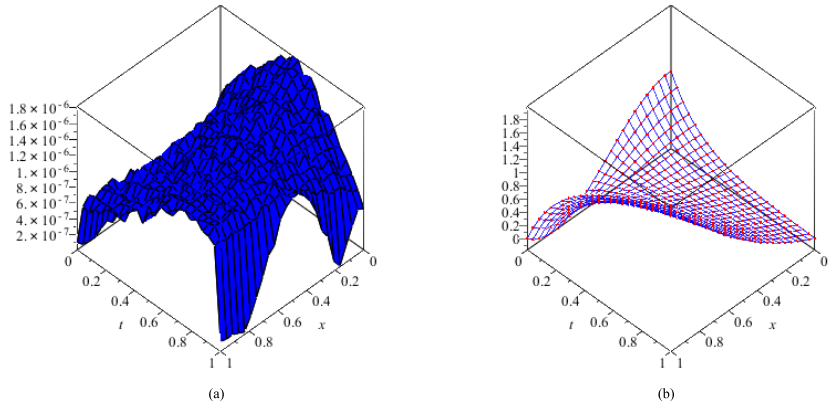
$$u(x,t) = x^2 + 4xt + e^{-t}\cos(\pi x). \tag{4.6}$$

In Table 5, we give the absolute errors at $t = 0.004$ for Laguerre-Gaussians radial basis functions with $dx = dt = 0.0833$ and $dx = dt = 0.05$ with shape parameter $\varepsilon = 0.9$. Similar to previous example, to compare our results we give the analytical and numerical solutions for homotopy perturbation method (HPM) [5]. The absolute errors of estimated solution, and the exact and estimated solutions are given in Figure 2.

TABLE 6. Absolute values of error for u from Example 3.

x	Method [5]		Present method with	
	u_{ex}	u_{hpm}	M=N=12, ε=0.9	M=N=16, ε=1.3
0.1	0.9429	0.9429	9.8069E–06	5.6883E–08
0.2	0.8340	0.8340	1.0682E–05	1.2380E–07
0.3	0.6675	0.6675	9.9480E–06	9.7971E–08
0.4	0.4646	0.4646	1.2549E–05	5.1661E–08
0.5	0.2520	0.2520	1.3E–06	4.4E–08
0.6	0.0594	0.0594	1.1261E–05	1.6136E–07
0.7	-0.0835	-0.0835	1.0286 E–06	1.2797E–07
0.8	-0.1500	-0.1500	1.0118E–05	5.8276E–08
0.9	-0.1189	-0.1189	8.9131E–06	9.2117E–08

FIGURE 3. (a) Absolute errors of the solution, (b) Analytical (line) and estimated (point) solutions with $dx = dt = 0.0833$ and $\varepsilon = 0.9$ for Example 3.



Example 3. Consider Eqs. (1.1)-(1.4) with $L = 1$, $T = 1$ and [5]

$$Q(x,t) = (\frac{\pi^2}{2})e^{-\frac{\pi^2}{2}t}\cos(\pi x) + x - 2, \tag{4.7}$$

$$f(x) = \cos(\pi x) + x^2,\ g_2(t) = t,\ g_3(t) = 2 + t, \tag{4.8}$$

with the exact solution

$$u(x,t) = x^2 + xt + e^{-\frac{\pi^2}{2}t}\cos(\pi x). \tag{4.9}$$

In Table 6, we give the absolute errors at $t = 0.004$ for Laguerre-Gaussians radial basis functions with $dx = dt = 0.0833$ with shape parameter $\varepsilon = 0.9$ and $dx = dt = 0.0625$

with shape parameter $\varepsilon = 1.3$. The absolute errors of our method are compared with the analytical and numerical solutions for homotopy perturbation method (HPM) [5]. The absolute errors of estimated solution, and the exact and estimated solutions are given in Figure 3.

5. Conclusion

A RBF-based numerical method proposed to solve the heat conduction problem with Dirichlet and Neumann boundary conditions. The Laguerre-Gaussians radial basis functions (LG-RBFs) on intervals $t \in [0, 1]$ and $x \in [0, 1]$ were employed. The method was based upon reducing the system into a set of algebraic equations. The proposed method was tested on several examples given in the literature. The obtained results showed that this approach can solve the problem effectively. Moreover, the method is more convenient for implementation in comparison to traditional techniques.

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