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Numerical solution of optimal control problems by using a new second kind Chebyshev wavelet

Mehdi Ramezani
Department of mathematics, Tafresh University,
Tafresh 39518 79611, Iran.
E-mail: ramezani@tafreshu.ac.ir

Abstract The main purpose of this paper is to propose a new numerical method for solving the optimal control problems based on state parameterization. Here, the boundary conditions and the performance index are first converted into an algebraic equation or in other words into an optimization problem. In this case, state variables will be approximated by a new hybrid technique based on new second kind Chebyshev wavelet.

Keywords. Second kind chebyshev wavelet, optimal control problems, numerical analysis.
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1. Introduction

To achieving an optimal control for the considered problems is not a simple task. Recently, optimal control is introduced as a mathematically challenging and practically important discipline [1, 2] . Traditional control techniques are based on model constructions. However, it may be difficult to construct an accurate enough model or to employ a lot of assumptions to solve a differential equation. There exist also several optimal control problems which are subject to constraints in state and/or control variables. Direct methods can obtain the optimal solution by direct minimization of the cost function (performance index), subject to constraints. Among these approaches, Pontryagins maximum principle method and dynamic programming introduced the best known methods for solving the optimal control problems [3, 4, 5, 6, 7, 8]. Because analytical solutions for optimal control problems cant always be solved accessible, finding an approximate solution can be a good idea to solve them. Numerical approaches have provided a personable field for researchers to the appearance of different numerical computational methods and efficient algorithms for solving optimal control problems (for details see [9, 10]). In this study, the Chebyshev wavelet method in [11, 12] is modified by the second kind Chebyshev, and an impressive iterative algorithm is proposed. In this method, only one unknown coefficient is achieved for finding a proper approximation; further iterations result better accuracy.

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2. Second Kind Chebyshev Wavelet

Generally, wavelets functions are generated from translation and dilation of a definite function called mother wavelet [11, 12, 13]. By considering the translation parameter (b) and dilation parameter (a) with a normalized time (t), wavelet functions can be described as below

$$\Psi_{ab}(t) = |a|^{-\frac{1}{2}} \Psi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R},\ a \neq 0\ , \tag{2.1}$$

Consider the second kind Chebyshev wavelets as $\Psi(t) = \Psi(m, n, t)$ where $n = 1, 2, ..., 2^k (k = 0, 1, 2, ...)$ and m illustrates the order of the second kind Chebyshev polynomial [12, 14, 15, 16, 17, 18]. By considering the illustrated cases, the hybrid second kind Chebyshev wavelet can be given as below

$$\Psi_{nm}(t) = \begin{cases} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \leq t \leq \frac{n}{2^k}, \\ 0, & \text{o.w.} \end{cases} \tag{2.2}$$

where

$$\alpha_m(t) = \begin{cases} \sqrt{2}, & m = 0, \\ 2, & m = 1, 2, \ldots. \end{cases} \tag{2.3}$$

In the equation above, $T_m(t)$ describes the second kind orthogonal Chebyshev polynomial of order m , [14, 19]. the orthogonality can be proved by the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$. The second kind Chebyshev polynomial can be presented as a three part recursive formula [12]

$$\begin{aligned} &T_0(t) = 1, \\ &T_1(t) = 2t, \\ &\quad\vdots \qquad\qquad\qquad\qquad , \ t \in [-1, 1], \\ &T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t). \end{aligned} \tag{2.4}$$

Therefore, the total Chebyshev wavelet approximation can be considered as below:

$$f(t) \simeq \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{nm} \Psi_{nm}(t). \tag{2.5}$$

If the infinite series in Eq. 2.5 are truncated, then Eq. 2.5 can be written as

$$f(t) \simeq \sum_{n=1}^{2^k} \sum_{m=0}^{m-1} f_{nm} \Psi_{nm}(t). \tag{2.6}$$

The second kind Chebyshev wavelet functions are defined in the interval and can be considered as orthogonal functions by the weight function as below

$$w_n(t) = w(2^{k+1}t - 2n + 1). \tag{2.7}$$

3. Solving Optimal Control Problems

Lets consider a nonlinear state equation as below [20],

$$u(\tau) = f(\tau, x(\tau), \dot{x}(\tau)), \tag{3.1}$$

where the initial conditions are:

$$x(t_0) = x_0, \quad x(t_1) = x_1, \tag{3.2}$$

here, $u(.) : [t_0, t_1] \to \mathbb{R}$ and $x(.) : [t_0, t_1] \to \mathbb{R}$ are the signal control and state variables respectively and f describes the continuously differentiable function with real-value. The main purpose of the optimal control problem is to minimize a definite performance index (PI) by achieving a proper control signal from the initial position $x(t_0) = x_0$ to the final position $x(t_1) = x_1$ within the time $(t_1 - t_0)$,

$$PI = \int_{t_0}^{t_1} L(\tau, x(\tau), u(\tau)) d\tau. \tag{3.3}$$

If $t_0 \neq 0$ or $t_1 \neq 1$, then for solving the problem with the second kind Chebyshev wavelet polynomials, we need the transformation below,

$$\tau = (t_1 - t_0)t + t_0, \tag{3.4}$$

after the conversation above, the optimal control variable, the initial conditions of the trajectory $x(t)$ and the performance index will be changed into:

$$u(t) = f\left((t_1 - t_0)\, t + t_0, x(t), \dot{x}(t)\right), \tag{3.5}$$

$$x(0) = x_0, \quad x(1) = x_1, \tag{3.6}$$

$$J(x) = (t_1 - t_0) \int_{-1}^{1} L\left((t_1 - t_0)\, t + t_0, x(t), u(t)\right) dt. \tag{3.7}$$

By considering Eq. (2.2) with $k = 0$ and the second kind Chebyshev wavelet basis as $\Psi_m(t) = \Psi_{1m}(t)$, the approximation for $x(.)$ can be achieved as below

$$x_1(t) = \sum_{m=0}^{2} a_m \Psi_m(t). \tag{3.8}$$

By considering the boundary conditions Eq. (3.6) we have

$$\begin{cases} x_1(0) = \frac{2}{\pi}(a_0 - a_1 + a_2), \\ x_1(1) = \frac{2}{\pi}(a_0 + a_1 + a_2), \end{cases} \tag{3.9}$$

the unknown coefficients a_0 and a_1, can be evaluated as below

$$\begin{cases} a_0 = \sqrt{\frac{\pi}{4}}(x_1 + x_0) - a_2, \\ \quad a_1 = \sqrt{\frac{\pi}{4}}(x_1 - x_0), \end{cases} \tag{3.10}$$

so from Eq. (3.8) we obtain

$$x_1(t) = \left(\frac{\sqrt{\pi}}{4}(x_1 + x_0) - a_2\right)\Psi_0(t) + \left(\frac{\sqrt{\pi}}{4}(x_1 - x_0)\right)\Psi_1(t) + a_2\Psi_2(t). \tag{3.11}$$

The above equations give us a chance to have just one uncertainty (a_2). Since, a robust approach has been performed. In the following, $x_2(t)$ can be approximated by

$$x_2(t) = x_1(t) + \sum_{m=1}^{3} a_m \Psi_m(t). \tag{3.12}$$

By substituting the boundary conditions $x_2(0) = x_1(0) = x_0$ and $x_2(1) = x_1(1) = x_1$, we have

$$\begin{aligned} &a_1\Psi_1(0) + a_2\Psi_2(0) + a_3\Psi_3(0) = 0, \\ &a_1\Psi_1(1) + a_2\Psi_2(1) + a_3\Psi_3(1) = 0, \end{aligned} \tag{3.13}$$

then unknown coefficients a_1 and a_2 are as

$$\begin{aligned} a_1 &= \frac{\Psi_2(0)\Psi_3(1) - \Psi_2(1)\Psi_3(0)}{\Psi_1(0)\Psi_2(1) - \Psi_1(1)\Psi_2(0)} a_3, \\ a_2 &= \frac{\Psi_1(0)\Psi_3(1) - \Psi_1(1)\Psi_3(0)}{\Psi_1(1)\Psi_2(0) - \Psi_1(0)\Psi_2(1)} a_3. \end{aligned} \tag{3.14}$$

Assuming a^* as the minimizer of the performance index $PI(a_3)$, $PI(a^*)$ includes the optimal control problem solution. Control and state variables can be also evaluated from a^* approximately. By considering the recurrence characteristics of the orthogonal functions, the more recurrence results the more precision. Lets assume the $(n+1)^{th}$ step approximation in below

$$x_{n+1}(t) = x_n(t) + \sum_{m=n}^{n+2} a_m \Psi_m(t). \tag{3.15}$$

By considering the boundary condition $x_{n+1}(0) = x_n(0) = x_0$ and $x_{n+1}(1) = x_n(1) = x_1$, we have

$$a_{n+2}\Psi_{n+2}(0) + a_{n+1}\Psi_{n+1}(0) + a_n\Psi_n(0) = 0, \tag{3.16}$$

$$a_{n+2}\Psi_{n+2}(1) + a_{n+1}\Psi_{n+1}(1) + a_n\Psi_n(1) = 0. \tag{3.17}$$

Assuming the Eqs. (3.16) and (3.17), the unknown coefficients a_n and $a_n + 1$, can be evaluated as below

$$a_n = \frac{\Psi_{n+1}(0)\Psi_{n+2}(1) - \Psi_{n+1}(1)\Psi_{n+2}(0)}{\Psi_n(0)\Psi_{n+1}(1) - \Psi_n(1)\Psi_{n+1}(0)} a_{n+2}, \tag{3.18}$$

and

$$a_{n+1} = \frac{\Psi_n(0)\Psi_{n+2}(1) - \Psi_n(1)\Psi_{n+2}(0)}{\Psi_n(1)\Psi_{n+1}(0) - \Psi_n(0)\Psi_{n+1}(1)} a_{n+2}. \tag{3.19}$$

By considering the above equations, the final recursive solution for the state variable is

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + a_{n+2}\Psi_{n+2}(t) \\ &+ \frac{\Psi_n(0)\Psi_{n+2}(1) - \Psi_n(1)\Psi_{n+2}(0)}{\Psi_n(1)\Psi_{n+1}(0) - \Psi_n(0)\Psi_{n+1}(1)} a_{n+2}\Psi_{n+1}(t) \\ &+ \frac{\Psi_{n+1}(0)\Psi_{n+2}(1) - \Psi_{n+1}(1)\Psi_{n+2}(0)}{\Psi_n(0)\Psi_{n+1}(1) - \Psi_n(1)\Psi_{n+1}(0)} a_{n+2}\Psi_n(t), \end{aligned} \tag{3.20}$$

after evaluation the Eq. (3.18), the recursive will be stopped when

$$|e_{n+1} - e_n| < \varepsilon, \tag{3.21}$$

as

$$e_{n+1} = PI(a^*_{n+1}). \tag{3.22}$$

4. Case Study

Assume the following example to minimize by $u(t)$ using

$$J = \int_0^1 \left(x(\tau) - \frac{1}{2}u(\tau)^2 \right) d\tau, \qquad \tau \in [0,1], \tag{4.1}$$

the state equation is

$$u(\tau) = \dot{x}(\tau) + x(\tau). \tag{4.2}$$

Also, the boundary conditions are

$$x(0) = 0, \quad x(1) = \tfrac{1}{2}(1 - \tfrac{1}{e})^2. \tag{4.3}$$

Here, the analytical solution can be achieved by the Pontryagin's maximum principle as below [6]

$$x(\tau) = 1 - \frac{1}{2}e^{\tau-1} + \left(\frac{1}{2e} - 1\right) e^{-\tau}, \tag{4.4}$$

$$u(\tau) = 1 - e^{\tau-1}, \tag{4.5}$$

by using Eqs. (3.18) and (3.19) , we have

$$x_1(t) = 9.027033336764101 a_2 t^2 + 9.027033336764101 a_2 t + 0.199788200446864 t, \tag{4.6}$$

$$\begin{aligned} u_1(t) = {} & 0.199788200446864t + 9.027033336764101(a_2t^2 + a_2t - a_2) \\ & +0.199788200446864. \end{aligned} \tag{4.7}$$

By substituting Eqs (4.6) and (4.7) into Eq. (4.1) we have

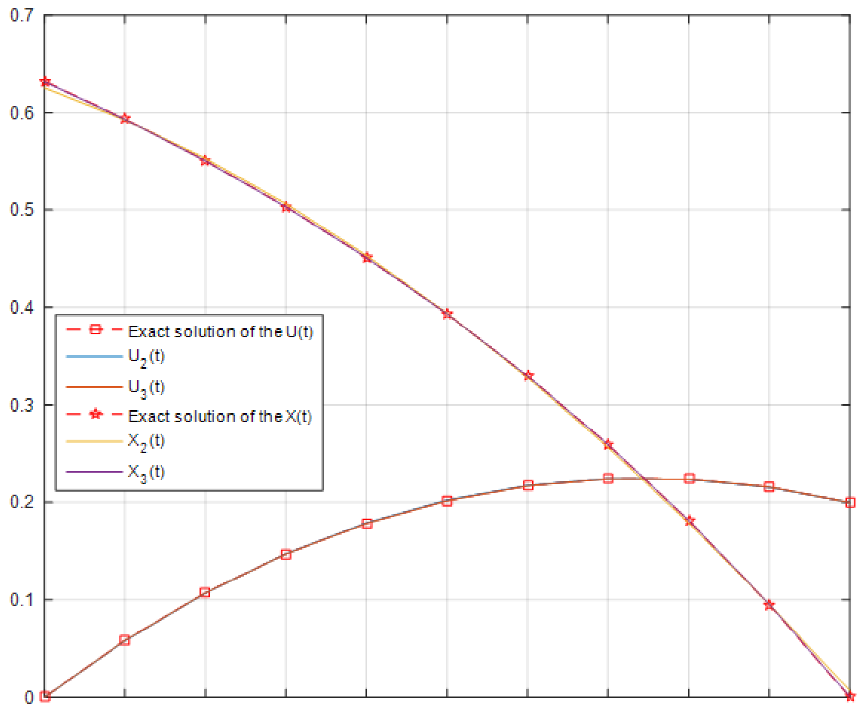
$$J(a_2) = -14.939343991559241 a_2^2 \text{ - } 1.354214327316854 a_2 + 0.053326221012670. \tag{4.8}$$

Table 1. Optimal cost functional J (PI) for different n in case study.

Iteration	Method	PI Value	Error
1	Proposed	0.0840152600	2.7e-5
1	Mehne [9]	0.05332622101	3e-2
1	Kafash [19]	0.0840152601	3e-5
2	Proposed	0.0840423344	3.2e-6
2	Mehne [9]	0.0840152600	3e-5
2	Kafash [19]	0.0840423344	3.2e-6
3	Proposed	0.0840455803	3.7e-8
3	Mehne [9]	0.08402496180	2e-5
3	Kafash [19]	0.0840455804	4e-8

By minimizing the Eq. (4.8), the minimum value is achieved as $a_2^* =$ - 0.045323754780732. The recurrence Eq. (3.13) can be used for enhancing the main solution performance in the next equations. The simulation results are shown in Table 1 and Figure 1.

Figure 1. Control and state variables solution for the case study which are compared with the exact analytical solution.



5. Conclusion

In this research, a new and robust algorithm is proposed for the optimal control problems. The proposed method is a hybrid second kind Chebyshev and wavelet function which has good characteristics of both functions. The basis of the approach is on state parameterization and because of reducing the uncertainties in the problem, it gives robust results toward the other methods. The system performance is compared by two different methods and gives better results for a selected case study.

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