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Superconvergence analysis of multistep collocation method for delay Volterra integral equations

P. Darania
Department of Mathematics, Faculty of Science,
Urmia University, P.O.Box 165, Urmia-Iran.
E-mail: p.darania@urmia.ac.ir

Abstract In this paper, we will present a review of the multistep collocation method for Delay Volterra Integral Equations (DVIEs) from [4] and then, we study the superconvergence analysis of the multistep collocation method for DVIEs. Some numerical examples are given to confirm our theoretical results.

Keywords. Delay integral equations, Multistep collocation method, Convergence and superconvergence analysis.

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1. Introduction

Many special cases of the differential and integral delay equation, can be encountered in applications: absorption of light by interstellar matter [1], analytic number theory [8], collection of current by the pantograph of an electric locomotive [9], nonlinear dynamical system [5], probability theory on algebraic structures [11], Cherenkov radiation [9], continuum mechanics [10], and the theory of dielectric materials [7]. Moreover, delay equation is an interesting example of a functional equation with a variable delay: sufficiently complicated to provide a clue to the behavior of more general classes of such equations, but also simple enough to be tractable by relatively straightforward means. For more detail see [7] and the references therein.

Consider the Volterra functional integral equation with delay function $\theta(t)$,

$$y(t) = \begin{cases} g(t) + \displaystyle\int_{t_0}^{t} k_1(t,s,y(s))ds + \int_{t_0}^{\theta(t)} k_2(t,s,y(s))ds, & t \in [t_0, T], \\ \phi(t), & t \in [\theta(t_0), t_0), \end{cases} \tag{1.1}$$

where $k_1 \in C(D \times \mathbb{R}), D = \{(t,s) : t_0 \le s \le t \le T\}$ and k_2 is assumed to be continuous in $D_\theta \times \mathbb{R}, D_\theta = \{(t,s) : \theta(t_0) \le s \le \theta(t)\}$, with $I = [t_0, T]$ and $\phi : [\theta(t_0), t_0] \to \mathbb{R}$ and $g : [t_0, T] \to \mathbb{R}$ are at least continuous on their domains.

The delay function θ satisfy in the following conditions:

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(D1) $\theta(t) = t - \tau(t)$, with $\tau(t) \geq \tau_0 > 0$ for $t \in I$;
(D2) θ is strictly increasing on I;
(D3) $\tau \in C^d(I)$ for some $d \geq 0$.

We will refer to the function $\tau = \tau(t)$ as the delay.

Definition 1.1. The points $\{\xi_\mu : \ \mu \geq 0\}$ generated by the recursion

$$\theta(\xi_{\mu+1}) = \xi_{\mu+1} - \tau(\xi_{\mu+1}) = \xi_\mu, \ \mu = 0, 1, ..., \ \xi_0 = t_0,$$

are called the primary discontinuity points associated with the lag function $\theta(t) = t - \tau(t)$.

Condition (D2) ensures that these discontinuity points have the (uniform) separation property

$$\xi_\mu - \xi_{\mu-1} = \tau(\xi_\mu) \geq \tau_0 > 0, \ \text{for all } \mu \geq 1.$$

Theorem 1.2. *Assume that the given functions in*

$$y(t) = \begin{cases} g(t) + \displaystyle\int_{t_0}^{t} k_1(t,s)y(s)ds + \int_{t_0}^{\theta(t)} k_2(t,s)y(s)ds, & t \in [t_0, T], \\ \phi(t), & t \in [\theta(t_0), t_0), \end{cases} \tag{1.2}$$

are continuous on their respective domains and that the delay function θ satisfies the above conditions (D1)-(D3). Then for any initial function $\phi \in C[\theta(t_0), t_0]$ there exists a unique (bounded) $y \in C(t_0, T]$ solving the Volterra functional integral equation (1.1) on $(t_0, T]$ and coinciding with ϕ on $[\theta(t_0), t_0]$. In general, this solution has a finite (jump) discontinuity at $t = t_0$:

$$\lim_{t \to t_0^+} y(t) \neq \lim_{t \to t_0^-} y(t) = \phi(t_0).$$

The solution is continuous at $t = t_0$ if, and only if, the initial function is such that

$$g(t_0) - \int_{t_0}^{\theta(t_0)} k_2(t_0, s)\phi(s)ds = \phi(t_0).$$

Proof. See [2]. □

It is well known that these equations typically have discontinuity in the solution or its derivatives at the initial point of integration domain. This discontinuity propagated along the integration interval giving rise to subsequent points, called singular points, which can not be determined priori and the solution derivatives in these points are smoothed out along the interval. Most of the known numerical methods for this type of equations are generally very sensitive to the singular points and therefore must have a process that detects these points and insert them into the mesh to guarantee the required accuracy.

In [2], H. Brunner applied collocation type methods for numerical solution of equation (1.1) and discussed about their connection with iterated collocation methods. V. Horvat, [6] had considered the collocation methods for Volterra integral equations with delay arguments. P. Darania [4], had considered the nonlinear Volterra integral equations with constant delays $\theta(t) = t - \tau,\ \tau > 0$, of the form

$$y(t) = \begin{cases} g(t) + (Vy)(t) + (V_\tau y)(t), & t \in I = [0, T], \\ \phi(t), & t \in [-\tau, 0), \end{cases} \tag{1.3}$$

where

$$(Vy)(t) = \int_0^t k_1(t, s, y(s))ds, \tag{1.4}$$

$$(V_\tau y)(t) = \int_0^{t-\tau} k_2(t, s, y(s))ds, \tag{1.5}$$

and the given functions, $\phi : [-\tau, 0] \to \mathbb{R}$, $g : I \to \mathbb{R}$, $k_1 : D \times \mathbb{R} \to \mathbb{R}, D = \{(t, s) : 0 \leq s \leq t \leq T\}$ and $k_2 : D_\tau \times \mathbb{R} \to \mathbb{R}, D_\tau = I \times [-\tau, T - \tau]$ are at least continuous on their domains.

In the present paper, we are briefly introduced the new families of multistep collocation method which has presented in [4, 3]. Also, we analyze the superconvergence analysis of the multistep collocation method when used to approximate smooth solutions of delay integral equations. Some numerical examples are given to confirm our theoretical results.

2. Preliminaries

Here, we recall the multistep collocation method that have been introduced in [4, 3].

2.1. Multistep collocation method. Let $t_n = nh$, $(n = 0, \ldots, N, \quad t_N = T, \quad h = \frac{\tau}{\tilde{r}}$ for som $\tilde{r} \in \mathbb{N})$ define a uniform partition for $I = [0, T]$, and let $\Omega_N := \{0 = t_0 < t_1 < \cdots < t_N = T\}, \sigma_0 := [t_0, t_1], \sigma_n := (t_n, t_{n+1}]$ $(1 \leq n \leq N-1)$. With a given mesh Ω_N, we associate the set of its interior points, $Z_N := \{t_n : n = 1, \ldots, N-1\}$. For a fixed $N \geq 1$ and, for given integer $m \geq 1$, the piecewise polynomial space $S_{m-1}^{(-1)}(Z_N)$ is defined by

$$S_{m-1}^{(-1)}(Z_N) := \{u : u|_{\sigma_n} = u_h \in \Pi_{m-1}, \quad 0 \leq n \leq N-1\},$$

where Π_{m-1} denotes the set of (real) polynomials of a degree not exceeding $m-1$.

Let $u_h = u|_{\sigma_n}, u \in S_{m-1}^{(-1)}(Z_N)$, for all $t \in \sigma_n$, we have

$$u_h(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j},\ s \in [0, 1],\ n = r, \ldots, N-1, \tag{2.1}$$

where $U_{n,j} = u_h(t_{n,j}), y_{n-k} = u_h(t_{n-k})$ and

$$\varphi_k(s) = \prod_{i=1}^{m} \frac{s - c_i}{-k - c_i} \cdot \prod_{\substack{i=0 \\ i \neq k}}^{r-1} \frac{s + i}{-k + i}, \qquad \psi_j(s) = \prod_{i=0}^{r-1} \frac{s + i}{c_j + i} \cdot \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{s - c_i}{c_j - c_i}. \tag{2.2}$$

Remark 2.1. In the following, we assume that the initial function $\phi(t)$ is such that

$$g(0) - \int_0^{-\tau} k_2(0,s)\phi(s)ds = \phi(0).$$

The collocation solution u_h will be determined by imposing the condition that u_h satisfies the integral equation (1.3) on the finite set $X_N = \{t_{n,j} = t_n + c_j h\}$

$$u_h(t) = \begin{cases} g(t) + (Vu_h)(t) + (V_\tau u_h)(t), & t \in [0,T], \\ \phi(t), & t \in [-\tau, 0), \end{cases} \tag{2.3}$$

where $\{c_j\}_{j=1}^m$, $0 \le c_1 < \cdots < c_m \le 1$, the set of collocation parameters. After some computations, the exact multistep collocation method is obtained by collocating both sides of (2.3) at the points $t = t_{n,j}$ for $j = 1, 2, \ldots, m$ and computing $y_{n+1} = u_h(t_{n+1})$:

$$\begin{cases} U_{n,j} = D_{n,j}, & j = 1, 2, \ldots, m, \\ y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum_{j=1}^{m} \psi_j(1) U_{n,j}, & n = r, r+1, \ldots, N-1, \end{cases} \tag{2.4}$$

where $D_{n,j} = D(t_{n,j})$ and

$$D(t_{n,j}) = g(t_{n,j}) + \begin{cases} (Vu_h)(t_{n,j}) + \Phi(t_{n,j}), & t_{n,j} - \tau < 0, \\ (Vu_h)(t_{n,j}) + (V_\tau u_h)(t_{n,j}), & t_{n,j} - \tau \ge 0, \end{cases} \tag{2.5}$$

$$\Phi(t_{n,j}) = \int_0^{t_{n,j}-\tau} k_2(t_{n,j}, s, \phi(s))ds, \; j = 1, 2, \ldots, m, \; n = 0, 1, \ldots, \tilde{r}-1, \tag{2.6}$$

and for $t_{n,j} - \tau < 0$, we have

$$\begin{aligned} (V_\tau u_h)(t_{n,j}) = \quad & -h \Bigg[\int_{c_j}^{1} k_2(t_{n,j}, t_{n-\tilde{r}} + sh, \phi(t_{n-\tilde{r}} + sh))ds \\ & + \sum_{i=n-\tilde{r}+1}^{-1} \int_0^1 k_2(t_{n,j}, t_i + sh, \phi(t_i + sh))ds \Bigg], \end{aligned} \tag{2.7}$$

and for $t_{n,j} - \tau \ge 0$, we get

$$\begin{aligned} (V_\tau u_h)(t_{n,j}) = \quad & h \Bigg[\sum_{i=0}^{n-\tilde{r}-1} \int_0^1 k_2(t_{n,j}, t_i + sh, u_h(t_i + sh))ds \\ & + \int_0^{c_j} k_2(t_{n,j}, t_{n-\tilde{r}} + sh, u_{n-\tilde{r}}(t_{n-\tilde{r}} + sh))ds \Bigg], \end{aligned} \tag{2.8}$$

and

$$(Vu_h)(t_{n,j}) = h\sum_{i=0}^{n-1}\int_0^1 k_1(t_{n,j}, t_i+sh, u_h(t_i+sh))ds \\ +h\int_0^{c_j} k_1(t_{n,j}, t_n+sh, u_h(t_n+sh))ds. \tag{2.9}$$

By using quadrature formulas with the weights w_l and nodes $d_l, l = 1, \ldots, \mu_1$, for integrating on $[0,1]$, and the weights $w_{j,l}$ and nodes $d_{j,l}, l = 1, \ldots, \mu_0$ for integrating on $[0, c_i]$, with positive integers μ_0 and μ_1, one can write

$$\begin{cases} Y_{n,j} = \bar{D}_{n,j}, & j = 1, 2, \ldots, m, \\ y_{n+1} = \displaystyle\sum_{k=0}^{r-1}\varphi_k(1)y_{n-k} + \sum_{j=1}^{m}\psi_j(1)Y_{n,j}, & n = r, r+1, \ldots, N-1, \end{cases} \tag{2.10}$$

where

$$\bar{D}(t_{n,j}) = g(t_{n,j}) + (\bar{V}u_h)(t_{n,j}) + (\bar{V}_\tau u_h)(t_{n,j}), \tag{2.11}$$

$$(\bar{V}u_h)(t_{n,j}) = h\sum_{i=0}^{n-1}\sum_{l=1}^{\mu_1} w_l k_1(t_{n,j}, t_i+d_l h, P_i(t_i+d_l h)) \\ +h\sum_{l=1}^{\mu_0} w_{j,l}k_1(t_{n,j}, t_n+d_{j,l}h, P_n(t_n+d_{j,l}h)), \tag{2.12}$$

and for $t_{n,j} - \tau < 0$, we have

$$(\bar{V}_\tau u_h)(t_{n,j}) = -h\left(\sum_{i=n-\tilde{r}+1}^{-1}\sum_{l=1}^{\mu_1} w_l k_2(t_{n,j}, t_i+d_l h, \phi(t_i+d_l h)) \right. \\ \left. +\sum_{l=1}^{\mu_1}\bar{w}_{j,l}k_2(t_{n,j}, t_{n-\tilde{r}}+\xi_{j,l}h, \phi(t_{n-\tilde{r}}+\xi_{j,l}h))\right), \tag{2.13}$$

and for $t_{n,j} - \tau \geq 0$, we get

$$(\bar{V}_\tau u_h)(t_{n,j}) = h\left(\sum_{i=0}^{n-\tilde{r}-1}\sum_{l=1}^{\mu_1} w_l k_2(t_{n,j}, t_i+d_l h, P_i(t_i+d_l h)) \right. \\ \left. +\sum_{l=1}^{\mu_0} w_{j,l}k_2(t_{n,j}, t_{n-\tilde{r}}+d_{j,l}h, P_{n-\tilde{r}}(t_{n-\tilde{r}}+d_{j,l}h))\right), \tag{2.14}$$

and $\xi_{j,l} := c_j + (1-c_j)d_l$, $\quad \bar{w}_{j,l} := (1-c_j)w_l$, $\quad j = 1, \ldots, m$, $\quad l = 1, \ldots, \mu_1$. Also, the discretized multistep collocation polynomial, denoted by

$$P_n(t_n+sh) = \sum_{k=0}^{r-1}\varphi_k(s)y_{n-k} + \sum_{j=1}^{m}\psi_j(s)Y_{n,j},\ \ s\in[0,1],\ \ n = r, \ldots, N-1.$$

$$(2.15)$$

For more detail see [4].

2.2. **Convergence.** Let $u_h \in S_{m-1}^{(-1)}(Z_N)$ denote the (exact) collocation solution to (1.3) defined by (2.4). In convergence analysis, we consider the linear test equation

$$y(t) = \begin{cases} g(t) + \int_0^t k_1(t,s)y(s)ds + \int_0^{t-\tau} k_2(t,s)y(s)ds, & t \in I, \\ \phi(t), & t \in [-\tau, 0), \end{cases} \tag{2.16}$$

where $k_1 \in C(D)$ and $k_2 \in C(D_\tau)$.

Theorem 2.2. *Let the given functions in (2.16) satisfy $g \in C^p(I),\ k_1 \in C^p(D),\ k_2 \in C^p(D_\tau),\ \phi \in C^p([-\tau, 0])$, and for $t \in [0, \tau]$ the integral*

$$\Phi(t) := \int_0^{t-\tau} k_2(t,s)\phi(s)ds, \tag{2.17}$$

is known exactly. Also, suppose that the starting error is

$$\| y - u_h \|_{\infty,[0,t_r]} = O(h^p), \tag{2.18}$$

and

$$\rho(\mathbf{A}) < 1, \tag{2.19}$$

where $p = m + r$ and ρ denotes the spectral radius and

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{r-1} \\ \hline \varphi_{r-1}(1) & \varphi_{r-2}(1), ..., \varphi_0(1) \end{array} \right]. \tag{2.20}$$

Then for all sufficiently small $h = \frac{\tau}{\tilde{r}}$, ($\tilde{r} \in \mathbb{N}$) the constrained mesh collocation solution $u_h \in S_{m-1}^{(-1)}(Z_N)$ to (2.16), satisfies

$$\| \mathcal{E} \|_\infty \le Ch^p, \tag{2.21}$$

where $\mathcal{E}(t) = y(t) - u_h(t)$ be the error of the exact collocation method (2.11) and C is positive constant not depending on h. This estimate holds for all collocation parameters $\{c_j\}$ with $0 \le c_1 < \cdots < c_m \le 1$.

Proof. See [4]. $\square$

Remark 2.3. [3]. The starting values $y_1, y_2, \ldots, y_r$, needed in (2.4) and (2.10), may be obtained by using a suitable starting procedure, based on a classical one step method has uniform convergence order of p.

Theorem 2.4. *Let the assumptions of Theorem 2.2 hold, except that the integrals*

$$\Phi(t) = \int_0^{t-\tau} k_2(t,s)\phi(s)ds,\ t = t_{n,j},\ n = 0, 1, ..., \tilde{r} - 1,$$

are now approximated by quadrature formulas $\bar{\Phi}(t)$, with corresponding quadrature errors $E_0(t) := \Phi(t) - \bar{\Phi}(t)$, such that

$$\|E_0(t)\| \le h^q, \tag{2.22}$$

for some $q > 0$. Then the collocation solution $u_h \in S^{(-1)}_{m-1}(Z_N)$ satisfies, for all sufficiently small $h > 0$,

$$\| \mathcal{E} \|_\infty \leq Ch^p, \tag{2.23}$$

with $p := \min\{m + r, q\}$, where C are finite constants not depending on h.

Proof. See [4]. □

3. Superconvergence

In this section, we analyze the superconvergence of the multistep collocation method when used to approximate smooth solutions of delay integral equations.

By the following theorem, we obtain local superconvergence in the interior points Z_N.

Theorem 3.1. *Suppose that the hypothesis of Theorem 2.2 hold with $p = 2m + r - 1$ and the collocation parameters $c_1, ..., c_m$ are the solution of system*

$$\begin{cases} c_m = 1, \\ \frac{1}{i+1} - \sum_{k=0}^{r-1} \beta_k(-k)^i - \sum_{j=1}^{m} \gamma_j(c_j)^i = 0, \quad i = m + r, ..., 2m + r - 2, \end{cases} \tag{3.1}$$

where

$$\beta_k = \int_0^1 \varphi_k(s)ds, \;\; \gamma_j = \int_0^1 \psi_j(s)ds. \tag{3.2}$$

Also, suppose that the delay integral

$$\Phi(t) = \int_0^{t-\tau} k_2(t, s, \phi(s))ds, \tag{3.3}$$

can be evaluated analytically and if $h = \frac{\tau}{\bar{r}}$ is sufficiently small. Then

$$\max_{1\leq n\leq N} |\mathcal{E}(t_n)| \leq Ch^p. \tag{3.4}$$

Proof. Without loss of generality, we assume that $T = t_N = M\tau$ for some $M \in \mathbb{N}$. The collocation equation (2.16) may be written in the form

$$u_h(t) = -\delta(t) + g(t) + \int_0^t k_1(t, s)u_h(s)ds + \int_0^{t-\tau} k_2(t, s)u_h(s)ds, \;\; t \in I, \tag{3.5}$$

where the defect function δ vanishes on X_N

$$\delta(t_{n,j}) = 0. \tag{3.6}$$

Also, we have $\delta(t) = 0$ for $t < 0$. The collocation error $\mathcal{E} = y - u_h$ solves the integral equation [2, 6]

$$\mathcal{E}(t) = \delta(t) + \int_0^t k_1(t, s)\mathcal{E}(s)ds + F(t), \;\; t \in I, \tag{3.7}$$

where

$$F(t) = \int_0^{t-\tau} k_2(t, s)\mathcal{E}(s)ds, \;\; t \in [\tau, T]. \tag{3.8}$$

For $t \in [0, \tau]$, we have $F(t) = 0$ and so the error equation (3.7) reduces to a classical Volterra equation, which is unique solution given by [2, 6]

$$\mathcal{E}(t) = \delta(t) + \int_0^1 R_1(t,s)\delta(s)ds, \tag{3.9}$$

where R_1 denotes the resolvent kernel associated with the given kernel k_1. As $T = M\tau$ for some positive integer M, we may set $\xi_\mu := \mu\tau,\ \mu = 0, ..., M$, and then for $t \in [\xi_\mu, \xi_{\mu+1}],\ 1 \leq \mu \leq M-1$ the collocation error $\mathcal{E}(t)$ governed by (3.7) can be expressed in the form [6, 2]

$$\mathcal{E}(t) = \delta(t) + \sum_{i=0}^{\mu} \int_0^{t-i\tau} R_{n,i}(t,s)\delta(s)ds, \tag{3.10}$$

where the function $R_{n,i}(t,s)$ depend on the given kernel functions $k_i(t,s),\ i = 1, 2$.

For $t = t_n \in [\xi_\mu + h, \xi_{\mu+1}]$, and $h = \frac{\tau}{\tilde{r}}$ for some $\tilde{r} \in \mathbb{N}$, we obtain

$$\mathcal{E}(t_n) = \delta(t_n) + h\sum_{i=0}^{\mu}\sum_{\nu=0}^{n-i\tilde{r}-1}\int_0^1 R_{n,i}(t_n, t_\nu + sh)\delta(t_\nu + sh)ds. \tag{3.11}$$

Now, let us consider the quadrature formula

$$\int_0^1 f(s)ds \approx \sum_{k=0}^{r-1}\beta_k f(-k) + \sum_{j=1}^{m}\gamma_j f(c_j),$$

for the computation of the integrals in (3.11), we have

$$\begin{aligned}\mathcal{E}(t_n) = \quad & \delta(t_n) + h\sum_{i=0}^{\mu}\sum_{\nu=0}^{n-i\tilde{r}-1}\left[\sum_{k=0}^{r-1}\beta_k R_{n,i}(t_n, t_{\nu-k})\delta(t_{\nu-k})\right. \\ & \left. + \sum_{j=1}^{m}\gamma_j R_{n,i}(t_n, t_{\nu,j})\delta(t_{\nu,j})\right] + h\sum_{i=0}^{\mu}\sum_{\nu=0}^{n-i\tilde{r}-1}E_{n,i,\nu},\end{aligned} \tag{3.12}$$

where $E_{n,i,\nu}$ the corresponding error terms. The hypothesis $c_m = 1$ assures that $t_{\nu-k}$ are collocation points for each ν. Since the defect function vanish in the collocation points, we have $\delta(t_{\nu,j}) = \delta(t_{\nu-k}) = 0$, hence

$$\mathcal{E}(t_n) = h\sum_{i=0}^{\mu}\sum_{\nu=0}^{n-i\tilde{r}-1}E_{n,i,\nu},\ 0 \leq \mu < n \leq \mu+1 \leq M,\ M\tau = T. \tag{3.13}$$

The quadrature errors in (3.13) can be bounded by $|E_{n,i,\nu}| \leq Ch^{2m+r-1}$ with some finite constant C not depending on h, (see [3] and [6]). Finally, because $M\tau = M\tilde{r}h = T = Nh$, we obtain $\max\limits_{1\leq n\leq N}|\mathcal{E}(t_n)| \leq Ch^{2m+r-1}$. □

Note that, the variation of constants formula (3.9) is the key to the establishing of (global and local) superconvergence results for collocation solutions to such equations [2].

Finally, we comment on the extension of the results in Theorem 3.1 to the nonlinear case. Instead of (3.7), the equation for the multistep collocation error $\mathcal{E}$ now has the form

$$\mathcal{E}(t) = \delta(t) + \int_0^t \{k_1(t,s,y(s)) - k_1(t,s,u_h(s))\}ds + F(t),\ t \in I, \tag{3.14}$$

where

$$F(t) = \int_0^{t-\tau} \{k_2(t,s,y(s)) - k_2(t,s,u_h(s))\}ds,\ t \in [\tau, T]. \tag{3.15}$$

Under appropriate differentiability and boundness conditions for k_1 and k_2, we obtain

$$k_i(t,s,y(s)) - k_i(t,s,u_h(s)) = \frac{\partial k_i}{\partial y}(t,s,y(s))\mathcal{E}(s) + \frac{1}{2}\frac{\partial^2 k_i}{\partial y^2}(t,s,z_i(s))\mathcal{E}^2(s),$$

where z_i is between y and u_h. The global convergence of $V(u_h), V_\tau(u_h), \bar{V}(u_h)$ and $\bar{V}_\tau(u_h)$ has already been establish [4]. So, we know that $\parallel \mathcal{E}^2 \parallel_\infty = O(h^{2(m+r)})$ for any $\{c_j\}$.

4. Presentation of results

In this section, two examples will be investigated to confirm our theoretical results. We consider the multistep collocation method with $m = 2, r = 3$, given by Examples 4.1 and 4.2 with the following choices of the collocation abscissas: $(c_1, c_2) = (0.7, 1)$ and from system (3.1), $(c_1, c_2) = (\frac{21}{38}, 1)$. The method have respectively order $p = 5$ (convergence) and $p = 6$ (superconvergence). The observed orders of convergence are computed from the maximum errors at the grid points. The starting values have been obtained from the known exact solutions. All computations are performed by the Mathematica® software.

Example 4.1. Consider the linear DVIEs as

$$y(t) = g(t) + \int_0^t (s+t+1)y(s)ds + \int_0^{t-\frac{1}{2}} (s+t^2+4)y(s)ds,\ t \in [0,1],$$

where $g(t)$ such that the exact solution is $y(t) = \sin t$.

Example 4.2. Consider the nonlinear DVIEs as

$$y(t) = g(t) + \int_0^t 2\cos(t-s)y^2(s)ds + \int_0^{t-\frac{1}{2}} 2\sin(t-s)y^2(s)ds,\ t \in [0,1],$$

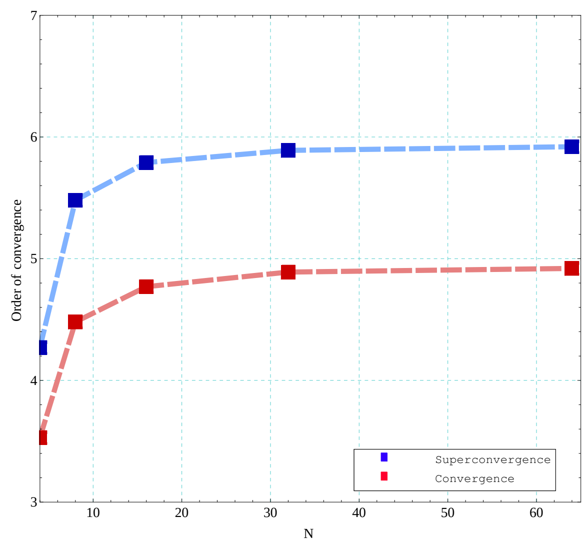
where $g(t)$ such that the exact solution is $y(t) = e^t$.

The maximum errors have been shown that for values of $r = 3, m = 2$ and for different values of N, at the grid points in the Table 1 and Table 2. The orders of convergence of the multistep collocation method for $r = 3$ and $m = 2$ are shown in Figure 1 and Figure 2 which they confirm the theoretical results of the Theorem 3.1

TABLE 1. Maximum errors $\|y - u_h\|_\infty$ for $r = 3$ and $m = 2$ in Example 4.1.

	$c_1 = 0.7, c_2 = 1$	$c_1 = \frac{21}{38}, c_2 = 1$
N	$\|\|y - u_h\|\|_\infty$	$\|\|y - u_h\|\|_\infty$
4	2.12002×10^{-6}	3.87888×10^{-7}
8	1.83053×10^{-7}	2.00000×10^{-8}
16	8.19697×10^{-9}	4.45114×10^{-10}
32	2.99995×10^{-10}	8.02836×10^{-12}
64	1.00999×10^{-11}	1.35225×10^{-13}

FIGURE 1. Orders of convergence of u_h for $r = 3$ and $m = 2$ in Example 4.1.



(the orders of convergence and superconvergence are $p = m + r$ and $p = 2m + r - 1$, respectively).

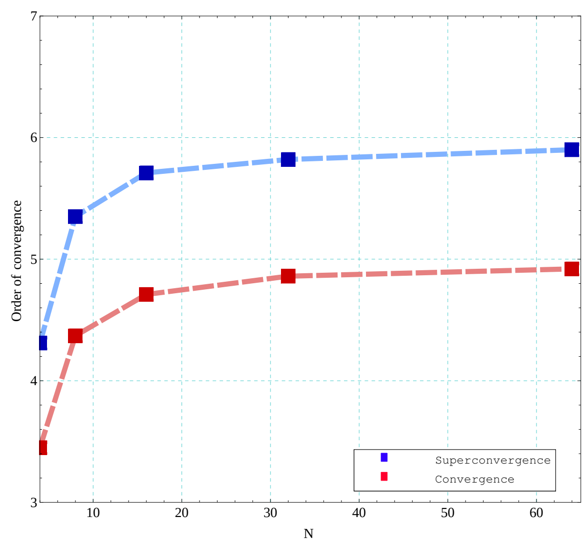
5. Conclusion

We have shown that the multistep collocation method yields an efficient and very accurate numerical method for the approximation of solutions to DVIEs. Furthermore, the multistep collocation method have uniform order $m + r$ for any choice of collocation parameters and local superconvergence order $2m + r - 1$ in mesh points

TABLE 2. Maximum errors $\|y - u_h\|_\infty$ for $r = 3$ and $m = 2$ in Example 4.2.

N	$c_1 = 0.7, c_2 = 1$ $\|\|y - u_h\|\|_\infty$	$c_1 = \frac{21}{38}, c_2 = 1$ $\|\|y - u_h\|\|_\infty$
4	1.15904×10^{-4}	5.06850×10^{-5}
8	1.22162×10^{-5}	2.54126×10^{-6}
16	5.89844×10^{-7}	6.19396×10^{-8}
32	2.23937×10^{-8}	1.18180×10^{-9}
64	7.68051×10^{-10}	2.07838×10^{-11}

FIGURE 2. Orders of convergence of u_h for $r = 2, 3$ and $m = 2$ in Example 4.2.



of collocation parameters.

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