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Application of high-order spectral method for the time fractional mobile/immobile equation

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Abstract In this paper, a numerical efficient method is proposed for the solution of time fractional mobile/immobile equation. The fractional derivative of equation is described in the Caputo sense. The proposed method is based on a finite difference scheme in time and Legendre spectral method in space. In this approach the time fractional derivative of mentioned equation is approximated by a scheme of order $\mathcal{O}(\tau^{2-\gamma})$ for $0 < \gamma < 1$. Also, we introduce the Legendre and shifted Legendre polynomials for full discretization. The aim of this paper is to show that the spectral method based on the Legendre polynomial is also suitable for the treatment of the fractional partial differential equations. Numerical examples confirm the high accuracy of proposed scheme.

Keywords. Time fractional, mobile/immobile (MIM) equation, finite difference, spectral method, Legendre collocation method, Lagrangian polynomial.

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1. Introduction

The partial differential equations with fractional derivatives have several applications in engineering and physics [1, 4, 17]. The existence, uniqueness, and structural stability of solutions of nonlinear fractional PDE have been discussed in [11]. The governing equation of transport was derived based on the Fick's law and is commonly called the advection-dispersion equation (ADE) (Bear in [3]). The ADE will predict a breakthrough curve (BTC) that can be described by a Gaussian distribution function from an instantaneously releasing solute source (Bear in [3]). However, extensive evidence has shown different findings. In summary, there are two striking features that cannot be explained by the ADE. First, the peak of the BTC arrives earlier than what is expected from the ADE (early arrival); second, the tailing of the BTC lasts much longer than what is expected (the long tail) (Berkowitz, [5]). There are several candidates of conceptual models that may be applicable for replacing the ADE. The first is the continuous time random walk (CTRW) as reported by Scher and Lax in [19], and later introduced to hydrology by Berkowitz and others [6] and [2]. The other one is the mobile/immobile model (MIM) approach [21]. The MIM approach is based on a simple hypothesis: not all the pore spaces in a geological medium contribute

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to global flow. Since the early work of [9], the idea of the MIM has become popular among hydrologists for studying transport in saturated and unsaturated zones, and in granular as well as fractured media.

Solute transport in rivers, streams and ground water is controlled by the physical features or heterogeneity in different reaches. While the advection-dispersion equation and its extensions (e.g., the mobile-immobile or transient storage models based on a second order dispersion term) have been successfully used in the past, recent research highlights the need for transport models that can better describe the heterogeneity and connectivity of spatial features within a general network perspective of solute transport[22] (you can see [4, 10, 12, 15]).

In this paper we consider the time fractional mobili/immobile equation to the following form [14, 22]

$$\begin{gathered}\beta_1 \frac{\partial u(x,t)}{\partial t} + \beta_2 \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = -\alpha \frac{\partial u(x,t)}{\partial x} + \theta \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ (x,T) \in [0\ h] \times [0\ T], \qquad 0 < \gamma < 1,\end{gathered} \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} &u(0,t) = 0, \\ & \qquad\qquad\qquad 0 < t < T, \\ &u(h,t) = 0, \end{aligned} \tag{1.2}$$

and initial condition

$$u(x,0) = g(x), \qquad 0 < x < h, \tag{1.3}$$

where $\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$ and is defined as follows [16, 18]

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\mathbf{x},s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \qquad \alpha \in (0,1), \tag{1.4}$$

in which $\beta_1, \beta_2 > 1$ and $\alpha, \theta > 0$.

Some Numerical methods have been developed for the solution of Eq. (1.1). For example, [22] presented and analyzed a finite difference scheme for this equation with time fractional derivative of variable order. Authors in this paper [22], have proved unconditionally stability and convergence. A numerical scheme for the time fractional MIM equation which has first order temporal accuracy and first order spatial accuracy is developed in [14]. Also, recently several researchers have been proposed some numerical high-order techniques for solving partial differential equations with fractional derivatives that the interested readers can study [13].

The structure of the remainder of this paper as follows: A numerical method for discretization of time fractional derivative based on a finite difference scheme presented in section 2. Full discretization, Legendre polynomials and shifted Legendre polynomials are introduced in section 3. In section 3, we construct a Legendre spectral collocation method for the spatial discretization of time fractional MIM equation. We report the numerical experiments of solving mentioned equation in Section 4. Section 5 ends this paper with a brief conclusion.

2. Discretization in time, a finite difference scheme

First, we introduce a finite difference approximation to discretize the time fractional derivative. Let $t = k\tau$, $k = 0, 1, \ldots, M$, where $\tau = \frac{T}{M}$ is the time step and

$$\delta_t u^k = \frac{u^k - u^{k-1}}{\tau},$$

Now, we express the two following Lemma for discretization of the time fractional derivative.

Lemma 2 [20]. Suppose $0 < \gamma < 1$ and $g(t) \in C^2[0, t_k]$, it holds that

$$\left| \frac{1}{\Gamma(1-\gamma)} \int_0^{t_k} \frac{g'(t)}{(t_k - t)^{\gamma}} dt - \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left[b_0 g(t_k) - \sum_{m=1}^{k-1} (b_{k-m-1} - b_{k-m}) g(t_m) - b_{k-1} g(t_0) \right] \right|$$

$$\leq \frac{1}{\Gamma(2-\gamma)} \left[\frac{1-\gamma}{12} + \frac{2^{2-\gamma}}{2-\gamma} - (1 + 2^{-\gamma}) \right] \max_{0 \leq t \leq t_k} |g''(t)| \tau^{2-\gamma},$$

where $b_m = (m+1)^{1-\gamma} - m^{1-\gamma}$.

Lemma 3 [8]. Let $0 < \gamma < 1$ and $b_m = (m+1)^{1-\gamma} - m^{1-\gamma}$, $m = 0, 1, \ldots$, then

$$1 = b_0 > b_1 > \ldots > b_m \to 0 \ , \ as \ m \to \infty.$$

Now, for the sake of simplification, we define the discrete fractioal differential operetor $\mathcal{P}_t^{\gamma}$ by

$$\mathcal{P}_t^{\gamma} u(x, t_k) = \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left[b_0 u(x, t_k) - \sum_{m=1}^{k-1} (b_{k-m-1} - b_{k-m}) u(x, t_m) - b_{k-1} g(x) \right],$$

Then from Lemma 1 and application of the standard first-order backward differentiation to the time derivative, we can obtain

$$\frac{\partial^{\gamma} u(x, t_k)}{\partial t^{\gamma}} = \mathcal{P}_t^{\gamma} u(x, t_k) + r_t^k, \tag{2.1}$$

$$\frac{\partial u(x, t_k)}{\partial t} = \frac{u(x, t_k) - u(x, t_{k-1})}{\tau} + \mathcal{O}(\tau),$$

where $r_t^k = \mathcal{O}(\tau^{2-\gamma})$.
Using $\mathcal{P}_t^\gamma$ as an approximation of $\dfrac{\partial^\gamma u(x,t_k)}{\partial t^\gamma}$ leads to the finite difference scheme to (1.1),

$$\beta_1 \delta_t u(x,t_k) + \beta_2 \mathcal{P}_t^\gamma u(x,t_k) = \alpha \frac{\partial u(x,t_k)}{\partial x} + \theta \frac{\partial^2 u(x,t_k)}{\partial x^2} + f(x,t_k) + R_t^k, \quad (2.2)$$

there is C constant where $R_t^k \leq C\tau$. Now, let u^k be an approximation to $u(x,t_k)$. Then by omitting the small term R_t^k, the scheme (2.2) can be rewritten into

$$\begin{aligned} &(\mu_1 + \mu_2)\, u^k + \tau\alpha \frac{\partial u^k}{\partial x} - \tau\theta \frac{\partial^2 u^k}{\partial x^2} \\ &\quad = \mu_1 u^{k-1} + \mu_2 \sum_{m=1}^{k-1} (b_{k-m-1} - b_{k-m})\, u^m + \mu_2 b_{k-1} u^0 + \tau f^k, \end{aligned} \quad (2.3)$$

in which

$$\mu_1 = \beta_1, \qquad \mu_2 = \beta_2 \frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)}.$$

To introduce the variational formulation of the problem (2.3), we define some functional spaces endowed with standard norms and inner products that will be used hereafter,

$$H^1(\Lambda) = \left\{ v \in L^2(\Lambda),\ \frac{dv}{dx} \in L^2(\Lambda) \right\},$$

$$H_0^1(\Lambda) = \left\{ v \in H^1(\Lambda),\ v|_{\partial\Lambda} = 0 \right\},$$

$$H^m(\Lambda) = \left\{ v \in L^2(\Lambda),\ \frac{d^k v}{dx^k} \in L^2(\Lambda) \quad \textit{for all positive integer } k \leq m \right\},$$

where $L^2(\Lambda)$ is the space of measurable functions whose square is Lebesgue integrable in Λ. The inner product of $L^2(\Lambda)$ is defined by

$$(u,v) = \int_\Lambda uv\,dx,$$

and norm $\|.\|_m$ of the space $H^m(\Lambda)$is defined by

$$\|v\|_m = \left(\sum_{k=0}^{m} \left\| \frac{d^k v}{dx^k} \right\|_0^2 \right)^{\frac{1}{2}}, \qquad \|v\|_2 = (v,v)^{\frac{1}{2}},$$

3. Full discretization

The variational (weak) formulation of the Eq. (2.3) subject to boundary and initial conditions reads,

find $u^k \in H_0^1(\Lambda)$**, such that for all** $v \in H_0^1(\Lambda)$

$$\left(\mu_1+\mu_2\right)\left(u^k,v\right)+\tau\alpha\left(\frac{\partial u^k}{\partial x},v\right)-\tau\theta\left(\frac{\partial^2 u^k}{\partial x^2},v\right)=$$

$$\mu_1\left(u^{k-1},v\right)+\mu_2\sum_{m=1}^{k-1}\left(b_{k-m-1}-b_{k-m}\right)\left(u^m,v\right)+\mu_2 b_{k-1}\left(g,v\right)+\tau\left(f^k,v\right), \tag{3.1}$$

3.1. **Legendre polynomials.** We present here a collection of the essential formulas for Legendre polynomials. The Legendre polynomials $L_k(z), k=0,1,\ldots,$ are the eigenfunctions of the singular Sturm-Liouville problem

$$\left(\left(1-z^2\right)L_k'(z)\right)'+k(k+1)L_k(z)=0,$$

$L_k(z)$ is even if k is even and odd if k is odd. If $L_k(z)$ is normalized so that $L_k(1)=1$, then for any k

$$L_k(z)=\frac{1}{2^k}\sum_{l=0}^{[k/2]}(-1)^l\left(\begin{array}{c}k\\ l\end{array}\right)\left(\begin{array}{c}2k-2l\\ k\end{array}\right)z^{k-2l},$$

where $[k/2]$ denotes the integral part of $k/2$. The Legendre polynomials satisfy the recursion relation

$$L_{k+1}(z)=\frac{2k+1}{K+1}zL_k(z)-\frac{k}{k+1}L_{k-1}(z),$$

where $L_0(z)=1$ and $L_1(z)=x$. The expansion of any $u\in L^2(-1,1)$ in terms of the L_k is

$$u(z)=\sum_{k=0}^{\infty}\widetilde{u}L_k(z),\qquad \widetilde{u}=\left(k+\frac{1}{2}\right)\int_{-1}^{1}u(z)L_k(z)dz,$$

For practical use of Legendre polynomials on the space interval of interest $x\in[0\ \ h]$ it is necessary to shift the defining domain by the following variable substitution

$$z=\frac{2x-h}{h},\qquad 0\le x\le h,$$

Let the shifted Legendre polynomials $L_i(\frac{2x-h}{h})$ be denoted by $L_i^h(x)$. $L_i^h(x)$ can be obtained as follows

$$L_{i+1}^h(x)=\frac{(2i+1)\,(2x-h)}{(i+1)\,h}L_i^h(x)-\frac{i}{i+1}L_{i-1}^h(x),$$

where $L_0^h(x)=1$ and $L_1^h(x)=\dfrac{2x-h}{h}$. The analytic form of the shifted Legendre polynomial $L_i^h(x)$ of degree i is given by

$$L_i^h(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(k!)^2h^k}.$$

Firstly, we define $\mathbb{P}_N(\Lambda)$ the space of all polynomial of degree $\leq N$ with respect to x. Then the discrete space, denoted by $\mathbb{P}_N^0$, is defined as follows $\mathbb{P}_N^0 = H_0^1(\Lambda)\bigcap\mathbb{P}_N(\Lambda)$. Now, we use the Legendre collocation spectral method, which consist in approximating the integral by using the Legendre Gauss type quadratures. also, we must introduce some more notations. we assume $\eta_j,\ \ j=0,1,2,\dots,N$, are the shifted Legendre-Gauss-Lobatto (LGL) points, i.e. zeros of $\frac{dL_N^h(x)}{dx}$ for $j=1,2,\dots,N-1$ where $\eta_0=0$ and $\eta_N=h$, also, let $w_j,\ \ j=0,1,\dots,N$, the shifted Legendre weights defined such that the following quadrature holds,

$$\int_0^h \varphi(x)dx = \sum_{j=0}^{N}\varphi(\eta_j)w_j, \qquad \forall\varphi(x)\in\mathbb{P}_{2N-1}(\Lambda).$$

ω_j multipliers are called as the Legendre-Gauss-Lobatto weights, which are positive and may be obtained as [7]

$$\omega_j = \frac{2}{n(n+1)}\frac{1}{\left(P_n(\hat{x}_j)\right)^2},\ j=0,\dots,N. \tag{3.2}$$

Also, we define the discrete inner product as follow, for any continuous function φ and ψ,

$$(\varphi,\psi)_N = \sum_{i=0}^{N}\varphi(\eta_i)\psi(\eta_i)w_i,$$

and the associated discreted norm $\|\varphi\|_N = (\varphi,\varphi)_N^{1/2}$. Now we consider the shifted Legendre collocation approximation as follows, find $u_N^k\in\mathbb{P}_N^0(\Lambda)$, such that

$$\mathcal{A}_N(u_N^k, v_N) = \mathcal{B}_N(u_N^k), \qquad for\ all\ v_N\in P_N^0(\Lambda). \tag{3.3}$$

where the bilinear form $\mathcal{A}_N(.,.)$ is defined by

$$\mathcal{A}_N\left(u_N^k, v_N\right) = (\mu_1+\mu_2)\left(u_N^k, v_N\right)_N + \tau\alpha\left(\frac{\partial u_N^k}{\partial x}, v_N\right)_N + \tau\theta\left(\frac{\partial u_N^k}{\partial x}, \frac{\partial v_N}{\partial x}\right)_N,$$

and the functional $\mathcal{B}_N(\cdot)$ is given by

$$\mathcal{B}_N(\cdot) = \mu_2\left(u_N^{k-1},\cdot\right)_N + \mu_2\sum_{m=1}^{k-1}\left(b_{k-m-1}-b_{k-m}\right)\left(u_N^m,\cdot\right)_N$$
$$+\mu_2 b_{k-1}(g,\cdot)_N + \tau\left(f^k,\cdot\right)_N,$$

Now, we consider problem (3.3) and express the function u_N^k in the Lagrangian interpolations based on the shifted Legendre-Gauss-Lobatto points $\eta_j, j=0,1,2,\dots,N$,

$$u_N^k(x) = \sum_{j=0}^{N}u_j^k\phi_i(x), \tag{3.4}$$

where $u_j^k = u_N(\eta_j)$, unknowns of the discrete solution and $\phi_i(x)$ is the Lagrangian polynomial defined in Λ that is holds

$$\phi_i(\eta_j) = \begin{cases} 1, & i = j, \\ \\ 0, & i \neq j, \end{cases}$$

By substituting (3.4) into (3.3), and into account homogeneous Dirichlet buondary condition, $u_0^k = u_N^k = 0$, then choosing each test function v_n to be $\phi_i(x)$, $i = 1, 2, \dots, N-1$, we can obtain,

$$(\mu_1 + \mu_2)\left(\sum_{j=1}^{N-1} u_j^k \phi_j, \phi_i\right) + \tau\alpha \left(\frac{d}{dx}\sum_{j=1}^{N-1} u_j^k \phi_j, \phi_i\right) + \tau\theta\left(\frac{d}{dx}\sum_{j=1}^{N-1} u_j^k \phi_j, \frac{d}{dx}\phi_i\right)$$

$$= \mu_2 \left(\sum_{j=1}^{N-1} u_j^{k-1}\phi_j, \phi_i\right) + \mu_2 \sum_{m=1}^{k-1} \left(b_{k-m-1} - b_{k-m}\right)\left(\sum_{j=1}^{N-1} u_j^m \phi_j, \phi_i\right)$$

$$+\mu_2 b_{k-1}\left(g, \phi_i\right) + \tau\left(f^k, \phi_i\right), \qquad i = 1, 2, \dots, N-1.$$

Using the definition of the discrete inner product $(.,.)_N$ to the above, we can obtain the following system,

$$\sum_{j=1}^{N-1} H_{ij} u_j^k = \mu_2 \sum_{j=1}^{N-1}\left[H_{ij}^1 u_j^{k-1} + \sum_{m=1}^{k-1}\left(b_{k-m-1} - b_{k-m}\right) H_{ij}^1 u_j^m\right] + \mu_2 b_{k-1} H_i^4 + H_i^5,$$

where

$$\begin{cases} H_{ij} = (\mu_1+\mu_2)H_{ij}^1 + \alpha H_{ij}^2 + \theta H_{ij}^3, & D_{ij} = \dfrac{d}{dx}\phi_j(\eta_i), \\ \\ H_{ij}^1 = \displaystyle\sum_{q=0}^{N} \phi_j(\eta_q)\phi_j(\eta_q) w_q, & H_i^4 = \displaystyle\sum_{q=0}^{N} g(\eta_q)\phi_i(\eta_q) w_q, \\ \\ H_{ij}^2 = \tau \displaystyle\sum_{q=0}^{N} D_{qj}\phi_j(\eta_q) w_q, & H_i^5 = \tau\displaystyle\sum_{q=0}^{N} f^k(\eta_q)\phi_i(\eta_q) w_q, \\ \\ H_{ij}^3 = \tau\displaystyle\sum_{q=0}^{N} D_{qj} D_{qi} w_q, & i, j \in \{0, 1, 2, \dots, N\}, \end{cases}$$

Therefore, at each time step, we obtain a linear algebraic equation system with different right hand side vector.

Remark. If instead of homogeneous boundary conditions (1.3) we have the following nonhomogeneous boundary conditions

$$u(0,t) = \varphi_1(t), \qquad u(h,t) = \varphi_2(t),$$

then we reformulate the problem (1.1)-(1.3) by applying the following transformation that makes the boundary conditions become homogeneous

$$v(x,t) = u(x,t) + (x-h)\varphi_1(t) - x\varphi_2(t).$$

4. Numerical results

In this section we present the numerical results of the proposed method on several test problems. We tested the accuracy of the method described in this paper by performing the mentioned method for different values of M, γ, τ and N. We performed our computations using Maple 15 software on a Pentium IV, 2800 MHz CPU machine with 2G byte of memory. we compute the vector error norms L_∞ and L_2.

4.1. **Test problem 1.**

We consider the following example[22]

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial^\gamma u(x,t)}{\partial t^\gamma} = -\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ (x,t) \in [0\ 1] \times [0\ T], \\ u(x,0) = 10x^2(1-x)^2, \qquad 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, \qquad 0 \le t \le T, \end{cases}$$

where

$$f(x,t) = 10\left(1 + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right)x^2(1-x)^2 + 10(1+t)\left(2 + 14x - 18x^2 + 4x^3\right),$$

The exact solution of above problem is

$$u(x,t) = 10(1+t)x^2(1-x)^2.$$

Also we assume that γ is dependent to x and t. In this problem, we let

$$\gamma = \gamma(x,t) = 1 - \frac{1}{2}e^{-xt}.$$

We solve this problem with the presented method in this article with several value of N, τ and γ for $h = 1$ at final time $T = 1$. The L_∞ error, numerical solution and exact solution are shown in Table 1. Table 1 shows that our results are better than the results presented in [22]. Figure 1 presents the graphs of approximate solution and absolute error obtained with $\gamma = 0.2$, $N = 4$ and $\tau = 1/10$.

4.2. **Test problem 2.**

We consider the following example

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial^\gamma u(x,t)}{\partial t^\gamma} = -\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ (x,t) \in [0\ 1] \times [0\ T], \\ u(x,0) = 0, \qquad 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, \qquad 0 \le t \le T, \end{cases}$$

FIGURE 1. Approximation solution of Test problem 1 (b) and error (a) with $\gamma = 0.2$, $N = 4$ and $\tau = 1/10$.

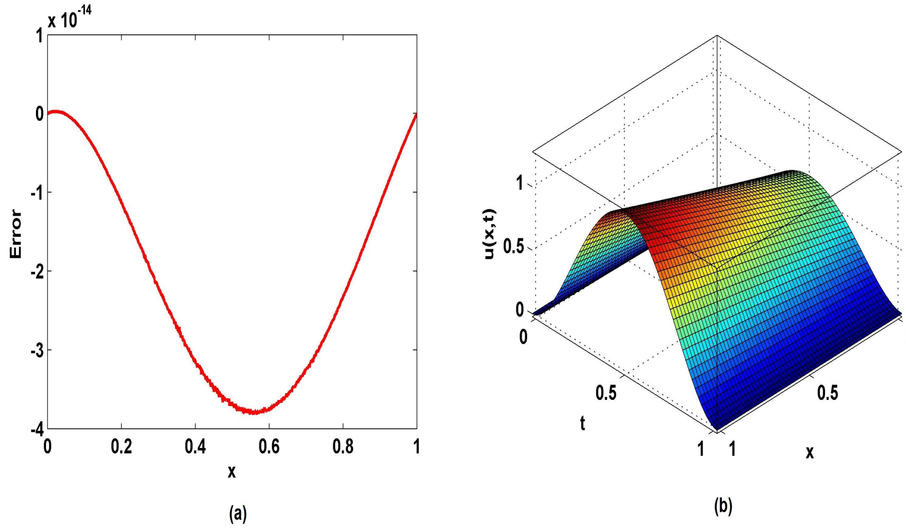


TABLE 1. Errors and numerical solution obtained for Test problem 1 with variable γ.

	Method of [22] with $\tau = h = 1/100$,		Present method with $N = 6, \tau = 1/10$,		
x	Num-sol	Error	Num-sol	Error	Exact-sol
0.1	0.16184371	0.00015629	0.1620000	2.0952×10^{-8}	0.1620000
0.2	0.51059931	0.00140069	0.5120000	2.1692×10^{-8}	0.5120000
0.3	0.87902481	0.00297519	0.8820000	2.1648×10^{-8}	0.8820000
0.4	1.14770234	0.00429766	1.1520000	2.1824×10^{-8}	1.1520000
0.5	1.24502781	0.00497219	1.2500000	2.1593×10^{-8}	1.2500000
0.6	1.14719659	0.00480341	1.1520000	2.1646×10^{-8}	1.1520000
0.7	0.87818473	0.00381527	0.8820000	2.1254×10^{-8}	0.8820000
0.8	0.50972531	0.00227469	0.5120000	2.1554×10^{-8}	0.5120000
0.9	0.16127925	0.00072075	0.1620000	2.1495×10^{-8}	0.1620000

where

$$f(x,t) = \left(1 + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right)\sin(\pi x) + t\left(\pi \cos(\pi x) + \pi^2 \sin(\pi x)\right),$$

The exact solution of above problem is

$$u(x,t) = t\sin(\pi x).$$

FIGURE 2. Approximation solution of Test problem 2 (b) and error (a) with $\gamma = 0.85$, $N = 20$ and $\tau = 1/10$.

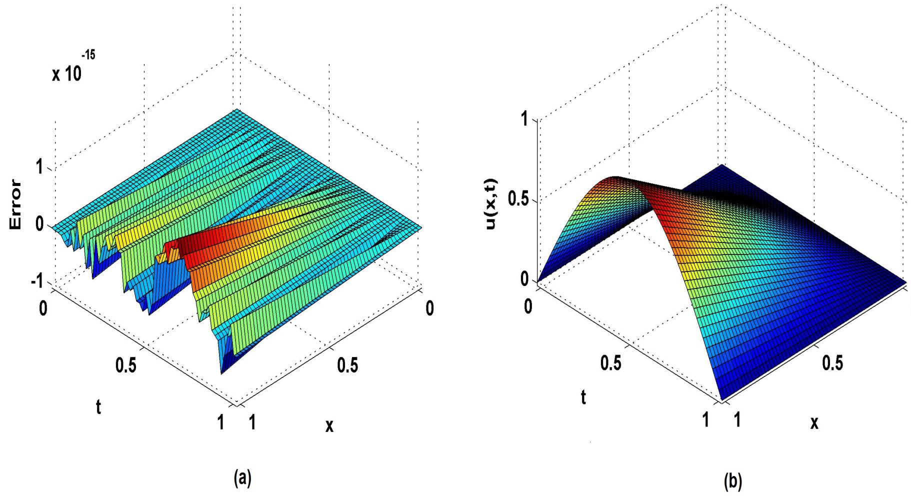


TABLE 2. Errors obtained for Test problem 2 with $N = 4$ and Digits=50.

	$\gamma = 0.25$		$\gamma = 0.75$	
τ	L_∞	L_2	L_∞	L_2
1/2	1.3374×10^{-3}	1.8547×10^{-3}	1.3328×10^{-3}	1.8525×10^{-3}
1/4	1.3360×10^{-3}	1.8542×10^{-3}	1.3295×10^{-3}	1.8513×10^{-3}
1/8	1.3355×10^{-3}	1.8541×10^{-3}	1.3283×10^{-3}	1.8509×10^{-3}
1/16	1.3354×10^{-3}	1.8540×10^{-3}	1.3278×10^{-3}	1.8508×10^{-3}
1/32	1.3353×10^{-3}	1.8539×10^{-3}	1.3276×10^{-3}	1.8507×10^{-3}

We solve this problem with the presented in this article with several value of N, τ and γ for $h = 1$ at final time $T = 1$. The L_∞ error and L_2 error are shown in Table 2, 3. Table 2 shows that the results of this numerical method is stable in time step. Also, Table 3 shows that the numerical results by increase of N are being better. Figure 2 presents the graphs of approximate solution and absolute error obtained with $\gamma = 0.85$, $N = 20$ and $\tau = 1/10$.

TABLE 3. Errors obtained for Test problem 2 with $\tau = 1/10$ and $Digits = 50$.

	$\gamma = 0.15$		$\gamma = 0.85$	
N	L_∞	L_2	L_∞	L_2
4	1.3379×10^{-3}	1.8555×10^{-3}	1.3279×10^{-3}	1.8514×10^{-3}
6	8.6784×10^{-6}	1.4306×10^{-5}	8.6753×10^{-6}	1.4302×10^{-5}
8	4.2723×10^{-8}	7.9009×10^{-8}	4.2717×10^{-8}	7.9002×10^{-8}
10	1.5452×10^{-10}	3.1662×10^{-10}	1.5451×10^{-10}	3.1660×10^{-10}
12	4.2584×10^{-13}	9.5289×10^{-13}	4.2582×10^{-13}	9.5286×10^{-13}
14	9.2079×10^{-16}	2.2235×10^{-15}	9.2077×10^{-16}	2.2235×10^{-15}
16	1.6006×10^{-18}	4.1323×10^{-18}	1.6007×10^{-18}	4.1323×10^{-18}
18	2.2829×10^{-21}	6.2537×10^{-21}	2.2828×10^{-21}	6.2536×10^{-21}
20	2.7173×10^{-24}	7.8508×10^{-24}	2.7172×10^{-24}	7.8507×10^{-24}

4.3. **Test problem 3.** We consider the following example

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial^\gamma u(x,t)}{\partial t^\gamma} = -\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (x,t) \in [0\ 1] \times [0\ 1], \\ u(x,0) = 0, \qquad 0 \le x \le 1, \\ u(0,t) = t, \quad u(1,t) = t^2 \exp(1), \qquad 0 \le t \le T, \end{cases}$$

with

$$f(x,t) = \mathrm{e}^x \left\{ 1 + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \right\},$$

in which the exact solution is

$$u(x,t) = t \exp(x).$$

Now, by considering Remark 1, the new problem with homogeneous boundary conditions is to the following form,

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} + \dfrac{\partial^\gamma v(x,t)}{\partial t^\gamma} = -\dfrac{\partial v(x,t)}{\partial x} + \dfrac{\partial^2 v(x,t)}{\partial x^2} + F(x,t), \quad (x,t) \in [0\ 1] \times [0\ 1], \\ v(x,0) = 0, \qquad 0 \le x \le 1, \\ v(0,t) = 0, \quad v(1,t) = 0, \qquad 0 \le t \le T, \end{cases}$$

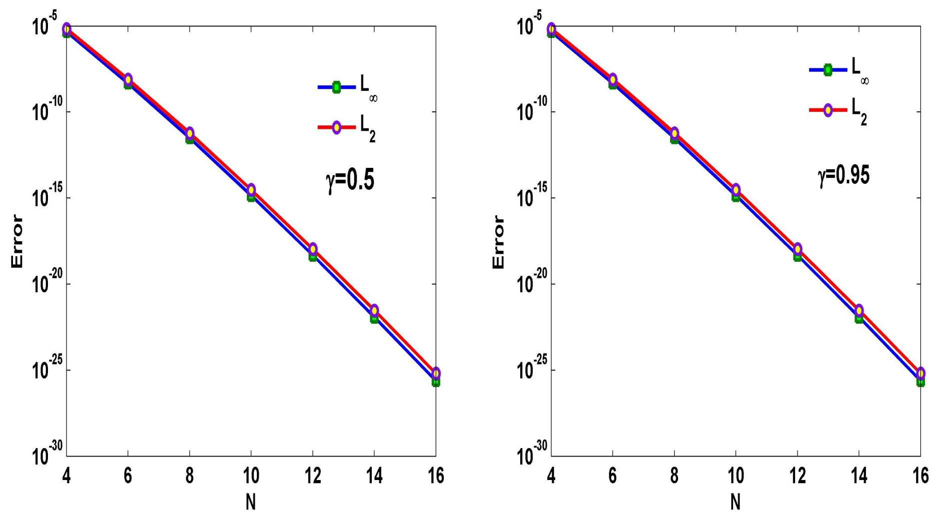
with

$$F(x,t) = \mathrm{e}^x \left\{ 1 + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \right\} + (1 - \exp(1))t$$
$$+ (x - 1 - x\exp(1)) \left\{ 1 + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \right\},$$

TABLE 4. Errors and computational orders obtained for Test problem 3 with $N = 4$ and Digits=50.

	$\gamma = 0.15$		$\gamma = 0.95$	
τ	L_∞	L_2	L_∞	L_2
1/2	4.6585×10^{-6}	6.7440×10^{-6}	4.6450×10^{-6}	6.7414×10^{-6}
1/4	4.6551×10^{-6}	6.7433×10^{-6}	4.6320×10^{-6}	6.7388×10^{-6}
1/8	4.6542×10^{-6}	6.7431×10^{-6}	4.6267×10^{-6}	6.7379×10^{-6}
1/16	4.6538×10^{-6}	6.7430×10^{-6}	4.6245×10^{-6}	6.7375×10^{-6}
1/32	4.6537×10^{-6}	6.7429×10^{-6}	4.6235×10^{-6}	6.7373×10^{-6}

FIGURE 3. Error as a function of the polynomial degree N for $\gamma = 0.5$ left panel and $\gamma = 0.95$ right panel for test problem 3.



where the exact solution is

$$v(x,t) = t(\exp(x) + x - 1 - x\exp(1)).$$

We solve this problem with the presented in this article with several value of N, τ and γ for $h = 1$ at final time $T = 1$. The L_∞ error and L_2 error are shown in Table 4. Table IV shows that the results of this numerical method is stable in time step. Figure 4 presents the error as a function of the polynomial degree N for $\gamma = 0.5$ left panel and $\gamma = 0.95$ right panel. Also, Figure 3 shows that the numerical results by increase of N are being better.

5. Conclusion

In this paper, we construct a finite difference/spectral method for the solution of time fractional mobile/immobile equation. The time fractional derivative of mentioned equation approximated by a scheme of order $\mathcal{O}(\tau^{2-\gamma})$ for $0 < \gamma < 1$ and for the spatial discretization, we use of the Legendre spectral collocation method where exponential convergence of this method had been proved. Comparing the numerical results with analytical solutions, reveals the applicability and efficiency of the proposed scheme.

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