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On the split-step method for the solution of nonlinear Schrödinger equation with the Riesz space fractional derivative

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Abstract The aim of this paper is to extend the split-step idea for the solution of fractional partial differential equations. We consider the multidimensional nonlinear Schrödinger equation with the Riesz space fractional derivative and propose an efficient numerical algorithm to obtain it's approximate solutions. To this end, we first discretize the Riesz fractional derivative then apply the Crank-Nicolson and a split-step methods to obtain a numerical method for this equation. In the proposed method there is no need to solve the nonlinear system of algebraic equations and the method is convergent and unconditionally stable. The proposed method preserves the discrete mass which will be investigated numerically. Numerical results demonstrate the reliability, accuracy and efficiency of the proposed method.

Keywords. Finite difference method, Riesz space fractional derivatives, Unconditional stability.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

In recent years there has been a growing interest in the field of fractional calculus [4, 6, 18, 19]. Fractional differential equations have attracted increasing attention because they have applications in various fields of science and engineering. Many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order. Some of the most applications are given in the book of Oldham and Spanier [17], the book of Podlubny [18] and the papers of Metzler and Klafter [15], Bagley and Trovik [3]. The fractional Schrödinger equation was discovered by Nick Laskin [13] as a result of extending the Feynman path integral, from the Brownian-like to Levy-like quantum mechanical paths. Some authors [8, 10] discussed the physical applications of fractional Schrödinger equation and obtained the exact solutions with several kinds of potentials. However, the exact solutions of fractional differential equations often contains some special functions, such as Fox functions, which are not studied very well. The numerical methods for fractional differential equations become important tools to understand the behaviors of the equations [20].

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The Riesz fractional operator for $1 < \alpha \leq 2$ is defined as follows [9, 12]

$$\frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} = -\frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{-\infty}^{\infty} |x-\xi|^{1-\alpha} u(\xi,t) d\xi, \tag{1.1}$$

where $\Gamma(.)$ is the gamma function. We consider the one-dimensional Schrödinger equation with Riesz space fractional

$$i\frac{\partial u(x,t)}{\partial t} = \kappa \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + \gamma u(x,t)|u(x,t)|^p + f(x,t), \quad a < x < b, \quad 0 \leq t \leq T, \tag{1.2}$$

with initial condition

$$u(x,0) = \phi(x), \quad a < x < b, \tag{1.3}$$

and boundary conditions

$$u(a,t) = u(b,t) = 0, \quad 0 \leq t \leq T, \tag{1.4}$$

where $p > 0$ is a real constant, $i^2 = -1$ and the complex wave function $u(x,t)$ may represent a probability amplitude, the amplitude of an electric field or the velocity field of a fluid. Some numerical methods for the solution of fractional Schrödinger equation are proposed in the literature. The authors of [20] proposed the Crank-Nicolson difference scheme for the coupled nonlinear Schrödinger equations with the Riesz space fractional derivative. The stability and convergence of this scheme are discussed in the L_2 norm. Wei et al. [24] presented and analyzed an implicit fully discrete local discontinuous Galerkin (LDG) finite element method for solving the one-dimensional time-fractional Schrödinger equation. Authors of [2] showed that it is possible to obtain numerical solutions to quantum mechanical problems involving a fractional Laplacian, using a collocation approach based on sinc functions, which discretizes the Schrödinger equation on a uniform grid. Homotopy perturbation method is proposed in [25] for the solution of fractional nonlinear Schrödinger equation. The meshless technique based on collocation and radial basis functions for solving the time fractional nonlinear Schrödinger equation arising in quantum mechanics is given in [16]. Another recently numerical methods for fractional Schrödinger equation are given in [14, 23], which both of them need to solve nonlinear system of algebraic equations.

In this paper we propose a split-step finite difference method for the solution of one and two dimensional nonlinear Schrödinger equation with the Riesz space fractional derivatives. To the best knowledge of author, there is no application of split-step scheme for the solution of nonlinear fractional partial differential equations. We apply the Crank-Nicolson and split-step methods to obtain a numerical algorithm for this equation. We prove that the proposed method is convergent and unconditionally stable. The method preserves the discrete mass which will be investigated numerically and there is no need to solve the nonlinear system of algebraic equations. The proposed method can be easily extended to the coupled and three-dimensional nonlinear Schrödinger equations with the Riesz space fractional derivatives.

The rest of this paper is organized as follows: In Section 2 we introduce the new method based on the finite difference method and a split-step scheme. In Section 3 we prove the unconditional stability and convergence of proposed method. We extend our approach to two dimensional nonlinear Schrödinger equation with the Riesz space fractional derivatives in Section 4. The results of numerical experiments are compared with analytical solution for confirming the good accuracy of the proposed scheme and investigating the conserved discrete mass quantities in Section 5. We conclude this article with a brief conclusive discussion in Section 6.

2. Derivation of proposed method

For positive integer numbers M and N, let $h = \frac{b-a}{M}$ denotes the step size of spatial variable, x, and $\tau = \frac{T}{N}$ denotes the step size of time variable, t. So we define

$$\begin{aligned} x_j &= a + jh \quad , \quad j = 0, 1, 2, ..., M, \\ t_k &= k\tau \quad , \quad k = 0, 1, 2, ..., N. \end{aligned}$$

The exact and approximate solutions at the point (x_j, t_k) are denoted by u_j^k and U_j^k respectively. First we state some definitions and properties of fractional derivatives which are used later.

Lemma 1 [5]. Let $g_k = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1)\Gamma(\frac{\alpha}{2}+k+1)}$ for $k = 0, \mp 1, \mp 2, ...$ and $\alpha > 1$, then

$$\begin{aligned} & g_0 \geq 0, \\ & g_{-k} = g_k \leq 0, \quad |k| \geq 1. \end{aligned} \tag{2.1}$$

Lemma 2 [5]. Let $f \in \mathbb{C}^5(\mathbb{R})$ and all derivatives up to order five belong to $L_1(\mathbb{R})$ and

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1)\Gamma(\frac{\alpha}{2}+k+1)} f(x-kh),$$

be the fractional centered difference, then

$$-h^{-\alpha} \Delta_h^\alpha f(x) = \frac{\partial^\alpha f(x)}{\partial |x|^\alpha} + \mathcal{O}(h^2), \tag{2.2}$$

when $h \to 0$ and $\frac{\partial^\alpha f(x)}{\partial |x|^\alpha}$ is the Riesz fractional derivative for $1 < \alpha \leq 2$.

If

$$f^*(x) = \begin{cases} f(x), & x \in [a, b], \\ 0, & x \notin [a, b], \end{cases}$$

such that $f^* \in \mathbb{C}^5(\mathbb{R})$ and all derivatives up to order five belong to $L_1(\mathbb{R})$, then from Lemma 2 we have [5]

$$\frac{\partial^\alpha f^*(x)}{\partial |x|^\alpha} = -h^{-\alpha} \sum_{k=-\infty}^{\infty} g_k f^*(x-kh) + O(h^2).$$

Since $f^*(x) = 0$ for $x \notin [a, b]$, we get

$$\frac{\partial^\alpha f(x)}{\partial |x|^\alpha} = -h^{-\alpha} \sum_{k=-\frac{b-a}{h}}^{\frac{x-a}{h}} g_k f(x - kh) + \mathcal{O}(h^2). \tag{2.3}$$

Now we want to use (2.3), Crank-Nicolson and split-step methods to get an algorithm to the solution of Schrödinger equation with the Riesz space fractional derivative. Split step methods, also known as fractional step methods, are efficient and extensively used for numerical solutions of differential equations, especially for higher dimensional ones [22]. The basic idea is based on splitting a complex problem into simpler sub-problems, whose sub-operators are chosen with respect to different physical processes. Then, each sub-equation is solved efficiently with suitable methods [21, 26]. In this method we decompose the problem into linear and nonlinear subproblems on each time step. To this end, we rewrite the fractional PDE (1.2) as follows

$$i\frac{\partial u}{\partial t} = (\mathcal{L} + \mathcal{N})u(x, t), \tag{2.4}$$

where $\mathcal{L}u(x, t) = \kappa \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x, t)$ is the linear part and $\mathcal{N}u(x, t) = \gamma|u(x, t)|^p u(x, t)$ is the nonlinear part of equation. The main idea in the split-step method, is solving the linear subproblem

$$i\frac{\partial u}{\partial t} = \mathcal{L}u(x, t), \tag{2.5}$$

and nonlinear subproblem

$$i\frac{\partial u}{\partial t} = \mathcal{N}u(x, t), \tag{2.6}$$

in a given sequential order. Using the standard Strang splitting idea for solving (2.4) we obtain [6, 21]

$$u(x, t + \tau) = e^{-i\mathcal{N}\tau/2} e^{-i\mathcal{L}\tau} e^{-i\mathcal{N}\tau/2} u(x, t). \tag{2.7}$$

After time-splitting, the nonlinear equation (2.6) can be solved exactly, while the linear equation (2.5) can be solved by using an appropriate numerical scheme. We can use the Crank-Nicolson (CN) difference scheme to the linear part. Also we approximate the Riesz fractional derivative using the fractional centered derivative approach. Using (2.3), we can discretize the Eq. (2.5) as follows

$$i\frac{U_j^{n+1} - U_j^n}{\tau} = -\frac{1}{2}\kappa h^{-\alpha} \sum_{k=j-M+1}^{j-1} g_k U_{j-k}^{n+1} - \frac{1}{2}\kappa h^{-\alpha} \sum_{k=j-M+1}^{j-1} g_k U_{j-k}^{n} + f_j^{n+\frac{1}{2}}, \tag{2.8}$$

or

$$U_j^{n+1} - \frac{i}{2}\tau\kappa h^{-\alpha} \sum_{k=j-M+1}^{j-1} g_k U_{j-k}^{n+1} = U_j^n + \frac{i}{2}\tau\kappa h^{-\alpha} \sum_{k=j-M+1}^{j-1} g_k U_{j-k}^{n} - i\tau f_j^{n+\frac{1}{2}},$$

$$j = 1, 2, ..., M-1, \quad n = 0, 1, ..., N-1, \tag{2.9}$$

where $f_j^{n+\frac{1}{2}} = f(x_i, t_{n+\frac{1}{2}}), \quad t_{n+\frac{1}{2}} = n\tau + \frac{\tau}{2}$. We can write the above difference scheme in matrix-vector form as follows

$$(I - A)U^{n+1} = (I + A)U^n - i\tau F^{n+\frac{1}{2}},$$

where $U^{n+1} = \left(U_1^n, U_2^n, ..., U_{M-1}^n\right)^T, \ F^{n+\frac{1}{2}} = \left(f_1^{n+\frac{1}{2}}, f_2^{n+\frac{1}{2}}, ..., f_{M-1}^{n+\frac{1}{2}}\right)$. Also A is the $(M-1) \times (M-1)$ symmetric matrix

$$A = \frac{i\tau\kappa}{2h^\alpha} \begin{pmatrix} g_0 & g_{-1} & g_{-2} & \cdots & g_{-M+1} \\ g_1 & g_0 & g_{-1} & \cdots & g_{-M+2} \\ g_2 & g_1 & g_0 & \cdots & g_{-M+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{M-1} & g_{M-2} & g_{M-3} & \cdots & g_0 \end{pmatrix}.$$

Now from relation (2.7) we can state the following split-step finite difference scheme for the solution of Eqs. (1.2)-(1.4) from time $t = t_n$ to $t = t_{n+1}$:

$$U_j^* = e^{-i\left(\gamma \left|U_j^n\right|^p \tau\right)/2} U_j^n, \qquad j = 0, 1, 2, ..., M, \tag{2.10}$$

$$U_j^{**} - \frac{i}{2}\tau\kappa h^{-\alpha} \sum_{k=j-M+1}^{j-1} g_k U_{j-k}^{**} = U_j^* + \frac{i}{2}\tau\kappa h^{-\alpha} \sum_{k=j-M+1}^{j-1} g_k U_{j-k}^{*} - i\tau f_j^{n+\frac{1}{2}},$$

$$j = 1, 2, ..., M-1, \quad U_0^{**} = U_M^{**} = 0, \tag{2.11}$$

$$U_j^{n+1} = e^{-i\left(\gamma \left|U_j^{**}\right|^p \tau\right)/2} U_j^{**}, \qquad j = 0, 1, 2, ..., M. \tag{2.12}$$

3. Stability and convergence of method

In this section we analyze the stability and convergence of algorithm (2.10)-(2.12). First we introduce the following notations:

$$u_j^{n+\frac{1}{2}} = \frac{u_j^{n+1} + u_j^n}{2}, \quad (u^n, v^n) = h \sum_{j=1}^{M-1} u_j^n \overline{v_j^n}, \quad \|u^n\|^2 = (u^n, u^n).$$

For stability analysis we assume that $f(x, t) = 0$. We can state the following lemma.

Lemma 3 [20]. The following relation holds

$$\mathrm{Im}\left(\sum_{j=1}^{M-1}\sum_{k=1}^{M-1} g_{j-k}\frac{U_k^{**}+U_k^{*}}{2}\overline{\frac{U_k^{**}+U_k^{*}}{2}}\right)=0,$$

where“Im” means taking the imaginary part.

Lemma 4 [5]. Let $\bar{u}$ be the exact solution of

$$i\frac{\partial u(x,t)}{\partial t}=\kappa\frac{\partial^{\alpha}u(x,t)}{\partial|x|^{\alpha}}+f(x,t), \tag{3.1}$$

and U be the solution of difference scheme (2.11), then we have

$$\|\bar{u}^n-U^n\|\le C_1T(\tau^2+h^2). \tag{3.2}$$

Lemma 5 [11]. Let u be the exact solution of (1.2)-(1.4) and v be the solution of Strang splitting scheme (2.7) then we have

$$\|v^n-u^n\|\le C_2\tau^2. \tag{3.3}$$

Theorem 1. The difference scheme (2.10)-(2.12) is unconditionally stable.

Proof. From Eq. (2.10) we can write

$$\|U^*\|=\|U^n\|,$$

also (2.12) gives

$$\left\|U^{n+1}\right\|=\|U^{**}\|,$$

so it is sufficient to show that

$$\|U^*\| \ =\|U^{**}\|.$$

We can rewrite expression (2.11) as follows

$$U_j^{**}-\frac{i}{2}\tau\kappa h^{-\alpha}\sum_{k=1}^{M-1}g_{j-k}U_k^{**}=U_j^{*}+\frac{i}{2}\tau\kappa h^{-\alpha}\sum_{k=1}^{M-1}g_{j-k}U_k^{*},$$

or

$$i\frac{U_j^{**}-U_j^{*}}{\tau}+\kappa h^{-\alpha}\sum_{k=1}^{M-1}g_{j-k}\frac{U_k^{**}+U_k^{*}}{2}=0.$$

Taking the inner product of above relation with $\frac{U^{**}+U^{*}}{2}$ gives

$$\left(i\frac{U^{**}-U^{*}}{\tau},\frac{U^{**}+U^{*}}{2}\right)+\kappa h^{-\alpha}\left(\sum_{j=1}^{M-1}\sum_{k=1}^{M-1}g_{j-k}\frac{U_k^{**}+U_k^{*}}{2}\overline{\frac{U_k^{**}+U_k^{*}}{2}}\right)=0,$$

Computing the imaginary part of above relation using Lemma 3, we obtain

$$\|U^*\| = \|U^{**}\|,$$

and as a result we have

$$\|U^n\| = \|U^0\|,$$

which shows the unconditional stability of method. □

Corollary 1. Difference scheme (2.10)-(2.12) for the numerical solution of Schrödinger equation with the Riesz space fractional derivative (1.2)-(1.4), preserves the discrete mass when $f(x,t) = 0$.

Theorem 2. The difference scheme (2.10)-(2.12) for the numerical solution of problem (1.2)-(1.4) is convergent with the convergence order $O(\tau^2 + h^2)$.

Proof. Let u be the exact solution of (1.2)-(1.4), v be the solution of Strang splitting method (2.7) and U be the solution of difference scheme (2.10)-(2.12). Note that the solution of Strang splitting scheme (2.7) is obtained from the following sequential subproblems

$$i\frac{\partial u}{\partial t} = \gamma|u(x,t)|^p u(x,t), \quad t \in [t^n, t^{n+1/2}], \tag{3.4}$$

$$i\frac{\partial u}{\partial t} = \kappa\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} + f(x,t), \quad t \in [t^n, t^{n+1}], \tag{3.5}$$

$$i\frac{\partial u}{\partial t} = \gamma|u(x,t)|^p u(x,t), \quad t \in [t^{n+1/2}, t^{n+1}]. \tag{3.6}$$

Since the nonlinear subproblems (3.4) and (3.6) are solved exactly, so from Lemma 4 we can obtain

$$\|v^n - U^n\| \le C_1 T(\tau^2 + h^2). \tag{3.7}$$

Lemma 5 gives

$$\|u^n - U^n\| \le \|v^n - u^n\| + \|v^n - U^n\| \le C(\tau^2 + h^2),$$

which completes the proof. □

4. Extension to two dimensional case

We can extend the proposed method in previous section to solve two-dimensional Schrödinger equation with fractional Riesz derivatives. Consider the following fractional partial differential equation

$$i\frac{\partial u(x,y,t)}{\partial t} = \kappa_1\frac{\partial^\alpha u(x,y,t)}{\partial|x|^\alpha} + \kappa_2\frac{\partial^\alpha u(x,y,t)}{\partial|y|^\alpha} + \gamma u(x,y,t)|u(x,y,t)|^p + f(x,y,t),$$

$$(x,y) \in \Omega, \quad t \in [0,T], \tag{4.1}$$

with initial condition

$$u(x, y, 0) = \phi(x, y), \quad (x, y) \in \Omega, \tag{4.2}$$

and boundary condition

$$u(x, y, t) = 0, \;\; (x, y) \in \partial\Omega, \quad 0 \le t \le T. \tag{4.3}$$

The split-step method which we proposed for the Eqs. (4.1)-(4.3), consists of solving the following sequential subproblems [21]

$$\begin{aligned}
&i\frac{\partial u(x,y,t)}{\partial t} = \gamma u(x,y,t)|u(x,y,t)|^p,\\
&i\frac{\partial u(x,y,t)}{\partial t} = \kappa_1 \frac{\partial^\alpha u(x,y,t)}{\partial |x|^\alpha} + \frac{1}{2} f(x,y,t),\\
&i\frac{\partial u(x,y,t)}{\partial t} = \kappa_2 \frac{\partial^\alpha u(x,y,t)}{\partial |y|^\alpha} + \frac{1}{2} f(x,y,t),\\
&i\frac{\partial u(x,y,t)}{\partial t} = \gamma u(x,y,t)|u(x,y,t)|^p.
\end{aligned} \tag{4.4}$$

If we solve the nonlinear subproblems exactly and use the Crank-Nicolsn scheme to the linear subproblems we obtain

$$U^*_{m,j} = e^{-i\left(\gamma |U^n_{m,j}|^p \tau\right)/2} U^n_{m,j}, \qquad m, j = 0, 1, 2, ..., M, \tag{4.5}$$

$$U^{**}_{m,j} - \frac{i}{2}\tau\kappa_1 h^{-\alpha} \sum_{k=m-M+1}^{m-1} g_k U^{**}_{m-k,j} = U^*_{m,j} + \frac{i}{2}\tau\kappa_1 h^{-\alpha} \sum_{k=m-M+1}^{m-1} g_k U^*_{m-k,j} - \frac{i}{2}\tau f^{n+\frac{1}{2}}_{m,j},$$

$$U^{***}_{m,j} - \frac{i}{2}\tau\kappa_2 h^{-\alpha} \sum_{k=j-M+1}^{j-1} g_k U^{***}_{m,j-k} = U^{**}_{m,j} + \frac{i}{2}\tau\kappa_2 h^{-\alpha} \sum_{k=j-M+1}^{j-1} g_k U^{**}_{m,j-k} - \frac{i}{2}\tau f^{n+\frac{1}{2}}_{m,j},$$

$$m, j = 1, 2, ..., M,$$

$$U^{***}_{m,j} = U^{**}_{m,j} = 0, \quad m, j = 0, M, \tag{4.6}$$

$$U^{n+1}_{m,j} = e^{-i\left(\gamma |U^{***}_{m,j}|^p \tau\right)/2} U^{***}_{m,j}, \qquad m, j = 0, 1, 2, ..., M. \tag{4.7}$$

5. Numerical results

In this section we present the numerical results of the new method on several test problems. We tested the accuracy and stability of the method described in this paper by performing the mentioned scheme for different values of h and τ. We performed our computations using **MATLAB 10** software on a Pentium IV, 2800 MHz CPU machine with 2 Gbyte of memory.
Also we calculated the computational order of the method presented in this article (denoted by C-order) with the following formula:

$$\frac{\log(\frac{E_1}{E_2})}{\log(\frac{h_1}{h_2})},$$

in which E_1 and E_2 are errors correspond to grids with mesh size h_1 and h_2 respectively.

5.1. **Test problem 1.** We consider the one dimensional Schrödinger equation with Riesz space fractional derivative

$$i\frac{\partial u(x,t)}{\partial t} = \kappa\frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + \gamma u(x,t)|u(x,t)|^4 + f(x,t), \quad 0 < x < 1, \quad 0 \le t \le 1,$$

with the initial condition

$$\phi(x) = x^2(1-x)^2,$$

and

$$f(x,t) = \alpha e^{it\alpha}x^2(1-x)^2 + \frac{1}{\Gamma(5-\alpha)}e^{it\alpha}x^{-\alpha}\left(\frac{1}{(1-x)^{\alpha}}(x-1)^2x^{\alpha}\left(12x^2-6x\alpha+(\alpha-1)\alpha\right)\right.$$

$$\left.+x^2\left(12(x-1)^2+(6x-7)\alpha+\alpha^2\right)+x^2\left(12(x-1)^2+(6x-7)\alpha+\alpha^2\right)\right)\sec\left(\frac{\pi\alpha}{2}\right)$$

$$-e^{it\alpha}x^{10}(1-x)^{10}.$$

The exact solution is given as follows

$$u(x,t) = e^{it\alpha}x^2(1-x)^2.$$

We show the absolute error, C-order and CPU time of applied method for solving this test problem with $h = \tau$ and $\alpha = 1.1$ in Table 1 and with $\alpha = 1.75$ in Table 2.

Table 1

Numerical results obtained for Test problem 1 with $\alpha = 1.1$ and $h = \tau$.

$h=\tau$	L_∞	C-order	CPU time(s)
$\frac{1}{5}$	7.5552×10^{-3}	–	0.006119
$\frac{1}{10}$	1.9446×10^{-3}	1.9580	0.009084
$\frac{1}{20}$	4.1771×10^{-4}	2.2189	0.014403
$\frac{1}{40}$	1.0307×10^{-4}	2.0189	0.037851
$\frac{1}{80}$	2.7207×10^{-5}	1.9216	0.157282

FIGURE 1. Real and imaginary parts of approximate solution for Test problem 1 with $h = \tau = 1/80$.

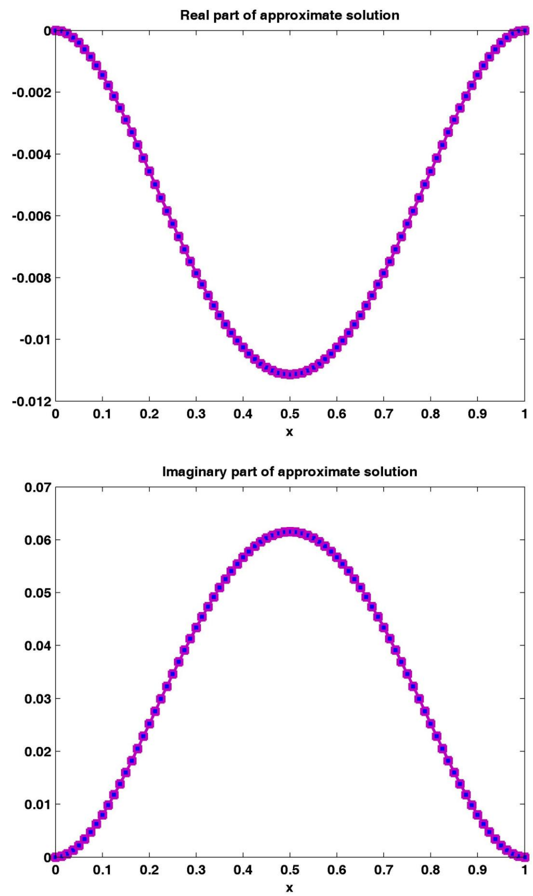


Table 2

Numerical results obtained for Test problem 1 with $\alpha = 1.75$ and $h = \tau$.

$h = \tau$	L_∞	C-order	CPU time(s)
$\frac{1}{10}$	3.9005×10^{-3}	$-$	0.007927
$\frac{1}{20}$	8.2339×10^{-4}	2.2440	0.015199
$\frac{1}{40}$	2.0792×10^{-4}	1.9855	0.038255
$\frac{1}{80}$	4.7193×10^{-5}	2.1394	0.149166
$\frac{1}{160}$	1.2004×10^{-5}	1.9751	0.831297

Tables 1, 2 present the good accuracy and CPU time of method and confirm that the proposed scheme has second-order of accuracy in both space and time components.

FIGURE 1 shows the real and imaginary parts of approximate solution with $h = \tau = 1/80$.

5.2. **Test problem 2.** We consider the one dimensional Schrödinger equation with Riesz space fractional derivative

$$i\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t)|u(x,t)|^2, \quad -30 < x < 30,$$

with the initial condition

$$\phi(x) = \text{sech}^2(x)\exp(2ix).$$

When $\alpha = 2$ the exact solution is given as

$$u(x,t) = \text{sech}^2(x - 4t)\exp(i(2x - 3t)).$$

For this test problem $f(x,t) = 0$, so the method preserves the discrete mass. To show this numerically, we define

$$M^n = \frac{\left|\,\|U^n\| - \|U^0\|\,\right|}{\|U^0\|}.$$

Table 3 presents the errors M^n with $h = 0.1$ and $\tau = 0.05$ and different values of t and α. As we see the method preserves the discrete mass for different values of t and α.

Table 3

Errors M^n for Test problem 2 with $h = 0.1$ and $\tau = 0.05$.

α	$t = 20$	$t = 40$	$t = 60$
1.1	3.2889×10^{-13}	6.7982×10^{-13}	9.9421×10^{-13}
1.4	2.5838×10^{-13}	5.1041×10^{-13}	7.3662×10^{-13}
1.7	2.2204×10^{-13}	3.1796×10^{-13}	5.2570×10^{-13}
1.9	1.5590×10^{-13}	2.4011×10^{-13}	3.7039×10^{-13}
2	7.4476×10^{-14}	1.2611×10^{-13}	1.9364×10^{-13}

FIGURE 2 shows the numerical solutions of Test problem 2 with $h = 0.1$, $\tau = 0.05$, $\alpha = 1.1, 1.5, 1.8, 2$ at time $t = 3$. When α becomes smaller, the shape of the soliton will change more quickly. When α tends to 2, the numerical solutions of the fractional equation are convergent to the solutions of the usual classical equation.

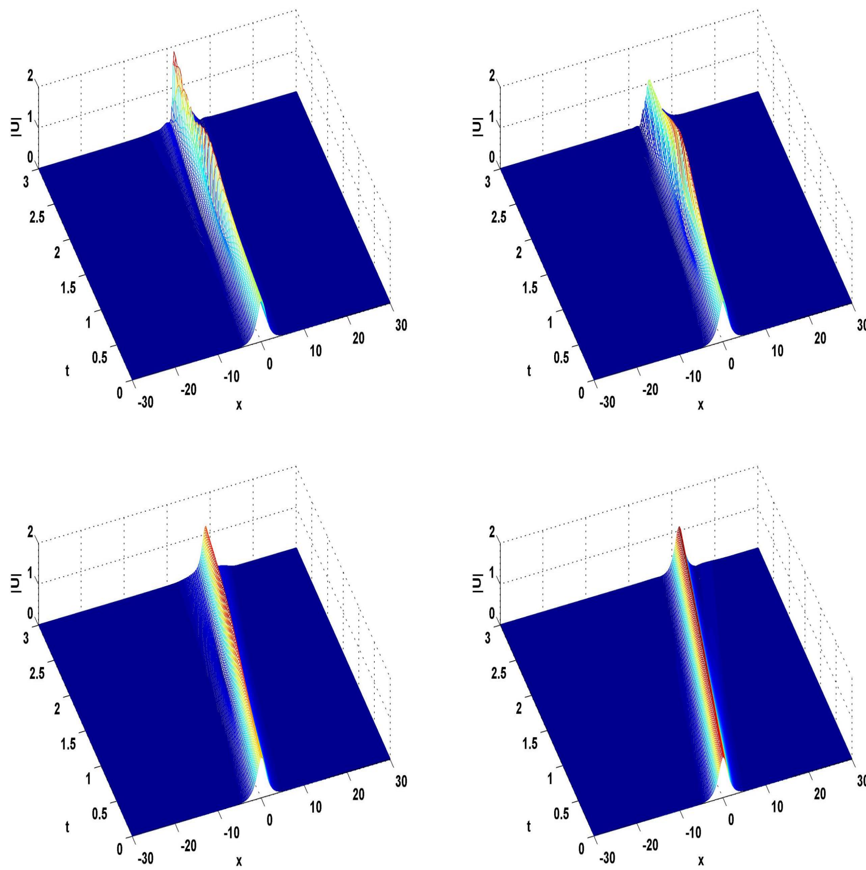
5.3. **Test problem 3.** We consider the two dimensional Schrödinger equation with Riesz space fractional derivatives

$$i\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^\alpha u(x,y,t)}{\partial |x|^\alpha} + \frac{\partial^\alpha u(x,y,t)}{\partial |y|^\alpha} + u(x,y,t)|u(x,y,t)|^2 + f(x,y,t), \quad 0 < x < 1,$$

with the initial condition

$$\phi(x,y) = x^2(1-x)^2y^2(1-y)^2,$$

FIGURE 2. Numerical solutions of Test problem 2 with $h = 0.1$, $\tau = 0.05$ with $\alpha = 1.1$ (top-left), $\alpha = 1.5$ (top-right), $\alpha = 1.8$ (down-left) and $\alpha = 2$ (down-right).



and

$$f(x,y,t) = \alpha e^{it\alpha} x^2(1-x)^2 y^2(1-y)^2 + \frac{1}{\Gamma(5-\alpha)} e^{it\alpha} x^{-\alpha} y^2 (1-y)^2 \left(\frac{1}{(1-x)^\alpha} (x-1)^2 x^\alpha (12x^2 - 6x\alpha \right.$$

$$+(\alpha-1)\alpha) + x^2 \left(12(x-1)^2 + (6x-7)\alpha + \alpha^2\right) + x^2 \left(12(x-1)^2 + (6x-7)\alpha + \alpha^2\right)\Big) \sec\left(\frac{\pi\alpha}{2}\right) +$$

$$+ \frac{1}{\Gamma(5-\alpha)} e^{it\alpha} y^{-\alpha} x^2 (1-x)^2 \left(\frac{1}{(1-y)^\alpha} (y-1)^2 y^\alpha \left(12y^2 - 6y\alpha + (\alpha-1)\alpha\right) \right.$$

$$\left. + y^2 \left(12(y-1)^2 + (6y-7)\alpha + \alpha^2\right) + y^2 \left(12(y-1)^2 + (6y-7)\alpha + \alpha^2\right)\right) \sec\left(\frac{\pi\alpha}{2}\right)$$

$$- e^{it\alpha} x^6 (1-x)^6 y^6 (1-y)^6.$$

The exact solution is given as

$$u(x,y,t) = e^{it\alpha}x^2(1-x)^2y^2(1-y)^2.$$

Table 4 shows the absolute error of applied method for different values of α, $h = 0.05$ and $\tau = 0.01$ and Table 5 presents the absolute error of applied method for different values of α, $h = 0.02$ and $\tau = 0.001$.

Table 4

Errors obtained for Test problem 3 with $h = 0.05$ and $\tau = 0.01$.

t	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.8$
$t = 0.5$	3.2204×10^{-5}	1.1611×10^{-4}	1.7771×10^{-4}
$t = 1$	2.9024×10^{-5}	9.0795×10^{-5}	1.9298×10^{-4}
$t = 2$	3.1545×10^{-5}	1.1779×10^{-4}	1.5500×10^{-4}
$t = 5$	2.5826×10^{-5}	5.3080×10^{-5}	2.1169×10^{-4}

Table 5

Errors obtained for Test problem 3 with $h = 0.02$ and $\tau = 0.001$.

t	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.8$
$t = 0.5$	3.7006×10^{-6}	1.2397×10^{-5}	2.1020×10^{-5}
$t = 1$	4.5712×10^{-6}	8.1640×10^{-6}	1.8986×10^{-5}
$t = 2$	3.0990×10^{-6}	1.3128×10^{-5}	1.7427×10^{-5}
$t = 5$	2.8425×10^{-6}	6.2316×10^{-6}	1.4689×10^{-5}

Tables 4, 5 show the good accuracy and efficiency of proposed method for the solution of this test problem.

5.4. **Test problem 4.** We consider the two dimensional Schrödinger equation with Riesz space fractional derivatives

$$i\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^\alpha u(x,y,t)}{\partial |x|^\alpha} + \frac{\partial^\alpha u(x,y,t)}{\partial |y|^\alpha} + u(x,y,t)|u(x,y,t)|^2, \quad 0 < x < 1,$$

with the initial condition

$$\phi(x,y) = e^{\left\{-\frac{(x-0.5)^2}{\beta} - \frac{(y-0.5)^2}{\beta}\right\}}.$$

We put $\beta = 0.02$ and as we see from FIGURE 3 the initial condition is a Gaussian pulse with unit hight centered at $x = 0.5$ and $y = 0.5$. FIGURE 3 presents the initial condition and numerical solutions of Test problem 4 with $h = \tau = 0.01$ and different values of α. Table 6 shows the errors M^n with $h = \tau = 0.01$ and different values of t and α. As we see the method preserves the discrete mass for different values of t and α.

FIGURE 3. Initial condition and numerical solutions of Test problem 4 with $h = \tau = 0.01$ and different values of α.

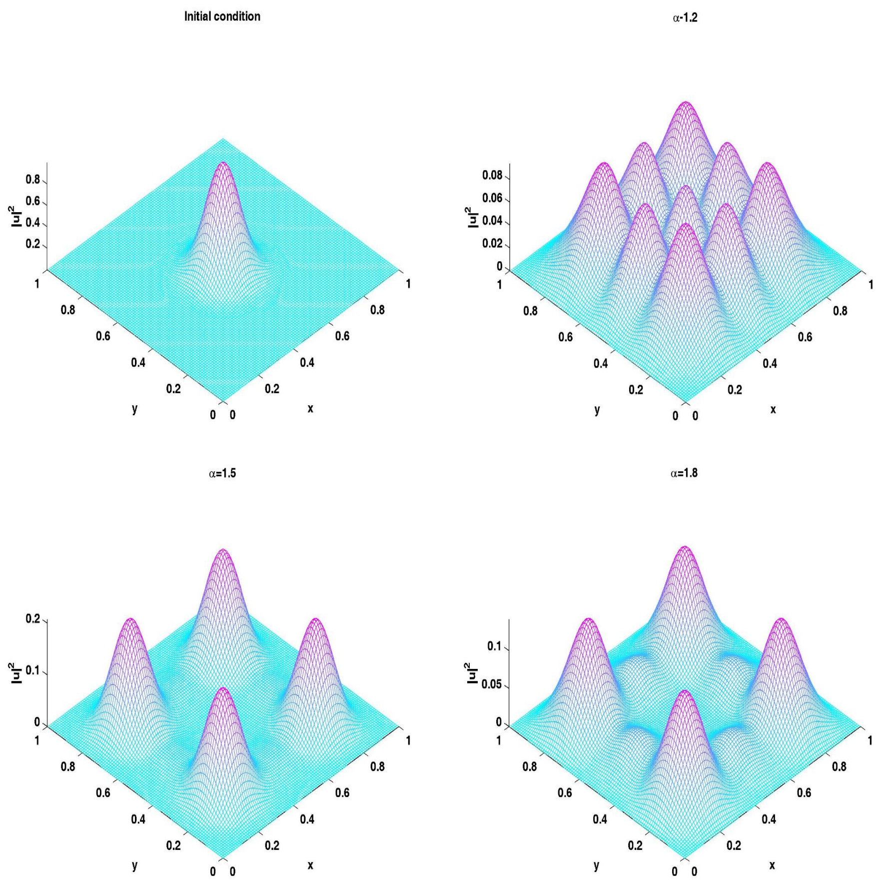


Table 6

Errors M^n for Test problem 4 with $h = \tau = 0.01$.

α	$t = 2$	$t = 5$	$t = 10$
1.1	3.1613×10^{-5}	3.2776×10^{-4}	1.0398×10^{-3}
1.4	3.7946×10^{-5}	2.1293×10^{-4}	8.1592×10^{-4}
1.7	3.9043×10^{-5}	2.5193×10^{-4}	9.5731×10^{-4}
1.9	3.6352×10^{-5}	2.3467×10^{-4}	8.9953×10^{-4}
2	3.9184×10^{-5}	2.3171×10^{-4}	9.1790×10^{-4}

6. Conclusion

We proposed a split-step finite difference method for the solution of Schrödinger equation with the Riesz space fractional derivative. After discretization of the Riesz fractional derivative, we applied the Crank-Nicolson and a split-step methods to obtain a numerical algorithm for this equation. We proved that the proposed method is unconditionally stable and convergent. Numerical results corroborated the theoretical results and efficiency of proposed scheme. The proposed method can be easily extended to the coupled and three-dimensional nonlinear Schrödinger equations with the Riesz space fractional derivatives.

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