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An approach based on statistical spline model for Volterra-Fredholm integral equations

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Abstract In this paper, an approach based on statistical spline model (SSM) and collocation method is proposed to solve Volterra-Fredholm integral equations. The set of collocation nodes is chosen so that the points yield minimal error in the nodal polynomials. Under some standard assumptions, we establish the convergence property of this approach. Numerical results on some problems are given to describe the introduced method. A comparison between the numerical results and those obtained from Lagrange and Taylor collocation methods demonstrates that the proposed method generates an approximate solution with minimal error.

Keywords. Statistical spline model, Volterra-Fredholm integral equations, Convergence analysis.

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1. Introduction

Integral equations are arisen in many real world applications in physics, biology and engineering such as air foil theory, elastic contact problems and molecular conduction [4, 9]. In recent years, integral equations have been a subject of extensive investigation and several numerical methods for solving these problems have been presented.

In this paper, the following Volterra-Fredholm integral equations are considered

$$y(h(x)) = f(x) + \lambda_1 \int_a^{h(x)} k_1(x,t)y(t)dt + \lambda_2 \int_a^b k_2(x,t)y(t)dt, \ \ (a \le x \le b), \quad (1.1)$$

and

$$y(x) = f(x) + \lambda_1 \int_a^{h(x)} k_1(x,t)y(t)dt + \lambda_2 \int_a^b k_2(x,t)y(h(t))dt, \ \ (a \le x \le b), \quad (1.2)$$

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where $f : [a, b] \to \mathbb{R}$, $h : [a, b] \to [a, \infty)$ and $k_i : [a, b] \times [a, b] \to \mathbb{R}$ $(i = 1, 2)$ are the known functions, h is invertible, $y : [a, b] \to \mathbb{R}$ is the unknown function, a and b are constants and λ_1 and λ_2 are real numbers so that $\lambda_1^2 + \lambda_2^2 \neq 0$ [10]. It is worth mentioning that when $h(x)$ is a linear polynomial, Eq. (1.2) is a functional integral equation with proportional delay [3, 5].

Recently, several numerical methods based on Lagrange collocation method [9], Taylor collocation method [10], least squares approximation [11] and Legendre collocation method [7] have been proposed in order to solve these Volterra-Fredholm integral equations.

The main contribution of this paper is to find an approximate solution of Eqs. (1.1) and (1.2) by using the statistical spline model. Hence, in Section 2 we summarize the relevant properties of statistical spline model. In Section 3, an algorithm along with its convergence analysis is provided for approximating the solution of integral equations (1.1) and (1.2). Finally, some numerical results are given to show the efficiency and effectiveness of SSM in practice.

2. Statistical spline model

In general, the usual form of a spline function is given by

$$S_{n,m}(f; x) = \begin{cases} p_{n_1}(x), & x \in I_1, \\ p_{n_2}(x), & x \in I_2, \\ \vdots & \vdots \\ p_{n_m}(x), & x \in I_m, \end{cases}$$

where p_{n_i}, $i = 1, 2, \ldots, m$, is a polynomial of degree $n_i \in \mathbb{Z}^+$, $n = \max\{n_i\}_{i=1}^m$ and $\{I_i\}_{i=1}^m$ are a specific partition of $[a, b]$. In usual form, $\{I_i\}_{i=1}^m$ are the predetermined sub-intervals and one should obtain the corresponding criteria by referring to the initial conditions of the problem.

In contrast to usual form, there is another type of spline function, such that instead of considering $\{p_{n_i}\}_{i=1}^m$ as the polynomials with unknown coefficients, the partition $\{I_i\}_{i=1}^m$ are unknown and must be derived. In this type of splines, since the obtained set $D = \{I_1, I_2, \ldots, I_m\}$ is a specific partition of $[a, b]$ so that $I_i \cap I_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, m$ and $\bigcup_{j=1}^{m} I_j = [a, b]$, hence D can be considered as a probability space in which the proportion of each sub-interval I_j is computed as

$$\mathbf{P_r}(I_j) = \frac{\ell(I_j)}{\ell([a, b])} = \frac{\ell(I_j)}{b - a}, \tag{2.1}$$

where ℓ is the length of each sub-interval. This type of spline function is called statistical spline model (see [6]). In continuation, we briefly describe how to construct the statistical spline model along with some further statistical concepts based on the same notations in [6].

Without loss of generality, we suppose that $n_1 = n_2 = \ldots = n_m = n$. Let $x_0 < x_1 < \ldots < x_n$ be $n + 1$ distinct points of $[a, b]$ and $f \in C^{n+1}[a, b]$. Then the

Lagrange interpolation is given by

$$f(x) = \sum_{i=0}^{n} f(x_i) L_i(x; x_i) + E_{n+1}(f; x; \{x_i\}),$$

where

$$L_i(x; x_i) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, \ i = 0, 1, \ldots, n,$$

and

$$E_{n+1}(f; x; \{x_i\}) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{j=0}^{n} (x - x_j), \ a < \xi_x < b.$$

Clearly, the absolute error of the Lagrange interpolation is bounded above by

$$|E_{n+1}(f; x; \{x_i\})| \leq \frac{M_{n+1}}{(n+1)!} \prod_{j=0}^{n} |x - x_j|,$$

where $M_{n+1} = \max_{a \leq x \leq b} \left| f^{(n+1)}(x) \right|$. Define

$$E^*_{n+1}(f; x; \{x_i\}) = \frac{M_{n+1}}{(n+1)!} \prod_{j=0}^{n} |x - x_j|.$$

This error bound depends on two parameters including M_{n+1} and $\prod_{j=0}^{n} |x - x_j|$. The first parameter is beyond control, but the second one is a polynomial of degree $n+1$ that only depends on $\{x_i\}_{i=0}^{n}$. Therefore, minimizing the aforementioned error bound only depends on the distribution of the points $\{x_i\}_{i=0}^{n}$. The key question that naturally arises here is:

What is the best choice for the points $\{x_i\}_{i=0}^{n}$ in order to minimize $\prod_{j=0}^{n} |x - x_j|$ as much as possible?

The statistical spline model of degree n can be a solution to this question.

In order to construct this model, consider k monic polynomials of degree $n+1$ as below

$$\bar{q}_{n+1,i}(x) = \prod_{j=0}^{n} (x - \ x_{j,i}), \qquad \text{for} \quad , \ i = 1, 2, \ldots, k.$$

where $\{x_{0,i}, x_{1,i}, \ldots, x_{n,i}\}$, for $i = 1, 2, \ldots, k$, are the sets of $n+1$ distinct points in $[a, b]$. Define

$$I_{n,i,j} := \{x \in [a,b] : |\bar{q}_{n+1,i}(x)| < |\bar{q}_{n+1,j}(x)|\,, \ \text{for} \ i, j = 1, 2, \ldots, k \text{ and } i \neq j\}\,.$$

Now, associated to the monic polynomial $\bar{q}_{n+1,i}(x)$, $i = 1, 2, \ldots, k$ define the subinterval $I_{n,i} = \cap_{j=1}^{k} I_{n,i,j}$, for $j \neq i$. This definition would eventually lead to a unique partition $D = \{I_{n,1}, I_{n,2}, \ldots, I_{n,k}\}$ for the main interval $[a, b]$. After constructing D, k-criterion statistical spline model is introduced by

$$S_{n,k}(f;x) = \begin{cases} p_{n,1}(x) = \sum_{j=0}^{n} f(x_{j,1}) L_j(x; x_{j,1}) & x \in I_{n,1}, \\ p_{n,2}(x) = \sum_{j=0}^{n} f(x_{j,2}) L_j(x; x_{j,2}) & x \in I_{n,2}, \\ \vdots & \vdots \\ p_{n,k}(x) = \sum_{j=0}^{n} f(x_{j,k}) L_j(x; x_{j,k}) & x \in I_{n,k}. \end{cases} \tag{2.2}$$

It is worth to point out the specific partition D for the main interval $[a, b]$ is determined so that the error bound for each criterion in different regions of $[a, b]$ is minimized.

Now, as an important case, let us restrict ourselves to the case in which $k = 3$ and $[a, b] = [-1, 1]$. In this case, the Chebyshev polynomials of the first and second kinds and the monic type of Legendre polynomial are considered for 3-criterion statistical spline model. We have the Chebyshev polynomials of the first kind

$$T_n(x) = \cos(n \text{ arccos } x) = 2^{n-1} \sum_{k=0}^{[n/2]} \frac{(-[n/2])_k (1/2 - [(n+1)/2])_k}{(-n+1)_k k!} x^{n-2k},$$

the Chebyshev polynomials of the second kind

$$U_n(x) = \frac{\sin((n+1) \text{ arccos } x)}{\sqrt{1-x^2}} = 2^n \sum_{k=0}^{[n/2]} \frac{(-[n/2])_k (1/2 - [(n+1)/2])_k}{(-n)_k k!} x^{n-2k},$$

and the monic type of Legendre polynomial

$$\bar{P}_n(x) = \sum_{k=0}^{[n/2]} \frac{(-[n/2])_k (1/2 - [(n+1)/2])_k}{(-n+1/2)_k k!} x^{n-2k},$$

where $(a)_k = \prod_{i=0}^{k-1} (a+i)$. These polynomials are the best selection of x_i's for minimizing the error bound in L_∞, L_1 and L_2 spaces, respectively [1, 2, 6, 8]. Thus, for constructing the statistical spline model of degree n, we choose

$$\begin{aligned} \bar{q}_{n+1,1}(x) &= 2^{-(n+1)} T_{n+1}(x), \\ \bar{q}_{n+1,2}(x) &= \bar{P}_{n+1}(x), \\ \bar{q}_{n+1,3}(x) &= 2^{-n} U_{n+1}(x) \end{aligned}$$

and

$$\{x_{j,l}\}_{j=0}^{n} = \{x \in [-1, 1] \mid \bar{q}_{n+1,l}(x) = 0\}, \ l = 1, 2, 3. \tag{2.3}$$

In order to determine the specific partition $D = \{I_{n,1}, I_{n,2}, I_{n,3}\}$, the following inequalities should be solved

$$\begin{cases} |\bar{q}_{n+1,3}(x)| \le |\bar{q}_{n+1,1}(x)| & x \in \alpha_n \subset [-1, 1], \\ |\bar{q}_{n+1,3}(x)| \le |\bar{q}_{n+1,2}(x)| & x \in \beta_n \subset [-1, 1], \\ |\bar{q}_{n+1,2}(x)| \le |\bar{q}_{n+1,1}(x)| & x \in \gamma_n \subset [-1, 1]. \end{cases}$$

For solving these inequalities numerically, an approach has been given in [6] for $n = 0, 1, \ldots, 6$. Thus, the statistical spline model is given by

$$S_{n,3}(x) = \begin{cases} p_{n,1}(x) = \sum_{i=0}^{n} f(x_{i,1}) L_i(x; x_{i,1}) & x \in I_{n,1} = \alpha'_n \cap \gamma'_n, \\ p_{n,2}(x) = \sum_{i=0}^{n} f(x_{i,2}) L_i(x; x_{i,2}) & x \in I_{n,2} = \beta'_n \cap \gamma_n, \\ p_{n,3}(x) = \sum_{i=0}^{n} f(x_{i,3}) L_i(x; x_{i,3}) & x \in I_{n,3} = \alpha_n \cap \beta_n, \end{cases} \tag{2.4}$$

where $\alpha'_n = [-1, 1] - \alpha_n$ (for example).

Remark 2.1. Note that, the interval $[-1, 1]$ can be converted into any arbitrary interval $[a, b]$ by using the transformation $x = \frac{b-a}{2}\, t + \frac{b+a}{2}$. Consequently, the specific partition $\{I_{n,1}, I_{n,2}, I_{n,3}\}$ and $\{x_{j,l}\}_{j=0}^{n}$, $l = 1, 2, 3$ can be defined over any $[a, b]$.

By referring to (2.1) we give some further statistical concepts into the definition of statistical spline model.

Remark 2.2. The expected value, the moment of order i at the neighborhood of any arbitrary polynomial $p(x)$ and the variance of the statistical spline (2.2) are respectively defined as follows [6]:

$$\begin{aligned} E\left(S_{n,k}(f;x)\right) &= \sum_{j=1}^{k} p_{n,j}(x) P_r(I_{n,j}), \\ \mu_i\left(S_{n,k}(f;x); p(x)\right) &= \sum_{j=1}^{k} \left(p_{n,j}(x) - p(x)\right)^i P_r(I_{n,j}), \\ \operatorname{var}\left(S_{n,k}(f;x)\right) &= \mu_2\left(S_{n,k}(f;x); 0\right) - E^2\left(S_{n,k}(f;x)\right). \end{aligned}$$

In next section, an algorithm based on statistical spline model for finding an approximate solution of Eqs. (1.1) and (1.2) along with its convergence analysis are provided.

3. Main results and description of the algorithm

We turn now to the construction of the numerical algorithm for solving integral equation (1.1) numerically. Obviously the integral equation (1.2) could either be solved using the same method. The main tool at our disposal is the ability to minimize the error in the nodal polynomial. We begin by constructing statistical spline model, on which the solution to (1.1) and (1.2) are sought, eligible.

For the sake of simplicity, we suppose that $y_{j,i}$ be the unknown approximate solution of $y(x)$ at $x_{j,i}$ for any $j = 0, 1, \ldots, n$ and $i = 1, 2, \ldots, k$. In addition, let $\{I_{n,1}, I_{n,2}, \ldots, I_{n,k}\}$ be the specific partition of $[a, b]$ determined by the rule as given in Section 2. Now, we define

$$S_{n,k}(y;x)=\begin{cases}\sum_{j=0}^{n} y_{j,1}L_j(x;x_{j,1}) & x\in I_{n,1},\\ \sum_{j=0}^{n} y_{j,2}L_j(x;x_{j,2}) & x\in I_{n,2}\\ \vdots & \vdots\\ \sum_{j=0}^{n} y_{j,k}L_j(x;x_{j,k}) & x\in I_{n,k}.\end{cases} \tag{3.1}$$

Then by substituting (3.1) into equations (1.1), we get

$$\begin{aligned}\delta_n(x) &= S_{n,k}(y,h(x))-f(x)-\lambda_1\int_a^{h(x)} k_1(x,t)S_{n,k}(y,t)dt\\ &- \lambda_2\int_a^b k_2(x,t)S_{n,k}(y,t)dt.\end{aligned} \tag{3.2}$$

If $S_{n,k}(y;x)=y(x)$, then $S_{n,k}(y;x)$ is the exact solution of (1.1). Otherwise, we find $y_{j,i}$ such that $\delta_n(x_{j,i})=0$ for collocation points $x_{j,i}$, $j=0,1,\dots,n$ and $i=1,2,\dots,k$, which are defined in Section 2.

Remark 3.1. For integral equation (1.2), the relation (3.2) changes to the following

$$\begin{aligned}\delta_n(x) &= S_{n,k}(y,x)-f(x)-\lambda_1\int_a^{h(x)} k_1(x,t)S_{n,k}(y,t)dt\\ &- \lambda_2\int_{h(a)}^{h(b)} \frac{1}{h'(h^{-1}(u))}k_2(x,h^{-1}(u))S_{n,k}(y,u)du,\end{aligned} \tag{3.3}$$

where $u=h(t)$.

Remark 3.2. According to Section 2, since the choice of $\{x_{0,i},x_{1,i},\dots,x_{n,i}\}$, for $i=1,2,\dots,k$ are arbitrary, therefore some of them can be repeated in each criterion. Suppose that N be the number of distinct collocation points $x_{j,i}$ for $j=0,1,\dots,n$ and $i=1,2,\dots,k$. It is obvious that, if $x_{s,\ell}=x_{j,i}$, then we put $y_{s,\ell}=y_{j,i}$.

Consequently, we derive a linear system of equations of order N. By solving this linear system of equations, N unknown coefficients $y_{j,i}$ for $j=0,1,\dots,n$ and $i=1,2,\dots,k$ are obtained.

Now we establish the convergence property of the proposed computational algorithm.

Theorem 3.3. *Let $y(x)$ be the exact solution of (1.1) for $h(x)=x$ and $S_{n,k}(y;x)$ be its approximate solution which is obtained from the proposed method. Also, suppose that f is a function defined on $[a,b]$ and $k_1(x,t)$ and $k_2(x,t)$ are sufficiently smooth functions on $[a,b]\times[a,b]$. Then*

$$\|y(x)-S_{n,k}(y;x)\|_\infty \le \frac{M.M_{n+1}}{(n+1)!}+\Gamma\max_{1\le i\le k}\{\|E_i\|_\infty\},$$

where $M = \max_{i=1,2,\ldots,k} \left\{ \max_{x \in I_{n,i}} \left| \prod_{j=0}^{n} (x - a_{j,i}) \right| \right\}$, $E_i = (y(a_{0,i}) - y_{0,i}, \ldots, y(a_{n,i}) - y_{n,i})^T$, $\Gamma = \max_{0 \le i \le n} \{\gamma_i\}$ *and* $\gamma_i = \max_{0 \le j \le n} \{L_j(x; a_{j,i})\}$.

Proof. Let

$$S^*_{n,k}(y;x) = \begin{cases} \sum_{j=0}^{n} y(a_{j,1}) L_j(x; a_{j,1}) & x \in I_{n,1}, \\ \sum_{j=0}^{n} y(a_{j,2}) L_j(x; a_{j,2}) & x \in I_{n,2} \\ \vdots & \vdots \\ \sum_{j=0}^{n} y(a_{j,1}) L_j(x; a_{j,k}) & x \in I_{n,k}, \end{cases} \tag{3.4}$$

be the exact solution of Eq. (1.1) that is obtained from the proposed computational algorithm. Thus, we have

$$\begin{aligned} \| y(x) - S_{n,k}(y;x) \|_\infty &= \| y(x) - S^*_{n,k}(y;x) + S^*_{n,k}(y;x) - S_{n,k}(y;x) \|_\infty \\ &\le \| y(x) - S^*_{n,k}(y;x) \|_\infty + \| S^*_{n,k}(y;x) - S_{n,k}(y;x) \|_\infty . \end{aligned} \tag{3.5}$$

Using the interpolation error formula, we have

$$\left| y(x) - S^*_{n,k}(y;x) \right| \le \begin{cases} \frac{M_{n+1}}{(n+1)!} \left| \prod_{j=0}^{n} (x - a_{j,1}) \right| & x \in I_{n,1}, \\ \frac{M_{n+1}}{(n+1)!} \left| \prod_{j=0}^{n} (x - a_{j,2}) \right| & x \in I_{n,2}, \\ \vdots & \vdots \\ \frac{M_{n+1}}{(n+1)!} \left| \prod_{j=0}^{n} (x - a_{j,k}) \right| & x \in I_{n,k}. \end{cases}$$

Suppose that $M = \max_{i=1,2,\ldots,k} \left\{ \max_{x \in I_{n,i}} \left| \prod_{j=0}^{n} (x - a_{j,i}) \right| \right\}$ and noting that M is the optimal error bound with respect to other Lagrange collocation methods, one may write

$$\left\| y(x) - S^*_{n,k}(y;x) \right\|_\infty \le \frac{M.M_{n+1}}{(n+1)!}. \tag{3.6}$$

On the other hand, from (3.1) and (3.4) we obtain:

$$S^*_{n,k}(y;x) - S_{n,k}(y;x) = \begin{cases} \sum_{j=0}^{n} (y(a_{j,1}) - y_{j,1}) L_j(x; a_{j,1}) = E_1^T.L_1 & x \in I_{n,1}, \\ \sum_{j=0}^{n} (y(a_{j,2}) - y_{j,2}) L_j(x; a_{j,2}) = E_2^T.L_2 & x \in I_{n,2}, \\ \vdots & \vdots \\ \sum_{j=0}^{n} (y(a_{j,k}) - y_{j,k}) L_j(x; a_{j,k}) = E_k^T.L_k & x \in I_{n,k}, \end{cases}$$

where $E_i = (y(a_{0,i}) - y_{0,i}, \ldots, y(a_{n,i}) - y_{n,i})^T$ and $L_i = (L_0(x; a_{1,i}), \ldots, L_n(x; a_{n,i}))^T$ for $i = 1, 2, \ldots, k$. Moreover

$$\max_{a \le x \le b} \left| S^*_{n,k}(y; x) - S_{n,k}(y; x) \right| = \max_{1 \le i \le k} \left\{ \max_{x \in I_{n,i}} \left| S^*_{n,k}(y; x) - S_{n,k}(y; x) \right| \right\}.$$

Therefore, for any $x \in I_{n,i}$, we get

$$\left| S^*_{n,k}(y; x) - S_{n,k}(y; x) \right| = \left| E_i^T . L_i \right| \le \|L_i\|_\infty \|E_i\|_\infty = \gamma_i \|E_i\|_\infty,$$

where $\gamma_i = \max_{0 \le j \le n} \{L_j(x; a_{j,i})\}$. So

$$\max_{a \le x \le b} \left| S^*_{n,k}(y; x) - S_{n,k}(y; x) \right| \le \max_{1 \le i \le k} \{\gamma_i \|E_i\|_\infty\} \le \Gamma \max_{1 \le i \le k} \{\|E_i\|_\infty\}, \tag{3.7}$$

where $\Gamma = \max_{0 \le i \le n} \{\gamma_i\}$. Now substituting the error bounds (3.6) and (3.7) into (3.5) completes the proof of the theorem. □

Similar to [10] we can state the following theorems for Eqs. (1.1) and (1.2).

Theorem 3.4. *Let $y(x)$ and $y(h(x))$ be the exact solutions of (1.1), $S_{n,k}(y; x)$ and $S_{n,k}(y; h(x))$ be their approximate solutions which are obtained from the proposed algorithm. Then*

$$\|y(h(x)) - S_{n,k}(y; h(x))\|_\infty \le M_1 \|y(x) - S_{n,k}(y; x)\|_\infty,$$

where $M_1 = \sup_{a \le x \le b} \left(|\lambda_1| \int_a^b |k_1(x,t)| dt + |\lambda_2| \int_a^b |k_2(x,t)| dt \right)$.

Proof. According to [10], let us consider

$$\begin{aligned} y(h(x)) &= f(x) + \lambda_1 \int_a^{h(x)} k_1(x,t) y(t) dt + \lambda_2 \int_a^b k_2(x,y) y(t) dt, \\ S_{n,k}(y; h(x)) &= f(x) + \lambda_1 \int_a^{h(x)} k_1(x,t) S_{n,k}(y; t) dt + \lambda_2 \int_a^b k_2(x,y) S_{n,k}(y; t) dt. \end{aligned}$$

Hence

$$\begin{aligned} & y\,(h(x)) - S_{n,k}\,(y; h(x)) \\ &= \lambda_1 \int_a^{h(x)} k_1(x,t)\,(y(t) - S_{n,k}(y;t))\,dt + \lambda_2 \int_a^b k_2(x,t)\,(y(t) - S_{n,k}(y;t))\,dt. \end{aligned}$$

Therefore

$$\begin{aligned} & |y(h(x)) - S_{n,k}(y; h(x))| \\ &\le |\lambda_1| \int_a^{h(x)} |k_1(x,t)|\, |y(t) - S_{n,k}(y;t)|\, dt + |\lambda_2| \int_a^b |k_2(x,t)|\, |y(t) - S_{n,k}(y;t)|\, dt \\ &\le |\lambda_1|\, \|y(t) - S_{n,k}(y;t)\|_\infty \int_a^{h(x)} |k_1(x,t)|\, dt + |\lambda_2|\, \|y(t) - S_{n,k}(y;t)\|_\infty \int_a^b |k_2(x,t)|\, dt \\ &\le M_1 \|y(t) - S_{n,k}(y;t)\|_\infty. \end{aligned}$$

□

Theorem 3.5. *Let $y(x)$ and $y(h(x))$ be the exact solutions of (1.2), $S_{n,k}(y;x)$ and $S_{n,k}(y;h(x))$ be their approximate solutions which are obtained from the proposed algorithm. Then*

$$\|y(x) - S_{n,k}(y;x)\|_\infty \leq M_2 \|y(h(x)) - S_{n,k}(y;h(x))\|_\infty,$$

where $M_2 = \sup_{a\leq x\leq b}\left(\frac{|\lambda_2|\int_a^b |k_2(x,t)|dt}{1-|\lambda_1|\int_a^b |k_1(x,t)|dt}\right)$.

Proof. This inequality can be easily proved by using similar approach to the proof of Theorem 3.3 in [10]. □

4. Numerical examples

In this section, we consider three examples as given in [10] to present the priority and efficiency of SSM with respect to Lagrange collocation method (LCM) and Taylor collocation method (TCM). 3-criterion statistical spline models of degree 2, 3 and 5 are applied to these examples. In this model, Chebyshev polynomials of the first and second kinds and the monic type of Legendre polynomials, which have been presented in Section 2, are employed. Due to (2.4) and Remark 2.1, we have

$$S_{2,3}(y;x) = \begin{cases} \sum_{i=0}^{2} y_{i,1} L_i(x;x_{i,1}), & I_1 = [0, 0.089] \cup [0.910, 1], \\ \sum_{i=0}^{2} y_{i,2} L_i(x;x_{i,2}), & I_2 = [0.089, 0.129] \cup [0.870, 0.910], \\ \sum_{i=0}^{2} y_{i,3} L_i(x;x_{i,3}), & I_3 = [0.129, 0.870], \end{cases}$$

$$S_{3,3}(y;x) = \begin{cases} \sum_{i=0}^{3} y_{i,1} L_i(x;x_{i,1}), & I_1 = [0,0.058] \cup [0.250,0.321] \cup [0.678,0.750], \cup [0.941,1] \\ \sum_{i=0}^{3} y_{i,2} L_i(x;x_{i,2}), & I_2 = [0.058,0.078] \cup [0.321,0.334] \cup [0.665,0.678], \cup [0.921,0.941] \\ \sum_{i=0}^{3} y_{i,3} L_i(x;x_{i,3}), & I_3 = [0.078, 0.250] \cup [0.334, 0.665] \cup [0.750, 0.921], \end{cases}$$

and

$$S_{5,3}(y;x) = \begin{cases} \sum_{i=0}^{5} y_{i,1} L(x;x_{i,1}), & I_1 = [0,0.031] \cup [0.095,0.164] \cup [0.345,0.377] \cup [0.621,0.654] \cup [0.835,0.904] \cup [0.968,1], \\ \sum_{i=0}^{5} y_{i,2} L(x;x_{i,2}), & I_2 = [0.031,0.036] \cup [0.166,0.171] \cup [0.379,0.381] \cup [0.618,0.625] \cup [0.828,0.833] \cup [0.963,0.968], \\ \sum_{i=0}^{5} y_{i,3} L(x;x_{i,3}), & I_3 = [0.036,0.095] \cup [0.164,0.166] \cup [0.171,0.345] \cup [0.377,0.379] \cup [0.381,0.618]. \cup [0.620,0.621] \cup [0.654,0.828] \cup [0.833,0.835] \cup [0.904,0.963]. \end{cases}$$

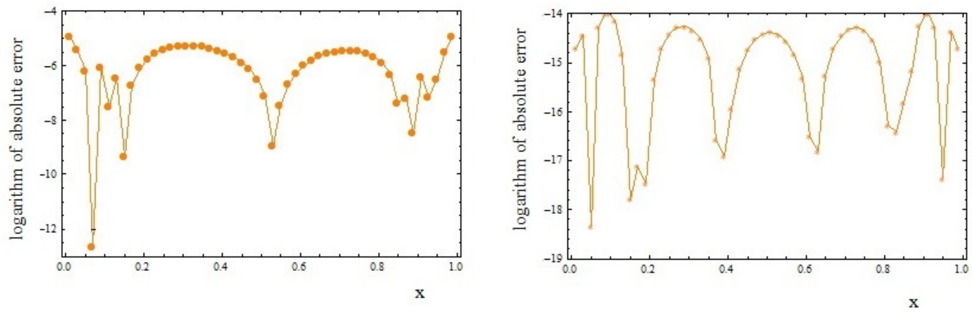
Using (2.3), we consider the collocation points in the following form

$$\{x_{j,l}\}_{j=0}^{n} = \{t \in [0,1] \mid \bar{q}_{n+1,l}(2t-1) = 0\}, \ l = 1, 2, 3.$$

Example 4.1. Consider the problem

$$y(x) = f(x) + \int_0^x e^t \cos(x) y(t) dt - \int_0^1 e^t \sin(x) y(t) dt, \tag{4.1}$$

FIGURE 1. The logarithm of absolute error of SSM for solving (4.1). • and ⋆ indicate the approximate solutions for n=2 and n=5, respectively.



where $f(x) = e^x - \frac{1}{2}\cos(x)(e^{2x}-1) + \frac{1}{2}\sin(x)(e^2-1)$. The exact solution is $y(x) = e^x$.

Table 1 shows the L_2-norm error of SSM, LCM and TCM. As observed, the obtained error for SSM is significantly less than those obtained for the other two methods. The logarithm of absolute error for SSM is shown in Figure 1.

TABLE 1. L_2-norm error using introduced method.

n	SSM	LCM	TCM
2	5.6846×10^{-4}	1.0589×10^{-2}	4.5148×10^{-2}
5	4.63035×10^{-8}	1.1211×10^{-6}	2.9791×10^{-4}

Example 4.2. Consider the problem

$$y(h(x)) = f(x) + \int_0^{h(x)} e^{x-t} y(t)dt - \int_0^1 e^{x+t} y(t)dt, \tag{4.2}$$

where $f(x) = h^2(x) - 4e^x + e^{x+1} + e^{x-h(x)}(h^2(x) + 2h(x) + 2)$. Obviously, $y(x) = x^2$ is the exact solution of this equation.

Since the exact solution is a quadratic polynomial, the interpolation error formula will be equal to zero, when we put $n = 2$ or $n = 3$. Thus, we can obtain the same numerical results as Table 2 in [10]. Tables 2, 3 and 4 show the L_2-norm error of SSM, LCM and TCM for $h(x) = x,\ x^2$ and $\sin(x)$, respectively. FIGURE 2 displays the logarithm of absolute error for SSM with $h(x) = x^2$.

TABLE 2. L_2-norm error using introduced method with $h(x) = x$.

n	SSM	LCM	TCM
2	1.2993×10^{-15}	1.5906×10^{-15}	1.5022×10^{-15}
3	1.1754×10^{-15}	5.7088×10^{-15}	1.6121×10^{-15}

Example 4.3. Consider the following integral equation

$$y(x) = f(x) + \int_0^{h(x)} e^{x+t} y(t)dt - \int_0^1 e^{x+h(t)} y(h(t))dt, \tag{4.3}$$

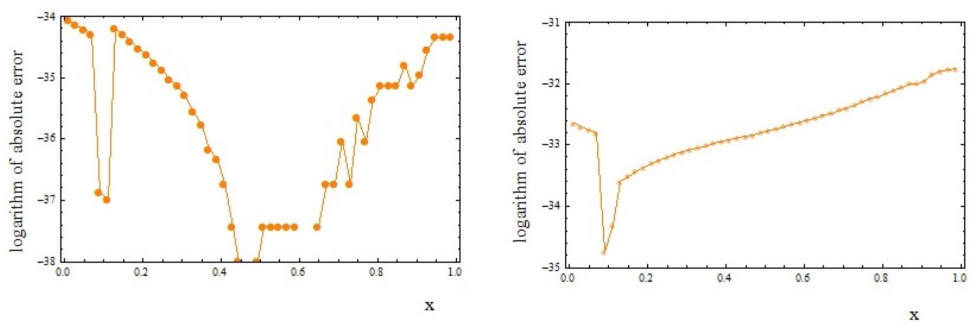
Table 3. L_2-norm error using introduced method with $h(x) = x^2$.

n	SSM	LCM	TCM
2	6.8823×10^{-16}	1.3986×10^{-15}	2.7804×10^{-15}
3	8.2785×10^{-15}	6.6122×10^{-15}	2.8216×10^{-15}

Table 4. L_2-norm error using introduced method with $h(x) = \sin(x)$.

n	SSM	LCM	TCM
2	1.1840×10^{-15}	2.7132×10^{-15}	1.5019×10^{-15}
3	8.4120×10^{-15}	1.4401×10^{-14}	1.6304×10^{-15}

Figure 2. The logarithm of absolute error of SSM for solving (4.2). For $h(x) = x^2$, $\bullet$ and $\star$ indicate the approximate solutions for $n = 2$ and $n = 5$, respectively.



where $f(x) = e^{-x} - e^{x}(h(x) - 1)$. The exact solution is $y(x) = e^{-x}$.

Tables 5, 6 and 7 show the L_2-norm error of SSM, LCM and TCM for $h(x) = x,\ \frac{x}{2}$ and $\ln(x + 1)$, respectively. As observed, the obtained error for SSM is much better than those obtained for the other two methods. In FIGURE 3, we draw the logarithm of absolute error for SSM with $h(x) = \ln(x + 1)$.

Table 5. L_2-norm error using introduced method with $h(x) = x$.

n	SSM	LCM	TCM
2	3.8993×10^{-4}	3.6864×10^{-3}	2.2354×10^{-2}
5	3.6560×10^{-8}	4.0319×10^{-7}	1.4135×10^{-4}

5. Conclusion

In this paper, using statistical spline model an approximate solution is derived to some Volterra-Fredholm integral equations. Statistical spline model is an algebraic method based on solving some inequalities of polynomial type to control the error value of interpolation formulas whose residue depends on a monic polynomial.

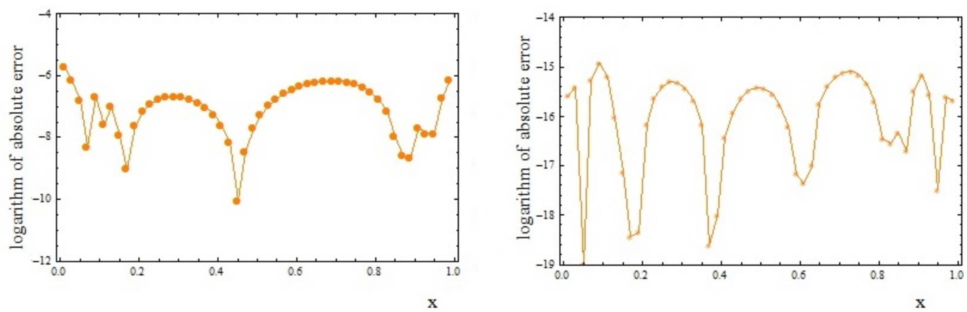
TABLE 6. L_2-norm error using introduced method with $h(x) = \frac{x}{2}$.

n	SSM	LCM	TCM
2	3.8600×10^{-4}	4.1529×10^{-3}	4.6564×10^{-2}
5	3.1415×10^{-8}	3.7054×10^{-7}	3.3616×10^{-4}

TABLE 7. L_2-norm error using introduced method with $h(x) = \ln(x+1)$.

n	SSM	LCM	TCM
2	3.1866×10^{-4}	3.2739×10^{-3}	3.5942×10^{-2}
5	2.2024×10^{-8}	4.3023×10^{-7}	3.0506×10^{-4}

FIGURE 3. The logarithm of absolute error of SSM for solving (4.3). For $h(x) = \ln(x+1)$, $\bullet$ and $\star$ indicate the approximate solutions for $n = 2$ and $n = 5$, respectively.



Hence, coupling the statistical spline model and collocation method cause an efficient numerical method to solve integral equations. Error analysis and numerical examples revealed the efficiency of the proposed method. The method can also be extended to solve integral equations of first kind or some inverse parabolic problems.

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