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Solving multi-order fractional differential equations by reproducing kernel Hilbert space method

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Abstract In this paper, we propose a relatively new semi-analytical technique to approximate the solution of nonlinear multi-order fractional differential equations (FDEs). We present some results concerning to the uniqueness of solution of nonlinear multi-order FDEs and discuss the existence of solution for nonlinear multi-order FDEs in reproducing kernel Hilbert space (RKHS). We further give an error analysis for the proposed technique in different reproducing kernel Hilbert spaces and present some useful results. The accuracy of the proposed technique is examined by comparing with the exact solution of some test examples.

Keywords. Multi-Order Fractional; Hilbert space; Reproducing kernel method; Uniqueness; Existence; Error analysis.

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1. Introduction

Fractional differential equations have gained considerable importance due to their frequent applications in various fields of science and engineering [1-2], integral equations [3], viscoelastic damping materials [4-5], bioengineering [6-8], solid mechanics [9], chaos [10-11], control theory [12] and finance [13-14]. It has been found that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of different substances. Due to these features, the fractional-order models become more practical and realistic than the classical integrad-order models, in which such effects are not taken into account. Finding exact solutions in closed forms for most differential equations of fractional order is a difficult task

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[15-16]. As a result, a number of methods have been proposed and applied successfully to approximate fractional differential equations, such as Adomian decomposition method [17], variational iteration method [18], homotopy analysis method [19] and collocation method [20]. Especially, Momani and Odibat [21-22], have applied He's variational iteration method to fractional differential equations. Meanwhile, various fractional order differential equation have been solved very recently including fractional advection-dispersion equations [23-24], reaction-diffusion system with fractional derivatives [25], fractional partial differential equations fluid mechanics [26] and fractional-order multi-point boundary value problem [27-29]. Fractional differential equations with multi-orders have been used to model different types [30-33].

In the present work, we are concerned with the numerical solution of the following multi-order fractional differential equations in a reproducing kernel space method **(RKSM)** :

$$\begin{aligned} \mathtt{D}_{*\tau}^{\alpha_n}\phi(\tau) &= f(\tau, \phi(\tau), \mathtt{D}_{*\tau}^{\alpha_1}\phi(\tau), \mathtt{D}_{*\tau}^{\alpha_2}\phi(\tau), \cdots, \mathtt{D}_{*\tau}^{\alpha_{n-1}}\phi(\tau)), \\ D^{(j)}\phi(0) &= 0,\ j = 0, 1, 2, \cdots, n-1\ n \in \mathcal{N}, \end{aligned} \tag{1.1}$$

where $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n \le n$ and f is a given function on $\mathbf{D} := [0, T] \times \mathbb{R}^n$ and $\mathtt{D}_{*\tau}^{\alpha_i}$ stands for the Caputo fractional derivative of order α_i. Recently, the **(RKSM)** [34-35], has been used for obtaining approximate solutions of differential and integral equations [36-45]. However, due to the multi-point initial value conditions in (1), especially for fractional differential equations, it is difficult to find the corresponding reproducing kernel space by applying traditional RKSM. The aim of this work is to extend the RKSM to derive the numerical solutions of (1). One important improvement is that we successfully construct a novel reproducing kernel space to overcome difficulties with the multi order FDEs. By using the new reproducing kernel functions, we present an efficient numerical algorithm to solve (1). We especially emphasis on the uniformly convergence of the approximate solution and error estimation of our algorithm are studied.
The rest of paper is organized as follows. In Section 2, we present some important definitions and preparations used in this paper. In Section 3, we construct and develop algorithms for solving nonlinear fractional differential equation. In Section 4 the proposed methods are applied to several examples. Also a conclusion is given in Section 5.

2. Fractional Calculus

We now give some basic definitions and properties of the fractional calculus theory, which are used in the following sections. For more details see [15-16]. For the finite derivative in $[0, T]$, we define the following fractional integral and derivatives.

Definition 2.1. (see [15-16]). A real function $\phi(\tau)$, $\tau > 0$, is said to be in the space $C_\alpha, \alpha \in \mathbb{R}$, if there exists a real number $p\,(> \alpha)$, such that $\phi(\tau) = \tau^p \phi_1(\tau)$, where $\phi_1(\tau) \in C[0, \infty)$, and it is said to be in the space $C_\alpha^m, m \in \mathcal{N} \bigcup \{0\}$, if and only if $\phi^{(m)}(\tau) \in C_\alpha$.

Definition 2.2. (see [15-16]). The (left sided) Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $\phi(\tau) \in C_\alpha$, $\alpha \geq -1$, is defined as

$$I_\tau^\alpha \phi(\tau) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{\phi(t)}{(\tau-t)^{1-\alpha}} dt, \ \alpha > 0, \\ \\ \phi(\tau), \ \alpha = 1, \end{cases} \tag{2.1}$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 2.3. (see [15-16]). The (left sided) Caputo fractional derivative of $\phi(\tau)$, $\phi(\tau) \in C_{-1}^m$, $m \in \mathcal{N} \bigcup \{0\}$, is defined as

$$\mathtt{D}_{*\tau}^\alpha \phi(\tau) \quad = \quad \begin{cases} I_\tau^{m-\alpha} D^m \phi(\tau), & m-1 < \alpha < m, \\ \\ D^m \phi(\tau), & \alpha = m. \end{cases} \tag{2.2}$$

Theorem 2.4. *(see [15-16]). Assuming that the continues functions $\phi(\tau)$ and $\varphi(\tau)$ have a fractional derivatives of order α, then*

$$D_{*\tau}^\alpha(\gamma\phi(\tau) + \eta\varphi(\tau)) = \gamma D_{*\tau}^\alpha \phi(\tau) + \eta D_{*\tau}^\alpha \varphi(\tau), \ \gamma, \eta \in \mathcal{C}, \tag{2.3}$$

$$D_{*\tau}^\alpha(\phi(\tau)\varphi(\tau)) \neq \varphi(\tau) D_{*\tau}^\alpha \phi(\tau) + \phi(\tau) D_{*\tau}^\alpha \varphi(\tau). \tag{2.4}$$

Theorem 2.5. *(see [15-16]). Assuming that the continues functions $\phi(\tau)$ and $\varphi(\tau)$ have a fractional derivatives of order α, then the following properties hold,*

$$I_\tau^\alpha D_{*\tau}^\alpha \phi(\tau) = \phi(\tau) - \sum_{k=0}^{m-1} \phi^{(k)}(0^+) \frac{\tau^k}{k!}, \ m-1 < \alpha \leq m, \ m \in \mathcal{N}, \tag{2.5}$$

$$D_{*\tau}^\alpha I_\tau^\alpha \phi(\tau) = \begin{cases} \phi(\tau), \ m-1 < \alpha \leq m, \ m \in \mathcal{N}, \\ \\ I_\tau^\alpha D_{*\tau}^\alpha \phi(\tau) + \phi(0), \ 0 < \alpha < 1. \end{cases} \tag{2.6}$$

3. The Reproducing Kernel Hilbert Space

We give some basic definitions and properties of the reproducing kernel and reproducing kernel Hilbert space, and then we introduce some reproducing kernel Hilbert spaces which are used in the proceeding sections.

3.1. **Basic Definitions and Theorems.**

Definition 3.1. (see [34-35]). Let H be a real Hilbert space of functions on a set U. Denote by $< \phi, \varphi >_H$ the inner product and let $\|\phi\| = \sqrt{< \phi, \phi >_H}$ be the norm in H, for ϕ, $\varphi \in H$. The real valued function $R : U \times U \longrightarrow \mathbb{R}$ is called a reproducing kernel of H if the followings are satisfied
(i) For every τ, $R_\tau(s) = R(\tau, s)$ as a function of s belongs to U.
(ii) The reproducing property: $\phi(\tau) =< \phi(.), R_\tau(.) >_H$ for all $\phi \in H$ and all $\tau \in U$.

Definition 3.2. (see [34-35]). A Hilbert space H of functions on a set U is called a reproducing kernel Hilbert space if there exists a reproducing kernel R of H.

Definition 3.3. (see [34-35]). If a Hilbert space H of functions on a set U possesses a reproducing kernel, then the reproducing kernel R is uniquely determined by the Hilbert space H.

3.2. **The Hilbert Space $H^m([0,T])$ and Reproducing Kernel.** We recall the following definitions from [34-35].

Definition 3.4. The inner product space $H^m([0,T])$ for function ϕ is defined as $H^m([0,T]) = \{\phi(\tau)|\phi, \phi^{'}, ..., \phi^{m-1}$are absolutely continuous real value functions, $\phi^{(m)} \in L^2([0,T]),\ D^{(i)}\phi(0) = 0, i = 0, 1, ..., m-1,\ \tau \in [0,T]\}$.
The inner product in $H^m([0,T])$ is of he form

$$\langle \phi(\tau), \varphi(\tau)\rangle_{H^m} = \sum_{0\leq i<m} D^{(i)}\phi(0)D^{(i)}\varphi(0) + \int_0^T D^{(m)}\phi(s)D^{(m)}\varphi(s)ds, \tag{3.1}$$

and norm in the space $H^m([0,T])$ is defined as

$$\|\phi\|_{H^m} = \sqrt{\langle \phi, \phi\rangle_{H^m}}, \tag{3.2}$$

where $\phi, \varphi \in H^m([0,T])$.
It can be proved that the inner product space $H^m([0,T])$ is a reproducing kernel Hilbert space [].

Theorem 3.5. *The inner product space $H^m([0,T])$ is a reproducing kernel Hilbert space. For each $\tau \in [0,T]$, there exists a unique element $R_\tau(s) \in H^m([0,T])$, for any $\phi(s) \in H^m([0,T])$ and each fixed $\tau \in [0,T]$, $s \in [0,T]$, such that $\langle \phi(s), R_\tau(s)\rangle_{H^m} = \phi(\tau)$. The reproducing kernel $R_\tau(s)$ can be represented by*

$$R_\tau(s) = \begin{cases} \sum_{i=0}^{2m+1} c_i(\tau)s^i,\ \tau < s, \\ \sum_{i=0}^{2m+1} d_i(\tau)s^i,\ s \leq \tau, \end{cases} \tag{3.3}$$

where, the coefficients $c_i(\tau),\ d_i(\tau),\ i = 0, 1, ..., 2m+1$, are determined by

$$\begin{cases} D^{(m)}R_\tau(0) - D^{(m+1)}R_\tau(0) = 0, \\ D^{(2m-i+1)}R_\tau(1) = 0,\ i = 0, 1, 2, ..., m, \\ \lim_{s\to\tau^+} \frac{D^i R_\tau(s)}{Ds^i} = \lim_{s\to\tau^-} \frac{D^i R_\tau(s)}{Ds^i},\ i = 1, 2, ..., 2m, \\ (-1)^{m+1}(\lim_{s\to\tau^+} \frac{D^{2m+1}R_\tau(s)}{Ds^{2m+1}} - \lim_{s\to\tau^-} \frac{D^{2m+1}R_\tau(s)}{Ds^{2m+1}}) = 1, \\ D^{(i)}\phi(0) = 0,\ i = 0, 1, 2, ..., m-1. \end{cases} \tag{3.4}$$

Definition 3.6. Suppose that B_1 and B_2 are two Banach space such that $B_1 \subseteq B_2$. Then, we say the space B_1 is continuously embedded in B_2 and write $B_1 \hookrightarrow B_2$, if

$$\|\phi\|_{B_2} \leq c\|\phi\|_{B_1},\ \forall \phi \in B_1. \tag{3.5}$$

Theorem 3.7. *Let l_1 and l_2 be non-negative integers, $l_1 > l_2$ and suppose that $U \subseteq \mathbb{R}$. Then the following statement holds*

$$H^{l_1}(U) \hookrightarrow H^{l_2}(U). \tag{3.6}$$

We now derive the following theorems concering to the norm $\|\phi\|$.

Theorem 3.8. *For any $\phi \in H^m$ we have the following statement*

$$\|\phi\|_\infty \leq \boldsymbol{const}\ \|\phi\|_{H^m}, \tag{3.7}$$

for some $\boldsymbol{const}$ *independent of* ϕ.
Proof. *We know*

$$\phi(\tau) = D^{(m-1)}\phi(0)D^{(m-1)}R_\tau(0) + \int_0^1 D^{(m)}\phi(s)D^{(m)}R_\tau(s)ds. \tag{3.8}$$

From (3.8), for some $\boldsymbol{const}_1$ *we obtain*

$$\begin{aligned} |\phi(\tau)| &\leq |D^{(m-1)}\phi(0)||D^{(m-1)}R_\tau(0)| + \int_0^1 |D^{(m)}\phi(s)||D^{(m)}R_\tau(s)|ds \\ &\leq |D^{(m-1)}\phi(0)||D^{(m-1)}R_\tau(0)| + (\int_0^1 |D^{(m)}\phi(s)|^2 ds)^{\frac{1}{2}}(\int_0^1 |D^{(m)}R_\tau(s)|^2 ds)^{\frac{1}{2}} \\ &\leq \boldsymbol{const}_1\ (|D^{(m-1)}\phi(0)| + (\int_0^1 |D^{(m)}\phi(s)|^2 ds)^{\frac{1}{2}}), \end{aligned} \tag{3.9}$$

and therefore

$$\begin{aligned} &\|\phi(\tau)\|_\infty^2 \\ &\leq \boldsymbol{const}_2\ (|D^{(m-1)}\phi(0)|^2 + \textstyle\int_0^1 |D^{(m)}\phi(s)|^2 ds + 2|D^{(m-1)}\phi(0)|(\int_0^1 |D^{(m)}\phi(s)|^2 ds)^{\frac{1}{2}}) \\ &\leq 2\boldsymbol{const}_2\ (|D^{(m-1)}\phi(0)|^2 + \textstyle\int_0^1 |D^{(m)}\phi(s)|^2 ds) \\ &= \boldsymbol{const}\ (|D^{(m-1)}\phi(0)|^2 + \textstyle\int_0^1 |D^{(m)}\phi(s)|^2 ds). \end{aligned} \tag{3.10}$$

This completes the proof.

4. Efficient Results for Multi-Order Fractional Differential Equations

A multi-order fractional differential equation can be presented in the following form [30-33]

$$\begin{aligned} \mathtt{D}_{*\tau}^{\alpha_n}\phi(\tau) &= f(\tau, \phi(\tau), \mathtt{D}_{*\tau}^{\alpha_1}\phi(\tau), \mathtt{D}_{*\tau}^{\alpha_2}\phi(\tau), \cdots, \mathtt{D}_{*\tau}^{\alpha_{n-1}}\phi(\tau)), \\ D^{(j)}\phi(0) &= 0,\ j = 0, 1, 2, \cdots, n-1\ n \in \mathcal{N}, \end{aligned} \tag{4.1}$$

where $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n \leq n$ and f is a given function on $\mathbf{D} := [0, T] \times \mathbb{R}^n$ and $\mathtt{D}_{*\tau}^{\alpha_i}$ indicates the Caputo fractional derivative of order α_i. In this section, we study the uniqueness of solutions for the the problem (4.1). Before proving the uniqueness, we present the following auxiliary lemmas which will be needed in the sequel.

Lemma 4.1. Suppose that function $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous. Then

$\phi \in B$ is a solution of (4.1) if and only if $\phi \in B$ is a solution of following equation.

$$\phi(\tau) = \frac{1}{\Gamma(\alpha_n)} \int_0^\tau \frac{f(t, \phi(t), \mathtt{D}_{*t}^{\alpha_1}\phi(t), \mathtt{D}_{*t}^{\alpha_2}\phi(t), \cdots, \mathtt{D}_{*t}^{\alpha_{n-1}}\phi(t))}{(\tau - t)^{1-\alpha_n}} dt. \tag{4.2}$$

Proof. Suppose, ϕ satisfies the initial value problem (4.1), then applying $I_\tau^{\alpha_n}$ to the both sides of (4.1) and using Theorem 3.8, it will be obvious that (4.1) is equivalent to (4.2).

Lemma 4.2. (see [33]) For any $\phi : (0, \infty) \to \mathbb{R}$ we have the following statement

$$\mathtt{D}_{*\tau}^{\alpha}\phi(\tau) \in C([0, T]). \tag{4.3}$$

Theorem 4.1. *Suppose that the following conditions are satisfied.*
(i) $\forall \tau \in [0, T], f \in C([0, T] \times \mathbb{R}^n)$,
(ii) there exists a constant $\kappa > 0$ *such that*

$$|f(\tau, u_1, u_2, ..., u_n) - f(\tau, v_1, v_2, ..., v_n)| \leq \kappa|u_1 - v_1| + ... + \kappa|u_n - v_n|,$$

for all $\tau \in [0, T]$, $(u_1, ..., u_n)$,$(v_1, ..., v_n) \in \mathbb{R}^n$,. *Then (4.2) has a unique solution.*
Proof. *Let* $C([0, T])$ *be the space of all continuous functions defined on* $[0, T]$. *In addition, we define the Banch space* $B = \{\phi(\tau)|\phi(\tau) \in C([0, T]), D_{*\tau}^{\alpha_i}\phi(\tau) \in C([0, T]), i = 1, 2, ..., n-1\}$ *with the norm*

$$\|\phi\|_\theta = \max_{\tau\in[0,T]} e^{-\theta\kappa\tau}|\phi(\tau)| + \sum_{i=1}^{n-1} \max_{\tau\in[0,1]} e^{-\theta\kappa\tau}|D_{*\tau}^{\alpha_i}\phi(\tau)|. \tag{4.4}$$

Let further $T : B \to B$ *be an operator defined as*

$$(T\phi)(\tau) = \frac{1}{\Gamma(\alpha_n)} \int_0^\tau \frac{f(t, \phi(t), D_{*t}^{\alpha_1}\phi(t), \cdots, D_{*t}^{\alpha_{n-1}}\phi(t))}{(\tau - t)^{1-\alpha_n}} dt. \tag{4.5}$$

For any $\phi_1, \phi_2 \in B$, *we have*

$$\begin{aligned}
&|(T\phi_2)(\tau) - (T\phi_1)(\tau)| \\
&\leq \frac{1}{\Gamma(\alpha_n)} \int_0^\tau \frac{|f(t, \phi_2(t), D_{*t}^{\alpha_1}\phi_2(t), \cdots, D_{*t}^{\alpha_{n-1}}\phi_2(t)) - f(t, \phi_1(t), D_{*t}^{\alpha_1}\phi_1(t), \cdots, D_{*t}^{\alpha_{n-1}}\phi_1(t))|}{(\tau - t)^{1-\alpha_n}} dt \\
&= \frac{1}{\Gamma(\alpha_n)} \int_0^\tau \frac{\kappa|\phi_2 - \phi_1| + \sum_{i=2}^n \kappa|D_{*t}^{\alpha_i}\phi_2 - D_{*t}^{\alpha}\phi_1|}{(\tau - t)^{1-\alpha_n}} dt \\
&= \frac{1}{\Gamma(\alpha_n)} \int_0^\tau \frac{\kappa_1 e^{+\theta\kappa t}(e^{-\theta\kappa t}|\phi_2 - \phi_1|) + \sum_{i=2}^n \kappa e^{+\theta\kappa t}(e^{-\theta\kappa t}|D_{*t}^{\alpha_i}\phi_2 - D_{*t}^{\alpha_i}\phi_1|)}{(\tau - t)^{1-\alpha_n}} dt \\
&\leq \frac{\kappa}{\Gamma(\alpha_n)} \int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau - t)^{1-\alpha_n}} dt \; \|\phi_2 - \phi_1\|_\theta.
\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&|D_{*\tau}^{\alpha_i}(T\phi_2)(\tau) - D_{*\tau}^{\alpha_i}(T\phi_1)(\tau)| \\
&\leq \frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{|f(t,\phi_2(t),D_{*t}^{\alpha_1}\phi_2(t),\cdots,D_{*t}^{\alpha_{n-1}}\phi_2(t)) - f(t,\phi_1(t),D_{*t}^{\alpha_1}\phi_1(t),\cdots,D_{*t}^{\alpha_{n-1}}\phi_1(t))|}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt \\
&= \frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{\kappa|\phi_2-\phi_1| + \sum_{i=2}^n \kappa|D_{*t}^{\alpha_i}\phi_2 - D_{*t}^{\alpha_i}\phi_1|}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt \\
&= \frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{\kappa e^{+\theta\kappa t}(e^{-\theta\kappa t}|\phi_2-\phi_1|) + \sum_{i=2}^n \kappa e^{+\theta\kappa t}(e^{-\theta\kappa t}|D_{*t}^{\alpha_i}\phi_2 - D_{*t}^{\alpha_i}\phi_1|)}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt \\
&\leq \frac{\kappa}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt\ \|\phi_2-\phi_1\|_\theta. \qquad (4.7)
\end{aligned}
$$

Further, from (4.6) and (4.7), we have

$$
\begin{aligned}
&e^{-\theta\kappa\tau}|(T\phi_2)(\tau) - (T\phi_1)(\tau)| \\
&\leq \frac{\kappa e^{-\theta\kappa\tau}}{\Gamma(\alpha_n)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n}}dt\ \|\phi_2-\phi_1\|_\theta. \\
&e^{-\theta\kappa\tau}|D_{*\tau}^{\alpha_i}(T\phi_2)(\tau) - D_{*\tau}^{\alpha_i}(T\phi_1)(\tau)| \\
&\leq \frac{\kappa e^{-\theta\kappa\tau}}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt\ \|\phi_2-\phi_1\|_\theta. \qquad (4.8)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|(T\phi_2)(\tau) - (T\phi_1)(\tau)|\|_\theta \\
&\leq \kappa e^{-\theta\kappa\tau}\|\phi_2-\phi_1\|_\theta[\frac{1}{\Gamma(\alpha_n)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n}}dt \\
&+\sum_{i=1}^{n-1}\frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt].
\end{aligned}
$$

So we get

$$
\begin{aligned}
&\|(T\phi_2)(\tau) - (T\phi_1)(\tau)|\|_\theta \\
&\leq \sup_{\tau\in[0,T]}\{\kappa e^{-\theta\kappa\tau}[\frac{1}{\Gamma(\alpha_n)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n}}dt + \sum_{i=1}^{n-1}\frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt\}\|\phi_2-\phi_1\|_\theta].
\end{aligned}
$$

Now, we show that

$$
\lim_{\theta\to\infty}\sup_{\tau\in[0,T]}\{\kappa e^{-\theta\kappa\tau}[\frac{1}{\Gamma(\alpha_n)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n}}dt + \sum_{i=1}^{n-1}\frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt\}] = 0.
$$

If $\theta\kappa(\tau - t) = w$, we have

$$\begin{aligned}
0 &\leq \kappa e^{-\theta\kappa\tau}[\frac{1}{\Gamma(\alpha_n)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n}}dt + \sum_{i=1}^{n-1}\frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt] \\
&\leq \kappa e^{-\theta\kappa\tau}[\frac{1}{\Gamma(\alpha_n)}\int_0^{\theta\kappa\tau} \frac{e^{+\theta\kappa\tau-w}w^{\alpha_n-1}}{(\theta\kappa)^{\alpha_n-1}}\frac{dw}{\theta\kappa} + \sum_{i=1}^{n-1}\frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^{\theta\kappa\tau} \frac{e^{+\theta\kappa\tau-w}w^{\alpha_n-\alpha_i-1}}{(\theta\kappa)^{\alpha_n-\alpha_i-1}}\frac{dw}{\theta\kappa}] \\
&= \kappa e^{-\theta\kappa\tau}[\frac{e^{+\theta\kappa\tau}}{(\theta\kappa)^{\alpha_n}\Gamma(\alpha_n)}\int_0^{\theta\kappa\tau} e^{-w}w^{\alpha_n-1}dw + \sum_{i=1}^{n-1}\frac{e^{+\theta\kappa\tau}}{(\theta\kappa)^{\alpha_n-\alpha_i}\Gamma(\alpha_n-\alpha_i)}\int_0^{\theta\kappa\tau} e^{-w}w^{\alpha_n-\alpha_i-1}dw] \\
&\leq \kappa e^{-\theta\kappa\tau}[\frac{e^{+\theta\kappa\tau}}{(\theta\kappa)^{\alpha_n}} + \sum_{i=1}^{n-1}\frac{e^{+\theta\kappa\tau}}{(\theta\kappa)^{\alpha_n-\alpha_i}}] \\
&= \kappa[\frac{1}{(\theta\kappa)^{\alpha_n}} + \sum_{i=1}^{n-1}\frac{1}{(\theta\kappa)^{\alpha_n-\alpha_i}}] \qquad (4.9)
\end{aligned}$$

and therefore we see

$$\lim_{\theta\to\infty}\sup_{\tau\in[0,1]}\{\kappa e^{-\theta\kappa\tau}[\frac{1}{\Gamma(\alpha_n)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n}}dt + \sum_{i=1}^{n-1}\frac{1}{\Gamma(\alpha_n-\alpha_i)}\int_0^\tau \frac{e^{+\theta\kappa t}}{(\tau-t)^{1-\alpha_n+\alpha_i}}dt\}] = 0.$$

By the Banach contraction principle, T has a unique fixed point which is a solution of (4.2) and the proof is completed now.

5. The Proposed Method

Consider, the model problem (4.2) as follows

$$\begin{aligned}
&\phi(\tau) - \frac{1}{\Gamma(\alpha_n)}\int_0^\tau \frac{f(t,\phi(t),\mathtt{D}_{*t}^{\alpha_1}\phi(t),\mathtt{D}_{*t}^{\alpha_2}\phi(t),\cdots,\mathtt{D}_{*t}^{\alpha_{n-1}}\phi(t))}{(\tau-t)^{1-\alpha_n}}dt \\
&= (\text{Ł}\phi)(\tau) - (\mathbf{N}\phi)(\tau) - g(\tau) = 0 \qquad (5.1)
\end{aligned}$$

where $g(\tau)$ is an analytic function, $\phi(\tau) \in H^r[0,1], (r \geq n+1)$ is an unknown function which should be determined, $\mathbf{N}$ and $\text{Ł} : H^r[0,1] \to H^1[0,1]$ are nonlinear and linear operators, respectively. Suppose further that there exists a nonnegative constant M such that $\|\text{Ł}\phi\|_{H^1} \leq M\|\phi\|_{H^r}$, for all $\phi \in H^r$, then Ł is a bounded linear operator. Then, we consider (5.1) as following

$$(\text{Ł}\phi)(\tau) = g(\tau) + (\mathbf{N}\phi)(\tau). \qquad (5.2)$$

We set $\rho_i(\tau) = R_\tau(s)|_{s=\tau_i}$ and $q_i(\tau) = (\text{Ł}^*\rho_i)(\tau),\ i = 1,2,\ldots,$ where $R_\tau(s)$ is the reproducing kernel of $H^1([0,1])$ and Ł^* is the adjoint operator of linear operator Ł.

Theorem 5.1. *(see [35]) Let $\{\tau_i\}_{i=1}^\infty$ be a countable dense subset in the domain $[0,1]$, then $\{q_i(\tau)\}_{i=1}^\infty$ is a complete system of $H^r([0,1])$ and $q_i(x) = \text{Ł}_s R_\tau(s)|_{s=\tau_i}$, where the subscript s in the operator Ł indicates that the operator Ł applies to the function of s.*
Now, the orthonormal system $\{\tilde{q}_i(x)\}_{i=1}^\infty$ of $H^r([0,1])$ can be derived by Gram-Schmidt

orthogonalization process applied to $\{q_i(x)\}_{i=1}^{\infty}$*, namely*

$$\overline{q}_i(\tau) = \frac{q_i(\tau) - \sum_{l=1}^{i-1} r_{li} q_l(\tau)}{r_{ii}}, \tag{5.3}$$

where the coefficient r_{li} *in the numerators is*

$$r_{li} = \langle q_l(\tau), q_i(\tau)\rangle_{H^r}, l \neq i \tag{5.4}$$

and the coefficients r_{kk} *in the denominators are chosen for normalization*

$$r_{ii} = \langle q_i(\tau) - \sum_{l=1}^{i-1} r_{li} q_l(\tau)\rangle_{H^r}. \tag{5.5}$$

We note that if

$$\overline{q}_i(\tau) = \sum_{l=1}^{i} c_{li} q_l(\tau), \tag{5.6}$$

then clearly, $c_{li} = -\frac{r_{li}}{r_{ii}}$ $c_{ii} = \frac{1}{r_{ii}}$.

Theorem 5.2. *(see [35]) Let* $\{\tau_i\}_{i=1}^{\infty}$ *be dense in* $[0,1]$*, and the solution of (5.1) be unique, then the exact solution of (5.1) will be*

$$\phi(\tau) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} c_{ik}[g(\tau_k) + (\boldsymbol{N}\phi)(\tau_k)]\tilde{q}_i(\tau). \tag{5.7}$$

Now, we give the approximation solution to $\phi(\tau)$ *in the reproducing kernel space* $H^r([0,1])$*. For simplicity of numerical computations, we truncate the series (33), to derive*

$$\phi_N(\tau) = \sum_{i=1}^{N}\sum_{k=1}^{i} c_{ik}[\xi_k + g(\tau_k)]\overline{q}_i(\tau), \tag{5.8}$$

which is the N*-term approximation to* $\phi(\tau)$*, where* $\xi_k = (\boldsymbol{N}\phi)(\tau_k)$*.*
To evaluate ξ_k *in (5.8), we define the function*

$$Q(\xi_1, \xi_2, ..., \xi_N) = \sum_{k=1}^{N}[\xi_k - (\boldsymbol{N}\phi_N)(\tau_k)]^2. \tag{5.9}$$

The objective is to minimize (5.9). Therefore, we can get $\Phi_N(\tau)$ *by finding* $\xi_k,\ k = 1,2,...,N$ *to minimum* $Q(\xi_1, \xi_2, ..., \xi_N)$*.*
The steps of our procedure for the approximate solution of problem (4.1) can be summarized as follows:
Input: *The suitable initial values* $\xi_k^{\{0\}},\ k = 1,2,...,N$*.*
Output: *The values of* $\xi_k^{\{p\}},\ k = 1,2,...,N$ *to minimize* $Q(\xi_1, \xi_2, ..., \xi_N)$*.*
(i) *Calculate the values of* $Q(\xi_1^{\{0\}}, \xi_2^{\{0\}}, ..., \xi_N^{\{0\}})$ *and*

$$\phi_N^{\{0\}}(\tau) = \sum_{i=1}^{N}\sum_{k=1}^{i} c_{ik}[\xi_k^{\{0\}} + g(\tau_k)]\overline{q}_i(\tau) \tag{5.10}$$

by using (5.9) and (5.7), respectively.
(ii) *For the prescribed tolerance ϵ, if $Q(\xi_1^{\{0\}}, \xi_2^{\{0\}}, ..., \xi_N^{\{0\}}) < \epsilon$, then stop the computations, if not, then calculate the value of $\xi_k^{\{1\}},\ k = 1, 2, ..., N$ as follows*

$$\xi_k^{\{1\}} = (\boldsymbol{N}\phi_N^{\{0\}})(\tau_k)\ k = 1, 2, ..., N. \tag{5.11}$$

(iii) *Calculate the value of $Q(\xi_1^{\{1\}}, \xi_2^{\{1\}}, ..., \xi_N^{\{1\}})$ by using (5.9).*
(iv) *If $Q(\xi_1^{\{1\}}, \xi_2^{\{1\}}, ..., \xi_N^{\{1\}}) < Q(\xi_1^{\{0\}}, \xi_2^{\{0\}}, ..., \xi_N^{\{0\}})$, then $\xi_k^{\{1\}} \to \xi_k^{\{0\}},\ k = 1, 2, ..., N$ (replace $\xi_k^{\{0\}}$ by $\xi_k^{\{1\}}$) and calculate*

$$\phi_N^{\{1\}}(\tau) = \sum_{i=1}^{N}\sum_{k=1}^{i} c_{ik}[\xi_k^{\{1\}} + g(\tau_k)]\overline{q}_i(\tau) \tag{5.12}$$

and return to step (ii). If not, then give up $\xi_k^{\{1\}},\ k = 1, 2, ..., N$, pick a new $\xi_k^{\{0\}},\ k = 1, 2, ..., N$, and return to step (i). Therefore we obtain the approximate values of $\xi_k^{\{p\}},\ k = 1, 2, ..., N$ such that $Q(\xi_1^{\{p\}}, \xi_2^{\{p\}}, ..., \xi_N^{\{p\}}) < \epsilon$. Consequently, the solution of (4.1) becomes

$$\phi_N^{\{p\}}(\tau) = \sum_{i=1}^{N}\sum_{k=1}^{i} c_{ik}[\xi_k^{\{p\}} + g(\tau_k)]\overline{q}_i(\tau). \tag{5.13}$$

6. Error Analysis

Now, we consider the error estimation for the RKHSM applied to the the following functional problem

$$\begin{aligned} \mathtt{D}_{*\tau}^{\alpha_n}\phi(\tau) &= f(\tau, \phi(\tau), \mathtt{D}_{*\tau}^{\alpha_1}\phi(\tau), \mathtt{D}_{*\tau}^{\alpha_2}\phi(\tau), \cdots, \mathtt{D}_{*\tau}^{\alpha_{n-1}}\phi(\tau)), \\ D^{(j)}\phi(0) &= 0,\ j = 0, 1, 2, \cdots, n-1\ n \in \mathcal{N},\ 0 \le \alpha_1 < \cdots < \alpha_n \le n \end{aligned} \tag{6.1}$$

In the following we will obtain the error bounds for the approximate solution of (6.1) in the reproducing kernel Hilbert spaces $H^{n+3}([0,1])$, $H^{n+4}([0,1])$ and $H^{n+5}([0,1])$.

6.1. Error analysis in $H^{n+3}([0,1])$.

Theorem 6.1. *Let $\Delta_N = \{0 = \tau_1 < \tau_2 < ... < \tau_N = 1\}$ be a partition of the interval $[0,1]$, and $\delta = \delta(\Delta_N) = \max_{1\le i\le N-1}\Delta\tau_i$. Also suppose that $\phi_N(\tau)$ be the approximate solution of the problem in the Hilbert space $H^{n+3}([0,1])$ and f is a given continuous function. The following relation hold,*

$$\|\phi(\tau) - \phi_N(\tau)\|_\infty \le \boldsymbol{const}\ \delta, \tag{6.2}$$

where ***const*** *is a real constant.*
Proof *First, we define the following residual function for any $\phi_N(\tau)$ as an approximate solution of the problem (6.1)*

$$\Xi_N(\tau) = \mathcal{D}_{*\tau}^{\alpha_n}\phi_N(\tau) - f(\tau, \phi(\tau), \mathcal{D}_{*\tau}^{\alpha_1}\phi_N(\tau), \cdots, \mathcal{D}_{*\tau}^{\alpha_{n-1}}\phi_N(\tau)) \neq 0. \tag{6.3}$$

So from the reproducing kernel Hilbert space method we have

$$\Xi_N(\tau_m) = 0,\ 1 \leq m \leq N. \tag{6.4}$$

The piecewise linear interpolant $p_{\Delta_N}(\Xi_N;\tau)$ of $\Xi_N(\tau)$ is defined by the following two requirements
(a) For each $i = 1, 2, ..., N$, $p_{\Delta_N}(\Xi_N;\tau)|_{[\tau_i,\tau_{i+1}]}$ is linear.
(b) For each $i = 1, 2, ..., N$, $p_{\Delta_N}(\Xi_N;\tau_i) = \Xi_N(\tau_i)$.
For a function $\Xi_N(\tau) \in H^4([\tau_i,\tau_{i+1}])$, let $p(\Xi_N;\tau)$ be its linear interpolant, since $\Xi_N(\tau_i) = \Xi_N(\tau_{i+1}) = 0$, then we have

$$p(\Xi_N;\tau) = \Xi_N(\tau_i)\frac{(\tau_{i+1}-\tau)}{\tau_{i+1}-\tau_i} + \Xi_N(\tau_{i+1})\frac{(\tau_i-\tau)}{\tau_{i+1}-\tau_i} = 0,\ \tau_i \leq \tau \leq \tau_{i+1}. \tag{6.5}$$

By the Taylor's theorem

$$\Xi_N(\tau) = (\tau-\tau_i)\frac{d}{d\tau}\Xi_N(\tau) + \int_{\tau}^{\tau_i}(t-\tau_i)\frac{d^2}{dt^2}\Xi_N(t)dt, \tag{6.6}$$

$$\Xi_N(\tau) = (\tau-\tau_{i+1})\frac{d}{d\tau}\Xi_N(\tau) + \int_{\tau}^{\tau_{i+1}}(t-\tau_{i+1})\frac{d^2}{dt^2}\Xi_N(t)dt. \tag{6.7}$$

Thus, from (6.6) and (6.7) we have

$$\begin{aligned} \Xi_N(\tau) &= \frac{\tau-\tau_i}{\tau_{i+1}-\tau_i}\int_{\tau}^{\tau_{i+1}}(t-\tau_{i+1})\frac{d^2}{dt^2}\Xi_N(t)dt \\ &+ \frac{\tau_{i+1}-\tau}{\tau_{i+1}-\tau_i}\int_{\tau}^{\tau_i}(t-\tau_i)\frac{d^2}{dt^2}\Xi_N(t)dt, \end{aligned} \tag{6.8}$$

and therefore for some $\boldsymbol{const}_1$

$$\int_{\tau_i}^{\tau_{i+1}}|\Xi_N(\tau)|^2 d\tau \leq \boldsymbol{const}_1 \int_{\tau_i}^{\tau_{i+1}}|\frac{d^2}{d\tau^2}\Xi_N(\tau)|^2 d\tau. \tag{6.9}$$

Using (6.9) we have

$$\begin{aligned} \int_{\tau_i}^{\tau_{i+1}}|\Xi_N(\tau)|^2 d\tau &= \Delta\tau_i\int_0^1|\Xi_N(\tau_i+t\Delta\tau_i)|^2 dt \\ &\leq \boldsymbol{const}_1\ \Delta\tau_i\int_0^1|\frac{d^2}{dt^2}\Xi_N(\tau_i+t\Delta\tau_i)|^2 dt \\ &= \boldsymbol{const}_1\ \Delta^4\tau_i\int_{\tau_i}^{\tau_{i+1}}|\frac{d^2}{d\tau^2}\Xi_N(\tau)|^2 d\tau. \end{aligned} \tag{6.10}$$

Therefore,

$$\|\Xi_N(\tau)\|_{L^2}^2 = \sum_{i=1}^{N-1}\int_{\tau_i}^{\tau_{i+1}}|\Xi_N(\tau)|^2 dt \leq \boldsymbol{const}_1\ \delta^4\|\frac{d^2}{d\tau^2}\Xi_N(\tau)\|_{L^2}^2, \tag{6.11}$$

Differentiating the equation (6.8) with respect to τ, we have the following equation

$$\begin{aligned}\Xi_N^{'}(\tau) &= \frac{1}{\tau_{i+1}-\tau_i}\int_{\tau}^{\tau_{i+1}}(t-\tau_{i+1})\frac{d^2}{dt^2}\Xi_N(t)dt\\ &- \frac{1}{\tau_{i+1}-\tau_i}\int_{\tau}^{\tau_i}(t-\tau_i)\frac{d^2}{dt^2}\widehat{\Xi}_N(t)dt, \end{aligned}\tag{6.12}$$

and therefore for some $\boldsymbol{const}_2$

$$\int_{\tau_i}^{\tau_{i+1}}|\Xi_N^{'}(\tau)|^2 d\tau \leq \boldsymbol{const}_2 \int_{\tau_i}^{\tau_{i+1}}|\frac{d^2}{d\tau^2}\Xi_N(\tau)|^2 d\tau, \tag{6.13}$$

Using (6.13), we have

$$\begin{aligned}\int_{\tau_i}^{\tau_{i+1}}|\Xi_N^{'}(\tau)|^2 d\tau &= \frac{1}{\Delta\tau_i}\int_0^1|\frac{d}{dt}\Xi_N(\tau_i+t\Delta\tau_i)|^2dt\\ &\leq \frac{\boldsymbol{const}_2}{\Delta\tau_i}\int_0^1|\frac{d^2}{dt^2}\Xi_N(\tau_i+t\Delta\tau_i)|^2dt\\ &= \boldsymbol{const}_2\ \Delta^2\tau_i\int_{\tau_i}^{\tau_{i+1}}|\frac{d^2}{d\tau^2}\Xi_N(\tau)|^2d\tau. \end{aligned}\tag{6.14}$$

Then, a similar argument shows

$$\|\Xi_N^{'}(\tau)\|_{L^2}^2 = \sum_{i=1}^{N-1}\int_{\tau_i}^{\tau_{i+1}}|\Xi_N^{'}(\tau)|^2d\tau \leq \boldsymbol{const}_2\ \delta^2\|\frac{d^2}{d\tau^2}\Xi_N(\tau)\|_{L^2}^2. \tag{6.15}$$

Furthermore, by using the theory of interpolation, it is straightforward to show that

$$\Xi_N^2(a) \leq \boldsymbol{const}_2\ \delta^4. \tag{6.16}$$

Then

$$\|\Xi_N(\tau)\|_{H^1} \leq \boldsymbol{const}_3\delta, \tag{6.17}$$

where $\boldsymbol{const}_3$ *is a constant.*
By using Theorem 5.2, it is easy to show that

$$\|\phi(\tau)-\phi_N(\tau)\|_{H^4} = \|\boldsymbol{L}^*\|\|\Xi_N(\tau)\|_{H^1} \leq \boldsymbol{const}_4\ \delta, \tag{6.18}$$

Then using Theorem 3.8, we can obtain the following error bound

$$\|\phi(\tau)-\phi_N(\tau)\|_\infty \leq \boldsymbol{const}_5\ \|\phi(\tau)-\phi_N(\tau)\|_{H^4} \leq \boldsymbol{const}\ \delta. \tag{6.19}$$

6.2. Error analysis in $H^{n+4}([0,1])$.

Theorem 6.2. *Let $\Delta_N = \{0=\tau_1<\tau_2<\ldots<\tau_N=1\}$ be a partition of the interval $[0,1]$, and $\delta=\delta(\Delta_N)=\max_{1\leq i\leq N-1}\Delta\tau_i$. Also suppose that $\phi_N(\tau)$ be the approximate solution of the above problems in the Hilbert space $H^5([0,1])$ and f be a given continuous function. The following relation hold,*

$$\|\phi(\tau)-\phi_N(\tau)\|_\infty \leq \boldsymbol{const}\ \delta^2, \tag{6.20}$$

*where **const** is a real constant.*
Proof *First, we define the following residual function for any $\phi_N(\tau)$ as an approximate solution of the problem*

$$\Xi_N(\tau) = D_{*\tau}^{\alpha_n}\phi_N(\tau) - f(\tau, \phi(\tau), D_{*\tau}^{\alpha_1}\phi_N(\tau), \cdots, D_{*\tau}^{\alpha_{n-1}}\phi_N(\tau)) \neq 0. \tag{6.21}$$

We know $\Xi_N(\tau_m) = 0,\ 1 \leq m \leq N$, on subinterval $[\tau_i, \tau_{i+1}]$, by applying the Roll's theorem to $\Xi_N(\tau)$, once more we get $\Xi_N^{'}(\varsigma_i) = 0,\ \varsigma_i \in [\tau_i, \tau_{i+1}],\ i = 1, 2, ..., N-1$. The piecewise linear interpolant $p_{\Delta_N}(\Xi_N^{'}; \tau)$ of $\Xi_N^{'}(\tau)$ is defined by the following two requirements
(a) For each $i = 1, 2, ..., N-2,\ p_{\Delta_N}(\Xi_N^{'}; \tau)|_{[\varsigma_i, \varsigma_{i+1}]}$ is linear.
(b) For each $i = 1, 2, ..., N-2,\ p_{\Delta_N}(\Xi_N^{'}; \varsigma_i) = \Xi_N^{'}(\varsigma_i)$.
For a function $\widehat{\Xi'}_N(\tau)$, let $p(\Xi_N^{'}; \tau)$ be its linear interpolant, since $\Xi_N^{'}(\varsigma_i) = \Xi_N^{'}(\varsigma_{i+1}) = 0$, then we have

$$p(\Xi_N^{'}; \tau) = \Xi_N^{'}(\varsigma_i)\frac{(\varsigma_{i+1} - \tau)}{\varsigma_{i+1} - \varsigma_i} + \Xi_N^{'}(\varsigma_{i+1})\frac{(\varsigma_i - \tau)}{\varsigma_{i+1} - \varsigma_i} = 0,\ \varsigma_i \leq \tau \leq \varsigma_{i+1}. \tag{6.22}$$

By the Taylor's theorem

$$\Xi_N^{'}(\tau) = (\tau - \varsigma_i)\frac{d^2}{d\tau^2}\Xi_N(\tau) + \int_\tau^{\varsigma_i}(t - \varsigma_i)\frac{d^3}{dt^3}\Xi_N(t)dt, \tag{6.23}$$

$$\Xi_N^{'}(\tau) = (\tau - \varsigma_{i+1})\frac{d^2}{d\tau^2}\Xi_N(\tau) + \int_\tau^{\varsigma_{i+1}}(t - \varsigma_{i+1})\frac{d^3}{dt^3}\Xi_N(t)dt. \tag{6.24}$$

Thus, from (6.23) and (6.24) we have

$$\begin{aligned}\Xi_N^{'}(\tau) &= \frac{\tau - \varsigma_i}{\varsigma_{i+1} - \varsigma_i}\int_\tau^{\varsigma_{i+1}}(t - \varsigma_{i+1})\frac{d^3}{dt^3}\Xi_N(t)dt \\ &+ \frac{\varsigma_{i+1} - \tau}{\varsigma_{i+1} - \varsigma_i}\int_\tau^{\varsigma_i}(t - \varsigma_i)\frac{d^3}{dt^3}\Xi_N(t)dt,\end{aligned} \tag{6.25}$$

and therefore for a $\boldsymbol{const}_1$

$$\int_{\varsigma_i}^{\varsigma_{i+1}}|\Xi_N^{'}(\tau)|^2 d\tau \leq \boldsymbol{const}_1 \int_{\varsigma_i}^{\varsigma_{i+1}}|\frac{d^3}{d\tau^3}\Xi_N(\tau)|^2 d\tau. \tag{6.26}$$

Using (6.26) we have

$$\begin{aligned}\int_{\varsigma_i}^{\varsigma_{i+1}}|\Xi_N^{'}(\tau)|^2 d\tau &= \frac{1}{\Delta\varsigma_i}\int_0^1|\frac{d}{dt}\Xi_N(\varsigma_i + t\Delta\varsigma_i)|^2 dt \\ &\leq \frac{\boldsymbol{const}_1}{\Delta\varsigma_i}\int_0^1|\frac{d^3}{dt^3}\Xi_N(\varsigma_i + t\Delta\varsigma_i)|^2 dt \\ &= \boldsymbol{const}_1\ \Delta^4\varsigma_i\int_{\varsigma_i}^{\varsigma_{i+1}}|\frac{d^3}{d\tau^3}\Xi_N(\tau)|^2 d\tau.\end{aligned} \tag{6.27}$$

Therefore,

$$\begin{aligned}\|\Xi_N'(\tau)\|^2_{L^2[\varsigma_1,\varsigma_{N-1}]} &= \int_{\varsigma_1}^{\varsigma_{N-1}} |\Xi_N'(\tau)|^2 d\tau \\ &= \sum_{i=1}^{N-2} \int_{\varsigma_i}^{\varsigma_{i+1}} |\Xi_N'(\tau)|^2 d\tau \leq \boldsymbol{const_2}\ \delta^4. \end{aligned} \tag{6.28}$$

Also, for $\tau \in [0, \varsigma_2]$, there exists $\varrho \in [\min\{\varsigma_1, \varsigma_2, \tau\}, \max\{\varsigma_1, \varsigma_2, \tau\}]$ such that

$$\Xi_N'(\tau) = (\tau - \varsigma_1)(\tau - \varsigma_2)\Xi_N[\varsigma_1, \varsigma_2, \tau, \tau]$$

where the quantity $\Xi_N[\varsigma_1, \varsigma_2, \tau, \tau]$ is a Newton divided difference of order three for function $\Xi_N(\tau)$.
From the above formula, we deduce

$$\|\Xi_N'(\tau)\|^2_{L^2[0,\varsigma_1]} \leq \|\Xi_N'(\tau)\|^2_{L^2[0,\varsigma_2]} \leq \boldsymbol{const_3}\ \delta^4. \tag{6.29}$$

A similar argument shows

$$\|\Xi_N'(\tau)\|^2_{L^2[\varsigma_{N-1},1]} \leq \|\Xi_N'(\tau)\|^2_{L^2[\varsigma_{N-2},1]} \leq \boldsymbol{const_4}\ \delta^4. \tag{6.30}$$

Using (6.28), (6.33) and (6.30), we have

$$\|\Xi_N'\|^2_{L^2[0,1]} \leq \boldsymbol{const_5}\ \delta^4, \tag{6.31}$$

where $\boldsymbol{const_5} = \boldsymbol{const_2} + \boldsymbol{const_3} + \boldsymbol{const_4}$.
Now, in each subinterval $[\tau_i, \tau_{i+1}]$, we have

$$\Xi_N(\tau) = \int_{\tau_i}^{\tau} \frac{d}{dt}\Xi_N(t)dt, \tag{6.32}$$

then, it holds

$$\|\Xi_N(\tau)\|^2_\infty \leq \boldsymbol{const_6}\ \delta^6. \tag{6.33}$$

Using (6.31) and (6.33) we have

$$\|\Xi_N(\tau)\|_{H^1} \leq \boldsymbol{const_7}\ \delta^2, \tag{6.34}$$

where $\boldsymbol{const_7}$ is a constant.
By using Theorem 3.7 and Theorem 3.8, we can obtain the following error bound

$$\|\phi(\tau) - \phi_N(\tau)\|_\infty \leq \boldsymbol{const}\ \delta^2, \tag{6.35}$$

6.3. Error analysis in $H^{n+5}([0,1])$.

Theorem 6.3. *Let $\Delta_N = \{0 = \tau_1 < \tau_2 < ... < \tau_N = 1\}$ be a partition of the interval $[a,b]$, and $\delta = \delta(\Delta_N) = \max_{1\leq i\leq N-1} \Delta\tau_i$. Also suppose that $\phi_N(\tau)$ be the approximate solution of the above problems in the Hilbert space $H^6([0,1])$ and f be a given continuous function. The following relation hold,*

$$\|\phi(\tau) - \phi_N(\tau)\|_\infty \leq \boldsymbol{const}\ \delta^3, \tag{6.36}$$

where $\boldsymbol{const}$ is a real constant.
Proof *First, we define the following residual function for any $\phi_N(\tau)$ as an approximate solution of the problem*

$$\Xi_N(\tau) = \mathcal{D}_{*\tau}^{\alpha_n}\phi_N(\tau) - f(\tau, \phi(\tau), \mathcal{D}_{*\tau}^{\alpha_1}\phi_N(\tau), \cdots, \mathcal{D}_{*\tau}^{\alpha_{n-1}}\phi_N(\tau)) \neq 0. \tag{6.37}$$

Since $\Xi_N(\tau_m) = 0,\ 1 \le m \le N,$ *on subinterval* $[\tau_i, \tau_{i+1}]$*, by applying the Roll's theorem to* $\Xi_N(\tau)$*, we get* $\Xi^{'}_N(\varsigma_i) = 0,\ \varsigma_i \in [\tau_i, \tau_{i+1}],\ i = 1, 2, ..., N-1,$ *by applying the Roll's theorem to* $\Xi^{'}_N(\tau)$*, we have* $\Xi^{''}_N(\sigma_i) = 0,\ \sigma_i \in [\varsigma_i, \varsigma_{i+1}],\ i = 1, 2, ..., N-2.$ *The piecewise linear interpolant* $p_{\Delta_N}(\Xi^{''}_N; \tau)$ *of* $\Xi^{''}_N(\tau)$ *is defined by the following two requirements*
(a) For each $i = 1, 2, ..., N-3,\ p_{\Delta_N}(\Xi^{''}_N; \tau)|_{[\sigma_i, \sigma_{i+1}]}$ *is linear.*
(b) For each $i = 1, 2, ..., N-3,\ p_{\Delta_N}(\Xi^{''}_N; \sigma_i) = \Xi^{''}_N(\sigma_i).$
For a function $\Xi^{''}_N(\tau)$*, let* $p(\Xi^{''}_N; \tau)$ *be its linear interpolant, Since* $\Xi^{''}_N(\sigma_i) = \Xi^{''}_N(\sigma_{i+1}) = 0,$ *then we have*

$$p(\Xi^{''}_N; \tau) = \Xi^{''}_N(\sigma_i)\frac{(\sigma_{i+1} - \tau)}{\sigma_{i+1} - \sigma_i} + \Xi^{''}_N(\sigma_{i+1})\frac{(\sigma_i - \tau)}{\sigma_{i+1} - \sigma_i} = 0,\ \sigma_i \le \tau \le \sigma_{i+1}. \tag{6.38}$$

By the Taylor's theorem

$$\Xi^{''}_N(\tau) = (\tau - \sigma_i)\frac{d^3}{d\tau^3}\Xi_N(\tau) + \int_{\tau}^{\sigma_i}(t - \sigma_i)\frac{d^4}{dt^4}\Xi_N(t)dt, \tag{6.39}$$

$$\Xi^{''}_N(\tau) = (\tau - \sigma_{i+1})\frac{d^3}{d\tau^3}\Xi_N(\tau) + \int_{\tau}^{\sigma_{i+1}}(t - \sigma_{i+1})\frac{d^4}{dt^4}\Xi_N(t)dt. \tag{6.40}$$

Thus, from (6.39) and (6.40) we obtain

$$\begin{aligned}\Xi^{''}_N(\tau) &= \frac{\tau - \sigma_i}{\sigma_{i+1} - \sigma_i}\int_{\tau}^{\sigma_{i+1}}(t - \sigma_{i+1})\frac{d^4}{dt^4}\Xi_N(t)dt \\ &+ \frac{\sigma_{i+1} - \tau}{\sigma_{i+1} - \sigma_i}\int_{\tau}^{\sigma_i}(t - \sigma_i)\frac{d^4}{dt^4}\Xi_N(t)dt,\end{aligned} \tag{6.41}$$

and for $\tau \in [\sigma_i, \sigma_{i+1}]$*, it follows that*

$$\begin{aligned}\Xi^{'}_N(\tau) &= \int_{\sigma_i}^{\tau}\Xi^{''}_N(\tau)d\tau \\ &= \int_{\sigma_i}^{\tau}\frac{s - \sigma_i}{\sigma_{i+1} - \sigma_i}\left(\int_{s}^{\sigma_{i+1}}(t - \sigma_{i+1})\frac{d^4}{dt^4}\Xi_N(t)dt\right)ds \\ &+ \int_{\sigma_i}^{\tau}\frac{\sigma_{i+1} - s}{\sigma_{i+1} - \sigma_i}\left(\int_{s}^{\sigma_i}(t - \sigma_i)\frac{d^4}{dt^4}\Xi_N(t)dt\right)ds,\end{aligned} \tag{6.42}$$

and consequently

$$\int_{\sigma_i}^{\sigma_{i+1}}|\Xi^{''}_N(\tau)|^2 d\tau \le \boldsymbol{const}_1\ \Delta^2\sigma_i\ \int_{\sigma_i}^{\sigma_{i+1}}|\frac{d^4}{d\tau^4}\Xi_N(\tau)|^2 d\tau. \tag{6.43}$$

Using (6.43), we obtain

$$\begin{aligned}\int_{\sigma_i}^{\sigma_{i+1}}|\Xi^{'}_N(\tau)|^2 d\tau &= \frac{1}{\Delta^3\sigma_i}\int_0^1|\frac{d^2}{dt^2}\Xi_N(\sigma_i + t\Delta\sigma_i)|^2 dt \\ &\le \frac{\boldsymbol{const}_1}{\Delta\sigma_i}\int_0^1|\frac{d^4}{dt^4}\Xi_N(\sigma_i + t\Delta\sigma_i)|^2 dt \\ &= \boldsymbol{const}_1\ \Delta^6\sigma_i\int_{\sigma_i}^{\sigma_{i+1}}|\frac{d^4}{d\tau^4}\Xi_N(\tau)|^2 d\tau.\end{aligned} \tag{6.44}$$

Therefore,

$$\begin{aligned}\|\Xi_N^{'}(\tau)\|^2_{L^2[\sigma_1,\sigma_{N-2}]} &= \int_{\sigma_1}^{\sigma_{N-2}} |\Xi_N^{'}(\tau)|^2 d\tau \\ &= \sum_{i=1}^{N-3} \int_{\sigma_i}^{\sigma_{i+1}} |\Xi_N^{'}(\tau)|^2 d\tau \leq \boldsymbol{const}_2\ \delta^6, \end{aligned} \tag{6.45}$$

Also, for $\tau \in [0, \varsigma_3]$*, there exists* $\varrho \in [\min\{\varsigma_1, \varsigma_2, \varsigma_3, \tau\}, \max\{\varsigma_1, \varsigma_2, \varsigma_3, \tau\}]$ *such that*

$$\Xi_N^{'}(\tau) = (\tau - \varsigma_1)(\tau - \varsigma_2)(\tau - \varsigma_3)\Xi_N[\varsigma_1, \varsigma_2, \varsigma_3, \tau, \tau]$$

where the quantity $\Xi_N[\varsigma_1, \varsigma_2, \varsigma_3, \tau, \tau]$ *is a Newton divided difference of fourth order for function* $\Xi_N(\tau)$*.*
From the above formula it holds

$$\|\Xi_N^{'}(\tau)\|^2_{L^2[0,\sigma_1]} \leq \|\Xi_N^{'}(\tau)\|^2_{L^2[0,\varsigma_2]} \leq \boldsymbol{const}_3\ \delta^6. \tag{6.46}$$

A similar argument shows

$$\|\Xi_N^{'}(\tau)\|^2_{L^2[\sigma_{N-1},1]} \leq \|\Xi_N^{'}(\tau)\|^2_{L^2[\varsigma_{N-3},1]} \leq \boldsymbol{const}_4\ \delta^6. \tag{6.47}$$

Using (6.45), (6.50) and (6.47), we get

$$\|\Xi_N^{'}\|^2_{L^2[0,1]} \leq \boldsymbol{const}_5\ \delta^6, \tag{6.48}$$

where $\boldsymbol{const}_5 = \boldsymbol{const}_2 + \boldsymbol{const}_3 + \boldsymbol{const}_4$*.*
Now, in each subinterval $[\tau_i, \tau_{i+1}]$*, we have*

$$\Xi_N(\tau) = \int_{\tau_i}^{\tau} \frac{d}{dt}\Xi_N(t)dt, \tag{6.49}$$

then, it follows that

$$\|\Xi_N(\tau)\|^2_{\infty} \leq \boldsymbol{const}_6\ \delta^8. \tag{6.50}$$

Using (6.48) and (6.50) we have

$$\|\Xi_N(\tau)\|_{H^1} \leq \boldsymbol{const}_7\ \delta^3, \tag{6.51}$$

where $\boldsymbol{const}_7$ *is a constant.*
By using Theorem 3.7 and Theorem 3.8, we can obtain the following error bound

$$\|\phi(\tau) - \phi_N(\tau)\|_{\infty} \leq \boldsymbol{const}\ \delta^3, \tag{6.52}$$

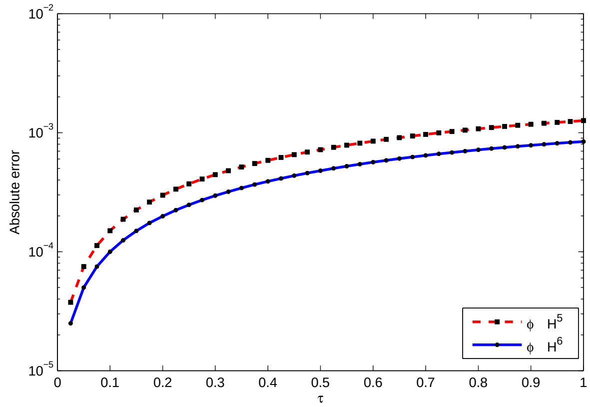
7. Numerical examples

In this section, some illustrative examples are considered to reveal the effectiveness and the accuracy of the proposed method for solving multi-order fractional differential equations. All of the computations have been performed by using the Maple software package.
In these examples, we report absolute error which is defined as:

$$e_N = |\phi(\tau) - \phi_N(\tau)|. \tag{7.1}$$

The numerical results in Tables 1 and 2 show that the approximate solution converge to the exact solution.

FIGURE 1. The absolute error using the technique given for $\alpha_2 = 2,\ \alpha_1 = 0$ in H^5 and H^6.



Example 7.1. Consider the following equation

$$\mathtt{D}^{\frac{3}{2}}_{*\tau}\phi(\tau) + 2D^1\phi(\tau) + 3\sqrt{\tau}\mathtt{D}^{\frac{1}{2}}_{*\tau}\phi(\tau) + (1-\tau)\phi(\tau) = g(\tau),$$
$$D^{(0)}\phi(0) = 0,\ D^{(1)}\phi(0) = 0, \tag{7.2}$$

where $g(\tau) = \frac{2}{\Gamma(\frac{3}{2})}\tau^{\frac{1}{2}} + 4\tau + \frac{4}{\Gamma(\frac{3}{2})}\tau^2 + (1-\tau)\tau^2$. The exact solution is $\phi(\tau) = \tau^2$. The absolute values of the errors are given in Table 1 for $\tau_i = \frac{i}{N}, i = 1, \dots, N$ for $N = 20, 25$. From the numerical results, it is clear that the approximate solutions are in good agreement with the exact solution. Also it is clear form Tables 1 that by increasing the value of r we get the better results.

TABLE 1. Absolute Error e_N for Example 1.

	$\phi \in H^5$		$\phi \in H^6$	
	$N = 20$	$N = 25$	$N = 20$	$N = 25$
0.1	6.207E-8	3.639E-8	6.667E-9	6.026E-9
0.2	7.730E-7	3.128E-7	5.779E-8	3.144E-8
0.3	8.277E-6	3.388E-6	4.294E-7	1.085E-7
0.4	3.065E-6	2.718E-6	6.237E-7	2.800E-7
0.5	4.791E-5	1.291E-5	5.328E-6	3.310E-6
0.6	8.511E-5	3.206E-5	6.494E-6	1.264E-6
0.7	9.213E-5	4.916E-5	7.865E-6	5.310E-6
0.8	5.615E-4	1.121E-4	8.768E-5	2.824E-5
0.9	6.214E-4	2.419E-4	7.754E-5	3.059E-5
1.0	8.360E-4	2.965E-4	6.354E-5	4.391E-5

TABLE 2. Numerical solution using the technique given for $0 \le \tau \le 1$ and for different α_1 and α_2 values.

	$\phi \in H^5$			$\phi \in H^6$		
	$\alpha_2 = 1.1$	$\alpha_2 = 1.4$	$\alpha_2 = 1.9$	$\alpha_2 = 1.3$	$\alpha_2 = 1.8$	$\alpha_2 = 1.9$
	$\alpha_1 = 0.5$	$\alpha_1 = 0.6$	$\alpha_1 = 0.8$	$\alpha_1 = 0.4$	$\alpha_1 = 0.2$	$\alpha_1 = 0.1$
0.1	0.095162	0.091184	0.096465	0.093435	0.099326	0.099662
0.2	0.181269	0.170796	0.185226	0.176480	0.195938	0.197658
0.3	0.259181	0.241989	0.266285	0.251071	0.288435	0.292745
0.4	0.329679	0.306409	0.339955	0.318451	0.375803	0.383862
0.5	0.393469	0.365174	0.406647	0.379573	0.457260	0.470073
0.6	0.451188	0.419115	0.466811	0.435206	0.532204	0.550561
0.7	0.503414	0.468880	0.520909	0.485991	0.600185	0.624611
0.8	0.550671	0.514991	0.569397	0.532471	0.660883	0.691618
0.9	0.593430	0.557879	0.612717	0.575111	0.714098	0.751076
1.0	0.632120	0.597901	0.651292	0.614315	0.714098	0.802582

Example 7.2. Now, let us consider the following equation

$$\mathtt{D}_{*\tau}^{\alpha_2}\phi(\tau) + \mathtt{D}_{*\tau}^{\alpha_1}\phi(\tau) = 0, \tag{7.3}$$

$$D^{(0)}\phi(0) = 0, \;\; D^{(1)}\phi(0) = 1. \tag{7.4}$$

where $1 < \alpha_2 \le 2$ and $0 \le \alpha_1 < 1$. The exact solution of this problem for $\alpha_2 = 2$ and $\alpha_1 = 0$ is $\phi(\tau) = \sin(\tau)$. Using the proposed method, we choose 40 points in [0,1], and calculate the absolute errors in H^5 and H^6, the computational errors are plotted in Figure 1. The results show that the approximate solutions are in a good agreement with the exact solution when $\alpha_2 = 2$ and $\alpha_1 = 0$. Table 2 shows the approximation values in some points $\tau \in [0, 1]$ for different α_2 and α_1.

Example 7.3. We consider the following multi-order fractional differential equation of the form [31]

$$\mu_1\mathtt{D}_{*\tau}^{\alpha_2}\phi(\tau) + \mu_2\mathtt{D}_{*\tau}^{\alpha_1}\phi(\tau) + \mu_3\phi^3(\tau) = g(\tau), \tag{7.5}$$

$$D^{(0)}\phi(0) = 0, \;\; 0 < \alpha_1 < \alpha_2 \le 1, \tag{7.6}$$

where

$$g(\tau) = \frac{2\mu_1\tau^{3-\alpha_2}}{\Gamma(4-\alpha_2)} + \frac{2\mu_2\tau^{3-\alpha_1}}{\Gamma(4-\alpha_1)} + \frac{\mu_3\tau^9}{27}. \tag{7.7}$$

The exact solution of this equation is $\phi(\tau) = \frac{\tau^3}{3}$. The absolute values of the errors are given in Table 3 for $\tau_i = \frac{i}{N}, i = 1, \ldots, N$ for $N = 40, 50$. From Table 3 we can see that the approximate solutions obtained by present method are in prefect agreement with the exact solution for $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{2}$, $\mu_1 = \mu_2 = \mu_3 = 1$.

Table 3. Absolute Error e_N for Example 3.

	$\phi \in H^4$		$\phi \in H^5$	
	$N = 40$	$N = 50$	$N = 40$	$N = 50$
0.1	4.204E-6	2.997E-6	3.785E-6	1.178E-7
0.2	6.545E-6	5.548E-6	5.345E-6	3.389E-6
0.3	3.744E-5	2.784E-5	4.342E-6	3.378E-6
0.4	4.162E-5	3.977E-5	2.234E-5	1.431E-5
0.5	5.489E-5	5.259E-5	4.125E-5	2.256E-5
0.6	6.234E-4	6.034E-4	5.907E-5	3.542E-5
0.7	8.981E-4	7.634E-4	6.675E-4	5.157E-4
0.8	5.554E-4	5.114E-4	5.456E-4	4.871E-4
0.9	6.412E-3	6.321E-3	2.341E-3	1.456E-3
1.0	5.631E-3	4.981E-3	3.641E-3	2.641E-3

8. Concluding Remarks

There are some main goals that we aimed by this work. The first is to present a relatively new semi-analytical technique to derive approximate analytical solution for nonlinear multi-order FDEs. The second is to addresse the sufficient conditions for uniqueness of solution and to study technique in different **RKHS**. Furthermore, the numerical tests are presented to show the accuracy of the proposed technique. The numerical results demonstrate the relatively rapid convergence of the proposed technique. We should also point out that the example studied in the paper shows that the technique is very effective and convenient for solving nonlinear multi-order FDEs.

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