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Fractional type of flatlet oblique multiwavelet for solving fractional differential and integro-differential equations

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Abstract The construction of fractional type of flatlet biorthogonal multiwavelet system is investigated in this paper. We apply this system as basis functions to solve the fractional differential and integro-differential equations. Biorthogonality and high vanishing moments of this system are two major properties which lead to the good approximation for the solutions of the given problems. Some test problems are discussed at the end of paper to show the efficiency of the proposed method.

Keywords. Integro-differential equations, fractional type of flatlet oblique multiwavelets, biorthogonal flatlet multiwavelet system.

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1. Introduction

Wavelets and multiwavelets are the interesting object in mathematical sciences which have many applications in various fields such as image processing, signal denoising, physics and etc [5, 11, 15, 17]. For instance they can be used as the basis functions to numerical solution of boundary value problems [1, 2, 4, 7, 14, 18].
A Biorthogonal Multiwavelet System(BMS) contains a pair of biorthogonal multiscaling functions and the corresponding pair of multiwavelets. BMS plays an important role when the original multiwavelet is not orthogonal. See [6, 9] for more information about constructions and samples of BMS. We use the flatlet multiwavelet and will

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present an algebraic tool to extend it to a fractional BMS in this paper and the computation of the operational matrix of derivative.
This paper is organized as follows: In Section 2, the flatlet scaling functions and multiwavelets on $[0,1]$ is given. In Section 3, we construct the fractional biorthogonal basis for the flatlet multiwavelets. Next we derive the operational matrix of fractional derivative which is applicable for numerical solution of the given boundary value problem. In Section 4, the proposed method is used to solve some boundary value problems. In Section 5, our computational results are reported and we demonstrate the accuracy of the proposed numerical scheme by some test problems. The paper ends with a brief conclusion in Section 6.

2. Preliminaries and notation

2.1. Brief view on wavelets.

Definition 2.1. The inner product of two functions $f(x)$ and $g(x)$ on $[0,1]$ with respect to the given nonnegative weight function $w(x)$ is defined as

$$\langle u, v\rangle_w = \int_0^1 u(x)v(x)w(x)dx.$$

Definition 2.2. A multiresolution analysis (MRA) is an infinite nested sequence of subspaces of L^2

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots,$$

with the following properties

i. $\bigcup_n V_n = L^2$,
ii. $\bigcap_n V_n = \{0\}$,
iii. $f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$ for all $n \in \mathbb{Z}$,
iv. There exists a vector function $\Phi = [\phi_1, \phi_2, \cdots, \phi_r]^T, \phi_k \in L^2$ such that

$$V_0 = \operatorname{span}\{\phi_l(x-k) : l = 1, 2, \cdots, r;\ k \in \mathbb{Z}\}.$$

According to definition (2.2) we can represent $\Phi(x)$ in terms of $\Phi(2x-k)$. Also corresponding to the sequence of subspases V_j, we can find a sequence of subspaces W_j which are complements of V_j in V_{j+1}, i.e.

$$V_{j+1} = V_j \dotplus W_j.$$

Also there exists a vector function $\Psi = [\psi_1, \psi_2, \cdots, \psi_r]^T$ such that

$$W_0 = \operatorname{span}\{\psi_j(x-k), j = 1, 2, \cdots, r;\ k \in \mathbb{Z}\}.$$

Therefor according to the third property of definition (2.2) for the r vector functions $\Phi(x)$ and $\Psi(x)$, we have the so-called two-scale relation as follows

$$\Phi(x) = [\mathbf{P}_1, \mathbf{P}_2, \cdots \mathbf{P}_k][\Phi(2x), \Phi(2x-1), \cdots, \Phi(2x-k)]^T, \tag{2.1}$$

$$\Psi(x) = [\mathbf{Q}_1, \mathbf{Q}_2, \cdots \mathbf{Q}_k][\Phi(2x), \Phi(2x-1), \cdots, \Phi(2x-k)]^T. \tag{2.2}$$

If the multiscaling function Φ be such that $\langle \Phi, \Phi^T \rangle = I_{r\times r}$, then we say Φ is orthogonal. If we have not the orthogonality condition, we can find another multiscaling vector function $\tilde{\Phi}$ such that

$$\langle \Phi, \tilde{\Phi}^T \rangle = I_{r\times r}.$$

In this case two multiscaling functions Φ and $\tilde{\Phi}$ are called biorghogonal. Note that we need more conditions to have orthogonal or biorthogonal system. One can refer to [12] for more discussion about multiwavelets and biorthogonality condition.

2.2. **Flatlet Multiwavelet System.** A flatlet multiwavelet system (FMS) in general consists of $m+1$ scaling functions and $m+1$ wavelets defined on $[0,1]$, called multiscaling functions and multiwavelets, respectively. The simplest example for the FMS is identical to the well known Haar wavelets. We can follow the same procedures as Haar wavelets to construct higher order FMS. The scaling functions in this system are defined as follows

$$\phi_i(x) = \begin{cases} 1 & \frac{i}{m+1} \le x < \frac{i+1}{m+1}, \\ & \\ 0 & o.w. \end{cases} \quad , i = 0, 1, ..., m. \tag{2.1}$$

We consider $\psi_0(x), \cdots, \psi_m(x)$ as the flatlet wavelets corresponding to flatlet scaling functions. They are constructed by using the two-scale equation which will be introduced next, for multiwavelet system. For simplicity, we put flatlet scaling functions and wavelets into two vector functions

$$\Phi(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_i(x) \\ \vdots \\ \phi_m(x) \end{bmatrix}, \Psi(x) = \begin{bmatrix} \psi_0(x) \\ \vdots \\ \psi_i(x) \\ \vdots \\ \psi_m(x) \end{bmatrix}. \tag{2.2}$$

The two-scale relation for FMS expressed as

$$\Phi(x) = \mathbf{P} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \Psi(x) = \mathbf{Q} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \tag{2.3}$$

where **P** and **Q** are $(m+1)\times 2(m+1)$ matrices. An efficient method was described in [9] for computing **P** and **Q**. As an example for the first order flatlet basis functions $(m=1)$ we get

$$\phi_0(x) = \begin{cases} 1 & for \quad 0 \le x < \frac{1}{2}\,, \\ 0 & o.w. \end{cases}$$
$$\phi_1(x) = \begin{cases} 1 & for \quad \frac{1}{2} \le x < 1\,, \\ 0 & o.w. \end{cases} \tag{2.4}$$

In this case **Q** and **P** are computed as

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{2.5}$$

The multiscaling functions and multiwavelet in FMS are not mutually orthogonal. So we need to construct biorthogonal flatlet multiwavelet system (BFMS) in order to reduce the computation cost in our approximations which is described in next section.

2.3. **The fractional derivative in the Caputo sense.** Here we recall a few essential concepts of the fractional calculus. There are different definitions of fractional differentiation of positive real order α, which are not necessarily equivalent to each other [10, 16]. We use the Caputo fractional derivative, which allows the utilization of initial and boundary conditions involving integer order derivatives.

Definition 2.3. The fractional-order derivative in Caputo sense is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt, \qquad n-1<\alpha<n, \quad n \in \mathbb{N}, \tag{2.6}$$

where $\alpha > 0$ is the order of the derivative, $\Gamma(\cdot)$ is the Gamma function and $n = [\alpha]+1$.

Note that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the differential operator of integer order. Also the Caputo's fractional differentiation is a linear operator:

$$D^{\alpha}(\nu f(x) + \mu g(x)) = \nu D^{\alpha} f(x) + \mu D^{\alpha} g(x), \tag{2.7}$$

where ν and μ are constants. Also, for the Caputo's derivative we have [10],

$$D^{\gamma} C = 0, \quad (C \text{ is a constant}), \tag{2.8}$$

$$D^{\gamma}x^{\alpha}=\begin{cases}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\gamma)}x^{\alpha-\gamma}, & \begin{array}{c}\alpha\in\mathbb{N}_0 \text{ and } \alpha\geq\lceil\gamma\rceil\\ \text{or}\\ \alpha\notin\mathbb{N} \text{ and } \alpha>\lfloor\gamma\rfloor,\end{array}\\ \\ 0, & \alpha\in\mathbb{N}_0 \text{ and } \alpha<\lceil\gamma\rceil,\end{cases} \tag{2.9}$$

where $\mathbb{N}=\{1,2,...\}$ and $\mathbb{N}_0=\{0,1,2,...\}$. Note that according to (2.9) we have

$$D^{\alpha}x^{k\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma\big((k-1)\alpha+1\big)}x^{(k-1)\alpha}. \tag{2.10}$$

Therefore the following matrix relation holds

$$D^{\alpha}\mathbf{X}_{\alpha}=\mathbf{D}_1\mathbf{X}_{\alpha}, \tag{2.11}$$

where

$$\mathbf{D}_1=\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 & 0\\ 0 & \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} & 0 & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & 0 & \frac{\Gamma(\alpha+1)}{\Gamma\big((r-1)\alpha+1\big)} & 0\end{bmatrix},\ \mathbf{X}_{\alpha}=\begin{bmatrix}1\\ x^{\alpha}\\ \vdots\\ x^{r\alpha}\end{bmatrix}.$$

Definition 2.4. A fractional polynomial p of degree n and order α is defined by

$$p_{n,\alpha(x)}=a_0+a_1x^{\alpha}+\cdots+a_nx^{n\alpha},$$

where n is an arbitrary natural number and α is a strictly positive real number.

3. Biorthogonal FMS

Now we use the same method as [9] to construct the biorthogonal FMS(BFMS). Let $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$ be dual scaling and wavelet vector functions in BFMS, respectively as

$$\tilde{\Phi}(x)=\begin{bmatrix}\tilde{\phi}_0(x)\\ \vdots\\ \tilde{\phi}_i(x)\\ \vdots\\ \tilde{\phi}_m(x)\end{bmatrix},\tilde{\Psi}(x)=\begin{bmatrix}\tilde{\psi}_0(x)\\ \vdots\\ \tilde{\psi}_i(x)\\ \vdots\\ \tilde{\psi}_m(x)\end{bmatrix}. \tag{3.1}$$

We use the change of variable $t = x^\alpha$ in the original definition of dual functions in [9] for the given positive real number α and introduce $\tilde{\phi}_i(x)$ and $\tilde{\psi}_i(x)$, $i = 0, 1, \cdots, m$, as

$$\tilde{\phi}_i(x) = \begin{cases} a_{i1} + a_{i2}x^\alpha + ... + a_{i,m+1}x^{m\alpha} & 0 \le x < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

$$\tilde{\psi}_i(x) = \begin{cases} b^1_{i1} + b^1_{i2}x^\alpha + ... + b^1_{i,m+1}x^{m\alpha} & 0 \le x < \frac{1}{2}, \\ b^2_{i1} + b^2_{i2}x^\alpha + ... + b^2_{i,m+1}x^{m\alpha} & \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.3}$$

with the biorthogonality conditions as

$$\langle \tilde{\phi}_i, \phi_j \rangle_w = \int_0^1 \tilde{\phi}_i(x)\phi_j(x)w(x)dx = \delta_{i,j}, \tag{3.4}$$

$$\langle \tilde{\psi}_i, \psi_j \rangle_w = \int_0^1 \tilde{\psi}_i(x)\psi_j(x)w(x)dx = \delta_{i,j}, \tag{3.5}$$

$$\langle \tilde{\psi}_i, \phi_j \rangle_w = \int_0^1 \tilde{\psi}_i(x)\phi_j(x)w(x)dx = 0, \tag{3.6}$$

$$i, j = 0, 1, \cdots, m,$$

where

$$w(x) = x^{\alpha-1}. \tag{3.7}$$

By using biorthogonality conditions (3.4)-(3.6), we can obtain the unknown coefficients $a_{i,j}$, $b^1_{i,j}$ and $b^2_{i,j}$ for $i, j = 0, \cdots, m$.

Note that we can write

$$\tilde{\Phi}(x) = \mathbf{A}\mathbf{X}_\alpha, \tag{3.8}$$

where $\mathbf{A}$ is the matrix of unknown coefficients and $\mathbf{X}_\alpha$ is defined in (2.10).

Theorem 3.1. *The system determined in* (3.4)-(3.6) *has unique solution.*

Proof. By changing the variable $t = x^\alpha$ and same way as [9] the theorem can easily be proved. □

Remark 3.2. According to [9] we see $\tilde{\Phi}$ and $\tilde{\Psi}$ hold in the following two-scale relation

$$\tilde{\Phi}(x) = \tilde{\mathbf{P}} \begin{bmatrix} \tilde{\Phi}(2x) \\ \tilde{\Phi}(2x-1) \end{bmatrix}, \tilde{\Psi}(x) = \tilde{\mathbf{Q}} \begin{bmatrix} \tilde{\Phi}(2x) \\ \tilde{\Phi}(2x-1) \end{bmatrix}, \tag{3.9}$$

where $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ are $(m+1) \times 2(m+1)$ matrices.

3.1. **Function approximation.** We can approximate a given function $f(x)$ defined in [0,1] by the flatlet scaling functions as

$$f(x) \simeq \sum_{i=0}^{m} \sum_{k=0}^{2^J-1} c_{i,k}\phi_i(2^J x - k),$$

for some m and J, where

$$c_{i,k} = \int_0^1 f(x)\tilde{\phi}_i(2^J x - k)w(x)dx\,,$$
$$i = 0, 1, ..., m,\ k = 0, 1, ..., 2^J - 1, \tag{3.10}$$

and $w(x)$ is defined in (3.7). Also, we can approximate $f(x)$ by the flatlet or dual flatlet multiwavelets, respectively, as

$$f(x) \simeq \sum_{i=0}^{m} c'_i\phi_i(x) + \sum_{i=0}^{m}\sum_{l=0}^{J}\sum_{k=0}^{2^l-1} d_{i,l,k}\psi_i(2^l x - k), \tag{3.11}$$

$$f(x) \simeq \sum_{i=0}^{m} \tilde{c}'_i\tilde{\phi}_i(x) + \sum_{i=0}^{m}\sum_{l=0}^{J}\sum_{k=0}^{2^l-1} \tilde{d}_{i,l,k}\tilde{\psi}_i(2^l x - k), \tag{3.12}$$

where

$$c'_i = \int_0^1 f(x)\tilde{\phi}_i(x)w(x)dx,$$
$$d_{i,l,k} = \int_0^1 f(x)\tilde{\psi}_i(2^l x - k)w(x)dx,$$
$$\tilde{c}'_i = \int_0^1 f(x)\phi_i(x)w(x)dx,$$
$$\tilde{d}_{i,l,k} = \int_0^1 f(x)\psi_i(2^l x - k)w(x)dx,$$
$$i = 0, \ldots, m;\ l = 0, \ldots, J;\ k = 0, \ldots, 2^l - 1.$$

We can write the expressions (3.11) and (3.12) respectively in the following matrix forms

$$f(x) \simeq \Theta^T\mathbf{f}, \tag{3.13}$$
$$f(x) \simeq \tilde{\Theta}^T\tilde{\mathbf{f}}, \tag{3.14}$$

where

$$\Theta(x)=\begin{bmatrix}\phi_0(x)\\ \vdots\\ \phi_m(x)\\ \psi_0(x)\\ \vdots\\ \psi_i(2^l x-k)\\ \vdots\\ \psi_m(2^J x-2^J+1)\end{bmatrix},\tilde{\Theta}(x)=\begin{bmatrix}\tilde{\phi}_0(x)\\ \vdots\\ \tilde{\phi}_m(x)\\ \tilde{\psi}_0(x)\\ \vdots\\ \tilde{\psi}_i(2^l x-k)\\ \vdots\\ \tilde{\psi}_m(2^J x-2^J+1)\end{bmatrix}, \tag{3.15}$$

and

$$\mathbf{f}=\left[c'_0,\cdots,c'_m,d_{0,0,0},\cdots,d_{i,l,k},\cdots,d_{m,J,2^J-1}\right]^T,$$
$$\tilde{\mathbf{f}}=\left[\tilde{c}'_0,\cdots,\tilde{c}'_m,\tilde{d}_{0,0,0},\cdots,\tilde{d}_{i,l,k},\cdots,\tilde{d}_{m,J,2^J-1}\right]^T.$$

Also using (3.9), $\tilde{\Theta}$ can expressed as

$$\tilde{\Theta}(x)=\mathbf{Q}'\Pi(x), \tag{3.16}$$

where

$$\mathbf{Q}'=\begin{bmatrix}\mathbf{I} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\ 0 & \tilde{\mathbf{Q}} & \cdots & 0 & \cdots & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \tilde{\mathbf{Q}} & \cdots & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & \tilde{\mathbf{Q}} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & 0\\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & \tilde{\mathbf{Q}}\end{bmatrix},$$

and

$$\Pi(x) = \begin{bmatrix} \tilde{\Phi}(x) \\ \tilde{\Phi}(2x) \\ \tilde{\Phi}(2x-1) \\ \vdots \\ \tilde{\Phi}(2^i x) \\ \tilde{\Phi}(2^i x - 1) \\ \vdots \\ \tilde{\Phi}(2^{i+1}x - 2^{i+1} + 2) \\ \tilde{\Phi}(2^{i+1}x - 2^{i+1} + 1) \\ \vdots \\ \tilde{\Phi}(2^{J+1}x - 2^{J+1} + 2) \\ \tilde{\Phi}(2^{J+1}x - 2^{J+1} + 1) \end{bmatrix},$$

$\mathbf{I}$ is the identity matrix with $m+1$ rows and columns and $\tilde{\mathbf{Q}}$ is defined in (3.9). Note that this matrix is a diagonal block matrix with $2^{J+1}-1$ nonzero elements. Using (3.16), we can rewrite (3.14) as

$$f(x) \simeq \Pi^T \cdot \mathbf{Q}'^T \cdot \mathbf{f}. \tag{3.17}$$

3.2. **The Operational Matrix of Fractional Derivative.** For simplicity and in order to reduce the computation cost we have to express the expansion of $D^\alpha f(x)$ in terms of the expansion of $f(x)$ which can be done by using the operational matrix of fractional derivative (OMFD). By using (3.14) let

$$D^\alpha f(x) \simeq \tilde{\Theta}^T \dot{\mathbf{f}}. \tag{3.18}$$

The OMFD $\mathbf{D}^\alpha$ connects two vectors $\mathbf{f}$ and $\dot{\mathbf{f}}$ by

$$\dot{\mathbf{f}} = \mathbf{D}^\alpha \mathbf{f}. \tag{3.19}$$

So OMFD helps us to express the coefficients of expansion $D^\alpha f(x)$ in terms of the coefficients of expansion $f(x)$. Now we express our method to determine the entire elements of OMFD. First from (2.11) and (3.8) we have

$$D^\alpha \Phi(x) = \mathbf{A} \cdot \mathbf{D}_1 \cdot \mathbf{X} \cdot \chi_{[0,1]}(x).$$

so

$$\begin{aligned} D^\alpha \Phi(x) &= \mathbf{A} \cdot \mathbf{D}_1 \cdot \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{X} \cdot \chi_{[0,1]}(x) \\ &= \mathbf{A} \cdot \mathbf{D}_1 \cdot \mathbf{A}^{-1} \cdot \Phi(x). \end{aligned}$$

Hence

$$D^\alpha \Phi(x) = \Delta_1 \cdot \Phi(x),$$

where $\Delta_1 = \mathbf{A} \cdot \mathbf{D}_1 \cdot \mathbf{A}^{-1}$. Applying this method to calculate derivative of each element of $\Pi(x)$ yields

$$D^\alpha \Phi(2^i x - l) = 2^{i\alpha} \cdot \Delta_1 \cdot \Phi(2^i x - l),$$
$$i = 0, 1, \cdots, J;\ l = 0, 1, \cdots, 2^i - 1,$$

therefore

$$D^\alpha \Pi(x) = \Delta \cdot \Pi(x), \tag{3.20}$$

where Δ is a block matrix with entries

$$\Delta_{i,j} = \begin{cases} 2^{[\log_2 i]\alpha} \Delta_1 & i = j \\ 0_{(m+1)\times(m+1)} & i \neq j \end{cases}, \quad i, j = 1, 2, \cdots, 2^{J+1} - 1.$$

Now, using (3.17) yields

$$D^\alpha f(x) \simeq \Pi^T \cdot \Delta^T \cdot \mathbf{Q}'^T \cdot \mathbf{f}, \tag{3.21}$$

Also the given function $g(x, y)$ could be approximated as

$$g(x, y) \simeq \tilde{\Theta}^T(y)\mathbf{G}\Theta(x), \tag{3.22}$$

where

$$[\mathbf{G}]_{i,j} = \int_0^1 \int_0^1 g(x, y)\tilde{\theta}_i(x)\theta_j(y)x^{\alpha-1}y^{\alpha-1}dxdy\ ,\ i, j = 1, 2, \cdots, N.$$

Here we use (3.14) for approximation of the functions because these basis functions are in terms of piecewise fractional polynomials of degree m. Hence it has higher order approximations than (3.13).

Remark 3.3. We use Gauss-Legendre quadrature for computation the coefficients of given functions.

4. Solving the integro-differential equations

In this section we solve the fractional integro-differential equations of the general form

$$D^\alpha u(x) = a(x)u(x) + g(x) + \int_0^1 k(x, t)F(u(t))dt, \tag{4.1}$$

where a, g, k and F are given functions. We first, approximate the given functions as

$$u(x) = \mathbf{u}^T \tilde{\Theta}(x),$$
$$k(x,t) = \Theta^T(t)\mathbf{K}\tilde{\Theta}(x),$$
$$F(u(t)) = \mathbf{f}^T \tilde{\Theta}(x), \tag{4.2}$$

where $\mathbf{u}$ and $\mathbf{f}$ are unknown vectors and $\mathbf{K}$ is $N \times N$ matrix. We start approximating the solution of (4.1) by using (3.21) and substituting (4.2) into (4.1) respectively, which results

$$\mathbf{u}^T \mathbf{Q}' \Delta\Pi(x) = a(x)\mathbf{u}^T \tilde{\Theta}(x) + g(x) + \int_0^1 \mathbf{f}^T \tilde{\Theta}(t)\Theta^T(t)\mathbf{K}\tilde{\Theta}(x)dt, \tag{4.3}$$

$$\mathbf{u}^T \tilde{\Theta}(0) = y_0, \tag{4.4}$$
$$\mathbf{u}^T \tilde{\Theta}(1) = y_1. \tag{4.5}$$

The biorthogonality condition yields

$$\mathbf{u}^T \mathbf{Q}' \Delta\Pi(x) = a(x)\mathbf{u}^T \tilde{\Theta}(x) + g(x) + \mathbf{f}^T \mathbf{K}\tilde{\Theta}(x), \tag{4.6}$$

which is an algebraic equation with $2N$ unknown coefficients. To determine these unknowns, we collocate (4.6) in $N-2$ evenly spaced nodes in $[0,1]$ as bellow

$$\mathbf{u}^T \mathbf{Q}' \Delta\Pi(x_i) = a(x_i)\mathbf{u}^T \tilde{\Theta}(x_i) + g(x_i) + \mathbf{f}^T \mathbf{K}\tilde{\Theta}(x_i), \tag{4.7}$$
$$x_i = \frac{i}{N}, \; i = 2, 3, \cdots, N-1.$$

The above equations together with two equations (4.4) and (4.5), constitutes a system with N linear equations. We need N equations to form a system of $2N$ equations with $2N$ unknowns. To this end we use the following relation

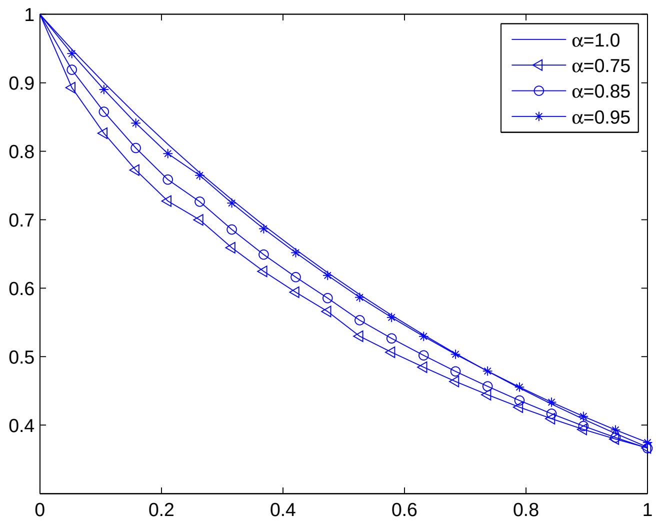
$$f\big(x, \mathbf{u}^T \tilde{\Theta}(x)\big) = \mathbf{f}^T \tilde{\Theta}(x), \tag{4.8}$$

next collocate it in N evenly spaced nodes in $[0,1]$. Hence we reach to a system with N linear and N nonlinear equations. By using the Newton iteration method, we can solve this system and obtain the unknown coefficients. In the next section some numerical examples are presented.

5. Numerical Experiments

To show the efficiency of our method, we apply it to some test problems.

FIGURE 1. Approximate solution of the unknown function in example 1, for $m = 3, J = 2$ and different values of α



Example 1. Consider the following problem,

$$D^{\alpha}y(x) = -y(x) + \int_0^1 y(t)dt + e^{-1} - 1, \;\; x \in [0, 1],$$
$$y(0) = 1. \tag{5.1}$$

When $\alpha = 1$, this problem has the exact solution $y(x) = e^{-x}$. Table 1 shows the absolute values of error in some points for $\alpha = 1$ comparing with [3, 13]. In Figure 1, we see the approximate solution tends to the exact solution where α tends to 1.

Example 2. The nonlinear problem

$$D^{\alpha}y(x) + (1 + x)y(x) = f(x) + 4\int_0^1 xty(t)^2dt, \quad x \in [0, 1]$$
$$y(0) = 1, \tag{5.2}$$

where $f(x) = -xe^{-x} - x + 3xe^{-2}$, for $\alpha = 1$ has the exact solution $y(x) = e^{-x}$. The absolute values of error for $\alpha = 1$ are shown in Table 2. Figure 2 shows approximate curve for some different values of α.

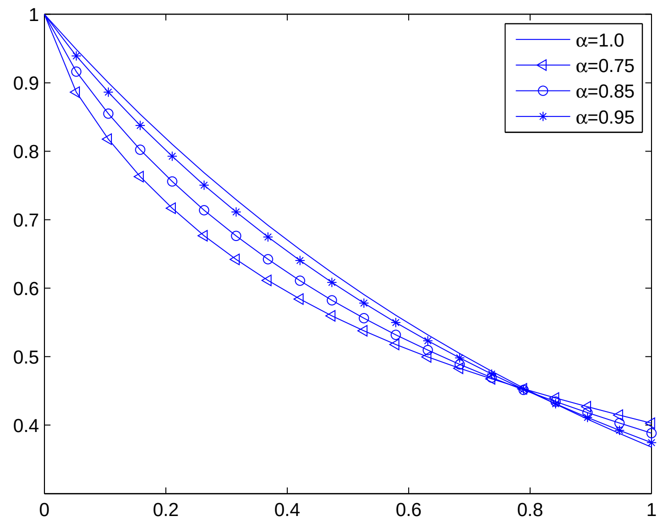
Table 1. Absolute values of error for Example 1

t_i	B-spline wavelet [13]	m=5,J=2	m=6,J=2
0.0	0.0	0.0	0.0
0.1	3.1×10^{-6}	1.9×10^{-8}	1.5×10^{-10}
0.2	2.7×10^{-6}	3.1×10^{-7}	2.5×10^{-10}
0.3	2.5×10^{-6}	3.7×10^{-7}	3.8×10^{-10}
0.4	2.5×10^{-6}	4.8×10^{-7}	4.6×10^{-10}
0.5	5.3×10^{-7}	2.9×10^{-7}	2.4×10^{-9}
0.6	2.1×10^{-6}	2.7×10^{-7}	2.2×10^{-9}
0.7	1.7×10^{-6}	2.6×10^{-7}	2.1×10^{-9}
0.8	1.5×10^{-6}	2.5×10^{-7}	2.0×10^{-9}
0.9	1.4×10^{-6}	2.4×10^{-7}	2.0×10^{-9}
1.0	0.0	4.3×10^{-7}	9.0×10^{-9}

Table 2. Absolute values of error

t_i	m=3, J=2	m=5, J=3	m=6, J=2
0.0	0.0	0.0	0.0
0.1	2.3×10^{-6}	2.8×10^{-8}	2.6×10^{-10}
0.2	4.7×10^{-6}	5.5×10^{-8}	4.9×10^{-10}
0.3	1.0×10^{-5}	7.3×10^{-8}	7.1×10^{-10}
0.4	8.1×10^{-6}	9.5×10^{-8}	8.9×10^{-10}
0.5	1.2×10^{-4}	1.3×10^{-7}	1.8×10^{-10}
0.6	6.9×10^{-5}	1.3×10^{-7}	3.4×10^{-10}
0.7	5.8×10^{-5}	8.9×10^{-8}	1.1×10^{-10}
0.8	5.8×10^{-5}	5.7×10^{-8}	7.1×10^{-11}
0.9	4.4×10^{-7}	2.6×10^{-8}	2.1×10^{-10}
1.000	0.0	0.0	0.0

FIGURE 2. Approximate solution of the unknown function in example 2, for $m = 3, J = 2$ and different values of α



Example 3. The following problem

$$D^{\frac{1}{2}}y(x) = \frac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xty(t)dt,$$
$$y(0) = 0,$$

has the exact solution $y(x) = x^2 - x$. We reach to the exact solution by taking $m \geq 4$ and arbitrary value for J.

6. Conclusion

The fractional type of biorthogonal flatlet multiwavelet system is constructed in this paper. We solve a second order integro-differential equation by employing the presented mathematical method. Some test problems are presented to show efficiency of proposed method.

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