

Generalized B-spline functions method for solving optimal control problems

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Abstract

In this paper we introduce a numerical approach that solves optimal control problems (OCPs) using collocation methods. This approach is based upon B-spline functions. The derivative matrices between any two families of B-spline functions are utilized to reduce the solution of OCPs to the solution of nonlinear optimization problems. Numerical experiments confirm our theoretical findings.

Keywords. Optimal control problem, B-spline functions, Derivative matrix, Collocation method.

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1. INTRODUCTION

Splines are ubiquitous in science and engineering. Sometimes they play a leading role as generators of paths or curves, but often they are hidden inside, for example, software packages for solving dynamic equations, in graphics, and in numerous other applications.

The rapid development of spline functions is due primarily to their great usefulness in applications. Classes of spline functions possess many good structural properties as well as excellent approximation powers.

It appears that the most turbulent years in the development of splines are over, and it is now generally agreed that they will become a firmly entrenched part of approximation theory and numerical analysis [21].

Solving an OCP is not easy. Because of the complexity of most applications, OCPs are most often solved numerically. Numerical methods for solving OCPs date back nearly five decades to the 1950s with the work of Bellman [1–3].

Over the years, various numerical methods have been proposed to solve OCPs. Authors of [18] presented the generalized gradient method to solve this problem. In

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[23] Chebyshev technique is used for the expansion of the state and control variables. A Fourier-based state parametrization approach for solving linear quadratic optimal control problems was developed in [24]. Razzaghi and Elnagar [20] proposed a method to solve the unconstrained linear-quadratic optimal control problem with the same number of state and control variables. Their approach is based on using the shifted Legendre polynomials to parameterize the derivative of each state variables. Also various types of the hybrid techniques with different polynomials are used to solve OCPs [8, 15, 17].

The approach in the current paper is different. A numerical technique based on collocation method is proposed for the solution of OCPs. We present a computational method for solving nonlinear constrained quadratic OCPs by using B-spline functions. These functions introduced first time by Curry and Schoenberg in [4].

The method is based on approximating the state variables and the control variables with a semiorthogonal linear B-spline functions [13]. Our method consists of reducing the OCP to a NLP by first expanding the state variable $x(t)$ and the control $u(t)$ as a B-spline functions with unknown coefficients. The main difference of this paper with our previous work [6] is using original derivative of B-spline functions. On the other hand we expand the state variable $x(t)$ as a B-spline functions with unknown coefficients instead of expanding the state rate $\dot{x}(t)$, and then we get $\dot{x}(t)$ using the derivative matrix $\mathbf{D}_{i,J}$. This matrix establishes the relationship between two B-spline functions with different orders.

In Section 2 we present some preliminaries in B-spline functions, and then describe the relationship between their derivatives required for our subsequent development. Section 3 is devoted to the formulation of OCPs. In Section 4, we apply the proposed method to OCPs, and in Section 5, we report our numerical Examples and demonstrate the accuracy of this method. We complete this paper with a brief conclusion.

2. REVIEW OF B-SPLINE FUNCTIONS

We use B-spline piecewise polynomials as basis functions for our numerical method. For this purpose, we first briefly review the properties of B-spline functions. The m th-order B-spline $N_m(t)$ has the knot sequence $\{\dots, -1, 0, 1, \dots\}$ and consists of polynomials of order m (degree $m - 1$) between the knots. Let $N_1(t) = \chi_{[0,1]}(t)$ be the characteristic function of $[0,1]$. Then for each integer $m \geq 2$, the m th-order B-spline is defined, inductively by [9]

$$\begin{aligned} N_m(t) &= (N_{m-1} * N_1)(t) = \int_{-\infty}^{\infty} N_{m-1}(t - \tau) N_1(\tau) d\tau \\ &= \int_0^1 N_{m-1}(t - \tau) d\tau. \end{aligned} \quad (2.1)$$

It can be shown [5] that $N_m(t)$ for $m \geq 2$ can be achieved using the following formula

$$N_m(t) = \frac{t}{m-1} N_{m-1}(t) + \frac{m-t}{m-1} N_{m-1}(t-1),$$

recursively, and $\text{supp}[N_m(t)] = [0, m]$.



The explicit expressions of $N_2(t)$ (linear B-spline function) and $N_3(t)$ (quadratic B-spline function) are [5, 9]:

$$N_2(t) = \begin{cases} t, & t \in [0, 1], \\ 2 - t, & t \in [1, 2], \\ 0, & \text{elsewhere}, \end{cases} \quad (2.2)$$

$$N_3(t) = \frac{1}{2} \begin{cases} t^2, & t \in [0, 1], \\ -2t^2 + 6t - 3, & t \in [1, 2], \\ t^2 - 6t + 9, & t \in [2, 3], \\ 0, & \text{elsewhere}. \end{cases} \quad (2.3)$$

From [13] we let $N_{i,j,k}(t) = N_i(2^j t - k)$, $i = 1, 2, 3$, $j, k \in \mathbb{Z}$ and $B_{i,j,k} = \text{supp}[N_{i,j,k}(t)] = \text{clos}\{t : N_{i,j,k}(t) \neq 0\}$. It is easy to see that

$$B_{i,j,k} = [2^{-j}k, 2^{-j}(i+k)], \quad i = 1, 2, 3, \quad j, k \in \mathbb{Z}.$$

Define the set of indices

$$S_{i,j} = \{k : B_{i,j,k} \cap (0, 1) \neq \emptyset\}, \quad i = 1, 2, 3, \quad j \in \mathbb{Z}.$$

Suppose $\mathfrak{m}_{i,j} = \min\{S_{i,j}\}$ and $\mathfrak{M}_{i,j} = \max\{S_{i,j}\}$, $i = 1, 2, 3$, $j \in \mathbb{Z}$. It is easy to see that $\mathfrak{m}_{1,j} = 0$, $\mathfrak{m}_{2,j} = -1$, $\mathfrak{m}_{3,j} = -2$, and $\mathfrak{M}_{1,j} = \mathfrak{M}_{2,j} = \mathfrak{M}_{3,j} = 2^j - 1$, $j \in \mathbb{Z}$.

The support of $N_{i,j,k}(t)$ may be out of interval $[0, 1]$, we need that these functions intrinsically defined on $[0, 1]$ so we put

$$N_{j,k}^i(t) = N_{i,j,k}(t)\chi_{[0,1]}(t), \quad i = 1, 2, 3, \quad j \in \mathbb{Z}, \quad k \in S_{i,j}. \quad (2.4)$$

2.1. The function approximation. Suppose

$$\Phi_{i,j}(t) = [N_{j,\mathfrak{m}_{i,j}}^i(t), N_{j,\mathfrak{m}_{i,j}+1}^i(t), \dots, N_{j,\mathfrak{M}_{i,j}}^i(t)]^T, \quad i = 1, 2, 3, \quad j \in \mathbb{Z}. \quad (2.5)$$

For a fixed $j = M$, a function $f(t) \in L^2[0, 1]$ may be represented by the B-spline functions as [13]

$$f(t) \simeq \sum_{k=-2}^{2^M-1} s_k N_{M,k}^3(t) = S^T \Phi_{3,M}(t), \quad (2.6)$$

where

$$S = [s_{-2}, s_{-1}, \dots, s_{2^M-1}]^T, \quad (2.7)$$

with

$$s_k = \int_0^1 f(t) \tilde{N}_{M,k}^3(t) dx, \quad k = -2, \dots, 2^M - 1,$$

where $\tilde{N}_{M,k}^3(t)$ are dual functions of $N_{M,k}^3(t)$ [12]. In similar to [6] by using (2.3), (2.4) and (2.6) we have

$$\int_0^1 \Phi_{3,M}(t) \Phi_{3,M}^T(t) dt = \mathbf{P}_M, \quad (2.8)$$



where $\mathbf{P}_M$ is a symmetric $(2^M + 2) \times (2^M + 2)$ matrix as follows

$$\mathbf{P}_M = 2^{-(M+1)} \begin{bmatrix} \frac{1}{10} & \frac{13}{60} & \frac{1}{60} & & & & \\ \frac{13}{60} & 1 & \frac{13}{30} & \frac{1}{60} & & & \\ \frac{1}{60} & \frac{13}{30} & \frac{11}{10} & \frac{13}{30} & \frac{1}{60} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \frac{1}{60} & \frac{13}{30} & \frac{11}{10} & \frac{13}{30} & \frac{1}{60} \\ & & & \frac{1}{60} & \frac{13}{30} & 1 & \frac{13}{60} \\ & & & & \frac{1}{60} & \frac{13}{60} & \frac{1}{10} \end{bmatrix}. \quad (2.9)$$

2.2. The derivative matrix. The differentiation of the vector $\Phi_{3,M}(t)$ can be expressed as [11]

$$\Phi'_{3,M}(t) = \mathbf{D}_M \Phi_{2,M}(t), \quad (2.10)$$

where

$$\mathbf{D}_M = 2^M \begin{bmatrix} -1 & & & & & \\ 1 & -1 & & & & \\ & \ddots & \ddots & & & \\ & & & 1 & -1 & \\ & & & & & 1 \end{bmatrix},$$

and $\mathbf{D}_M$ is a $(2^M + 2) \times (2^M + 1)$ matrix.

3. PROBLEM STATEMENT

Find the optimal control $\mathbf{u}^*(t)$, and the corresponding optimal state $\mathbf{x}^*(t)$, which minimize the performance index

$$J = \frac{1}{2} \mathbf{x}^T(t_f) \mathbf{Z} \mathbf{x}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (\mathbf{x}^T(t) \mathbf{Q}(t) \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{R}(t) \mathbf{u}(t)) dt, \quad (3.1)$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad (3.2)$$

$$\Psi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) = 0, \quad (3.3)$$

$$\mathbf{g}_i(\mathbf{x}(t), \mathbf{u}(t), t) \leq 0, \quad i = 1, 2, \dots, w, \quad (3.4)$$

where $\mathbf{Z}$ and $\mathbf{Q}(t)$ are positive semidefinite matrices, $\mathbf{R}(t)$ is a positive definite matrix, t_0 and t_f are known initial and terminal time respectively, $\mathbf{x}(t) \in \mathbb{R}^l$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^q$ is the control vector, and $\mathbf{f}$ and $\mathbf{g}_i$, $i = 1, 2, \dots, w$, are nonlinear functions. This problem is defined on the time interval $[t_0, t_f]$. Certain numerical techniques



(like B-spline functions) require a fixed time interval, such as $[0, 1]$. The independent variable can be mapped to the interval $[0, 1]$ via the affine transformation

$$\tau = \frac{t - t_0}{t_f - t_0}. \quad (3.5)$$

Using Eq. (3.5), this problem can be redefined as follows. Minimize the performance index

$$J = \frac{1}{2} \mathbf{x}^T(1) \mathbf{Z} \mathbf{x}(1) + \frac{1}{2} (t_f - t_0) \int_0^1 (\mathbf{x}^T(\tau) \mathbf{Q}(\tau) \mathbf{x}(\tau) + \mathbf{u}^T(\tau) \mathbf{R}(\tau) \mathbf{u}(\tau)) d\tau, \quad (3.6)$$

subject to the constraints

$$\frac{d\mathbf{x}}{d\tau} = (t_f - t_0) \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau), \quad (3.7)$$

$$\Psi(\mathbf{x}(0), \mathbf{x}(1)) = 0, \quad (3.8)$$

$$\mathbf{g}_i(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) \leq 0, \quad i = 1, 2, \dots, w, \quad \tau \in [0, 1]. \quad (3.9)$$

4. THE PROPOSED METHOD

Consider the following assumptions: let

$$\mathbf{x}(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_l(t)]^T, \quad (4.1)$$

$$\dot{\mathbf{x}}(t) = [\dot{\mathbf{x}}_1(t), \dot{\mathbf{x}}_2(t), \dots, \dot{\mathbf{x}}_l(t)]^T, \quad (4.2)$$

$$\mathbf{u}(t) = [\mathbf{u}_1(t), \mathbf{u}_2(t), \dots, \mathbf{u}_q(t)]^T, \quad (4.3)$$

$$\hat{\Phi}_1(t) = I_l \otimes \Phi_{3,M}(t), \quad (4.4)$$

$$\hat{\Phi}_2(t) = I_l \otimes \mathbf{D}_M \Phi_{2,M}(t), \quad (4.5)$$

$$\hat{\Phi}_3(t) = I_q \otimes \Phi_{3,M}(t), \quad (4.6)$$

where I_l and I_q are $l \times l$ and $q \times q$ dimensional identity matrices, $\Phi_{3,M}(t)$ is $(2^M + 2) \times 1$ vector, $\otimes$ denotes Kronecker product [14], $\hat{\Phi}_1(t)$ and $\hat{\Phi}_2(t)$ are matrices of order $l(2^M + 2) \times l$ and $\hat{\Phi}_3(t)$ is a matrix of order $q(2^M + 2) \times q$. Assume that each $\mathbf{x}_i(t)$ and each $\mathbf{u}_j(t)$, $i = 1, 2, \dots, l$, $j = 1, 2, \dots, q$, can be written in terms of B-spline functions as

$$\mathbf{x}_i(t) \simeq \Phi_{3,M}^T(t) \mathbf{X}_i,$$

$$\dot{\mathbf{x}}_i(t) \simeq \Phi_{2,M}^T(t) \mathbf{D}_M^T \mathbf{X}_i,$$

$$\mathbf{u}_j(t) \simeq \Phi_{3,M}^T(t) \mathbf{U}_j.$$

Then using Eqs. (4.4), (4.5) and (4.6) we have

$$\mathbf{x}(t) \simeq \hat{\Phi}_1^T(t) \mathbf{X}, \quad (4.7)$$

$$\dot{\mathbf{x}}(t) \simeq \hat{\Phi}_2^T(t) \mathbf{X}, \quad (4.8)$$

$$\mathbf{u}(t) \simeq \hat{\Phi}_3^T(t) \mathbf{U}, \quad (4.9)$$



where $\mathbf{X}$ and $\mathbf{U}$ are vectors of order $l(2^M + 2) \times 1$ and $q(2^M + 2) \times 1$, respectively, given by

$$\begin{aligned}\mathbf{X} &= [\mathbf{X}_1^T, \mathbf{X}_2^T, \dots, \mathbf{X}_l^T]^T, \\ \mathbf{U} &= [\mathbf{U}_1^T, \mathbf{U}_2^T, \dots, \mathbf{U}_q^T]^T.\end{aligned}$$

4.1. The performance index approximation. We have replaced Eqs. (4.7)-(4.9) in Eq. (3.6) and get

$$\begin{aligned}J &= \frac{1}{2} \mathbf{X}^T \hat{\Phi}_1(1) \mathbf{Z} \hat{\Phi}_1^T(1) \mathbf{X} \\ &\quad + \frac{1}{2} (t_f - t_0) \mathbf{X}^T \left(\int_0^1 \hat{\Phi}_1(t) \mathbf{Q}(t) \hat{\Phi}_1^T(t) dt \right) \mathbf{X} \\ &\quad + \frac{1}{2} (t_f - t_0) \mathbf{U}^T \left(\int_0^1 \hat{\Phi}_3(t) \mathbf{R}(t) \hat{\Phi}_3^T(t) dt \right) \mathbf{U},\end{aligned}\tag{4.10}$$

this means that

$$\begin{aligned}J &= \frac{1}{2} \mathbf{X}^T (\mathbf{Z} \otimes \Phi_{3,M}(1) \Phi_{3,M}^T(1)) \mathbf{X} \\ &\quad + \frac{1}{2} (t_f - t_0) \mathbf{X}^T \left(\int_0^1 \mathbf{Q}(t) \otimes \Phi_{3,M}(t) \Phi_{3,M}^T(t) dt \right) \mathbf{X} \\ &\quad + \frac{1}{2} (t_f - t_0) \mathbf{U}^T \left(\int_0^1 \mathbf{R}(t) \otimes \Phi_{3,M}(t) \Phi_{3,M}^T(t) dt \right) \mathbf{U}.\end{aligned}\tag{4.11}$$

For problems with time-varying performance index, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are functions of time and

$$\int_0^1 \mathbf{Q}(t) \otimes \Phi_{3,M}(t) \Phi_{3,M}^T(t) dt, \quad \int_0^1 \mathbf{R}(t) \otimes \Phi_{3,M}(t) \Phi_{3,M}^T(t) dt,$$

can be evaluated numerically. For time-invariant problems, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are constant matrices and can be brought out from the integrals. In this case Eq. (4.11) can be rewritten as

$$\begin{aligned}J(\mathbf{X}, \mathbf{U}) &= \frac{1}{2} \mathbf{X}^T (\mathbf{Z} \otimes \Phi_{3,M}(1) \Phi_{3,M}^T(1)) \mathbf{X} \\ &\quad + \frac{1}{2} (t_f - t_0) \mathbf{X}^T (\mathbf{Q} \otimes \mathbf{P}_M) \mathbf{X} \\ &\quad + \frac{1}{2} (t_f - t_0) \mathbf{U}^T (\mathbf{R} \otimes \mathbf{P}_M) \mathbf{U},\end{aligned}\tag{4.12}$$

where

$$\mathbf{P}_M = \int_0^1 \Phi_{3,M}(t) \Phi_{3,M}^T(t) dt,$$

that this matrix is obtained in Eq. (2.9).



4.2. The system constraints approximation. The system constraints are approximated with our method as follows:

By substituting Eqs. (4.7)-(4.9) in the system constraints (3.7)-(3.9) we get

$$\widehat{\Phi}_2^T(t)\mathbf{X} = (t_f - t_0)\mathbf{f}(\widehat{\Phi}_1^T(t)\mathbf{X}, \widehat{\Phi}_3^T(t)\mathbf{U}, t), \quad (4.13)$$

$$\Psi(\widehat{\Phi}_1^T(0)\mathbf{X}, \widehat{\Phi}_1^T(1)\mathbf{X}) = 0, \quad (4.14)$$

$$\mathbf{g}_i(\widehat{\Phi}_1^T(t)\mathbf{X}, \widehat{\Phi}_3^T(t)\mathbf{U}, t) \leq 0, \quad i = 1, 2, \dots, w. \quad (4.15)$$

We collocate Eqs. (4.13) and (4.15) at Newton-cotes nodes t_k ,

$$t_k = \frac{k-1}{2^M+1}, \quad k = 1, 2, \dots, 2^M + 2. \quad (4.16)$$

The OCP has now been reduced to a parameter optimization problem which can be stated as follows. Find $\mathbf{X}$ and $\mathbf{U}$ so that $J(\mathbf{X}, \mathbf{U})$ is minimized (or maximized) subject to Eq. (4.14) and

$$\widehat{\Phi}_2^T(t_k)\mathbf{X} = (t_f - t_0)\mathbf{f}(\widehat{\Phi}_1^T(t_k)\mathbf{X}, \widehat{\Phi}_3^T(t_k)\mathbf{U}, t_k), \quad (4.17)$$

$$\mathbf{g}_i(\widehat{\Phi}_1^T(t_k)\mathbf{X}, \widehat{\Phi}_3^T(t_k)\mathbf{U}, t_k) \leq 0, \quad i = 1, 2, \dots, w, \quad k = 1, 2, \dots, 2^M + 2. \quad (4.18)$$

5. ILLUSTRATIVE EXAMPLES

In this section we give some computational results of numerical experiments with methods based on two preceding sections, to support our theoretical discussion. All problems were programmed in MAPLE, running on a Pentium 4, 2.4-GHz PC with 4 GB of RAM. Also we solved the obtained NLP that is minimize (or maximize) $J(\mathbf{X}, \mathbf{U})$ subject to Eqs. (4.14), (4.17) and (4.18), using "NLPsolve" command in MAPLE software.

Example 1. Consider the problem [7, 25]

$$\min J = \int_0^1 (x^2(t) + u^2(t))dt$$

s.t.

$$\dot{x}(t) = u(t),$$

$$u(t) \leq 1,$$

$$x(0) = \frac{1+3e}{2(1-e)}.$$

For comparison our method with [25] and [7], we give Table 1, in which the optimal performance indices obtained by using our method and those listed in [25] and [7] are shown. In Table 1, N is the number of mesh-points for the Chebyshev-Legendre collocation method. In Figure 1, exact and approximated results of the optimal control variable obtained from B-spline functions with $M = 8$ are reported.

Example 2. This example is studied by using rationalized Haar functions [19] and hybrid of block-pulse and Bernoulli polynomials [17] and hybrid of block-pulse and



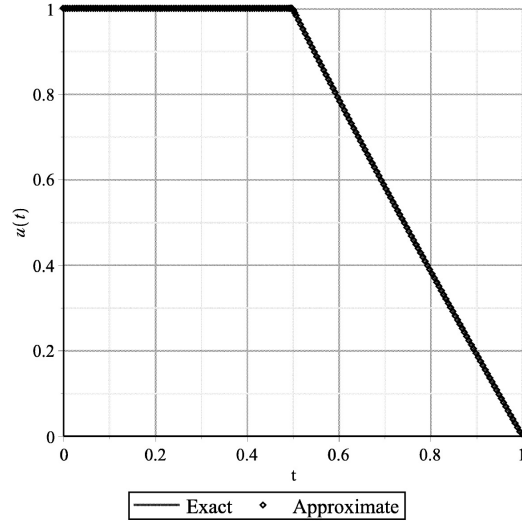


FIGURE 1. Exact value and approximation of optimal control variable using B-spline functions for Example 1 with $M = 8$.

TABLE 1. Estimated and exact values of J for Example 1

Methods	Parameters	J
Result in [7]	$N = 9$	5.5879725
	$N = 11$	5.5879712
	$N = 13$	5.5879712
CL method [25]	$N = 9$	5.5879399
	$N = 11$	5.5879527
	$N = 13$	5.5879526
Present method	$M = 8$	8.5879530
	$M = 9$	8.5879551
	$M = 10$	8.5879557
Exact		8.5879557

Legendre polynomials [15]. Find the control vector $u(t)$ which minimizes

$$J = \frac{1}{2} \int_0^1 (x_1^2(t) + u^2(t)) dt,$$



subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix},$$

and subject to the following inequality control constraint

$$|u(t)| \leq 1.$$

In Table 2, K in [19] is the order of Rationalized Haar functions, and N and M in [17] are the order of block-pulse functions and Bernoulli polynomials, respectively. In Figure 2, the control and state variables with the absolute value of constraint's errors for two methods of B-spline functions with $M = 7$, are reported.

TABLE 2. Results for Example 2

Methods	Parameters	J
Rationalized Haar functions [19]	$K = 4$	8.07473
	$K = 8$	8.07065
Hybrid of block-pulse and Bernoulli [17]	$N = 4, M = 2$	8.07058
	$N = 4, M = 3$	8.07055
B-spline functions [6]	$M = 6$	8.07058
	$M = 7$	8.07056
Present method	$M = 6$	8.07053
	$M = 7$	8.07055
	$M = 8$	8.07055

Example 3. In order to show the effectiveness of the proposed method, the continuous stirred-tank chemical reactor as a benchmark problem in the class of nonlinear optimal control problems is considered. This example is adapted from [10]. The state equations of the system are highly nonlinear and coupled:

$$\dot{x}_1(t) = -2(x_1(t) + 0.25) + (x_2(t) + 0.25) \exp\left(\frac{25x_1(t)}{x_1(t) + 2}\right) - (x_1(t) + 0.25)u(t),$$

$$\dot{x}_2(t) = 0.5 - x_2(t) - (x_2(t) + 0.5) \exp\left(\frac{25x_1(t)}{x_1(t) + 2}\right).$$

The initial conditions are

$$x_1(0) = 0.05, \quad x_2(0) = 0.$$

The performance index to be minimized is given by

$$J = \int_0^{0.78} (x_1^2(t) + x_2^2(t) + 0.1u^2(t)) dt,$$



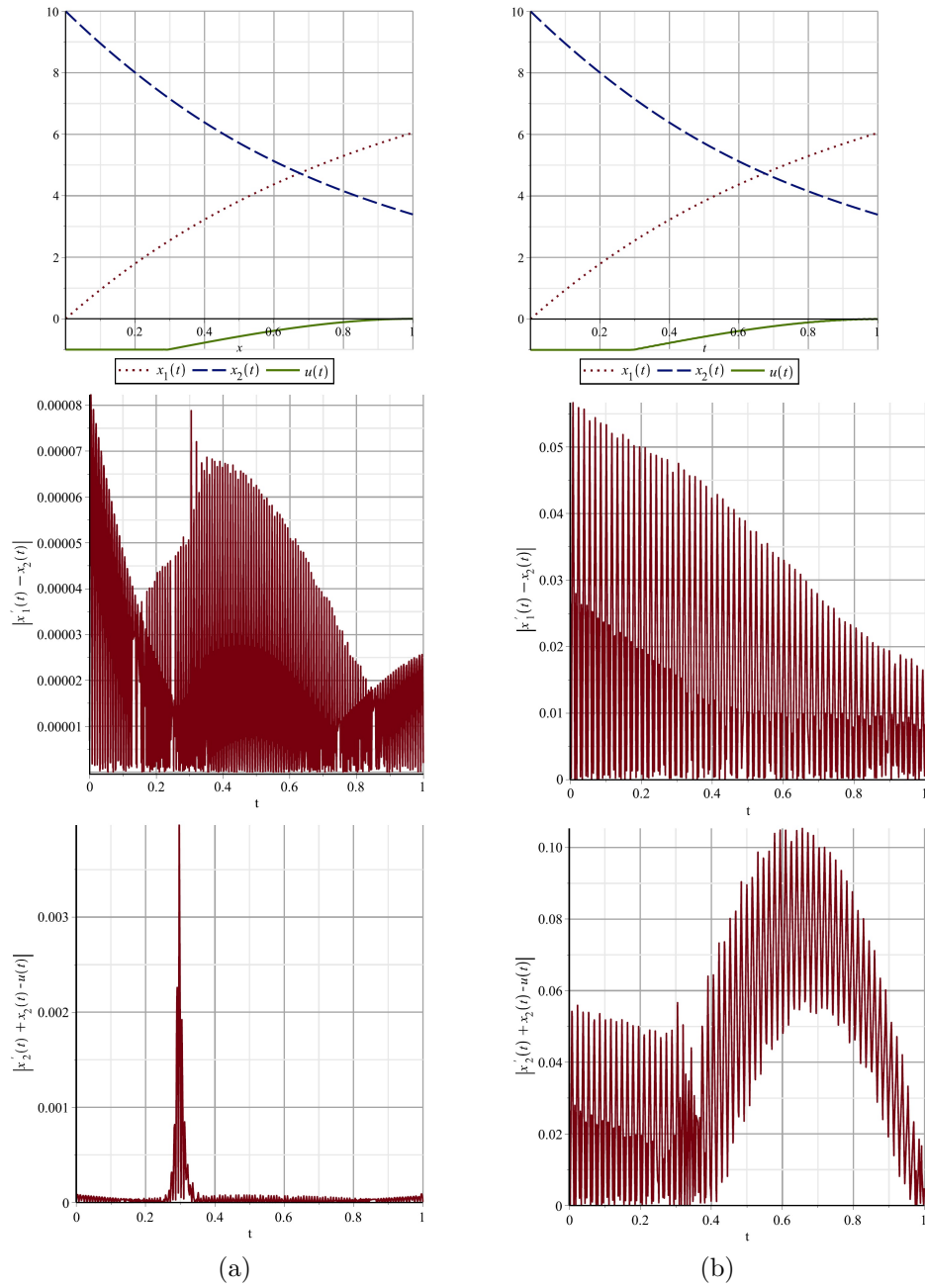


FIGURE 2. State and control variables and the constraint errors $|\dot{x}_1(t) - x_2(t)|$ and $|\dot{x}_2(t) + x_2(t) - u(t)|$ for Example 2 using present method (a) and using method of [6], (b) with $M = 7$.



where $x_1(t)$ is the deviation from the steady-state temperature, $x_2(t)$ is the deviation from the steady-state concentration and $u(t)$ is the normalized control variable that represent the effect of the flow-rate of the cooling fluid on chemical reactor. The performance index indicates that the desired objective is to maintain the temperature and concentration close to their steady-state values without expending large amount of control effort.

This problem was solved using three different numerical methods: steepest-descent, variation of extremals, and quasilinearization [10]. In Table 3, the solutions obtained by the proposed method for different values of M are compared with existing solutions.

TABLE 3. Results for Example 3

Methods	Parameters	J
Steepest descent [10]		0.02668
Variation of extremals [10]		0.02660
Quasilinearization [10]		0.02660
ChFD scheme [16]	$N = 3, M = 10$	0.026603355455
	$N = 3, M = 12$	0.026603355459
	$N = 4, M = 10$	0.026603355460
	$N = 4, M = 12$	0.026603355460
Present method	$M = 7$	0.026604322981
	$M = 8$	0.026603298510
	$M = 9$	0.026603298505
	$M = 10$	0.026603298505

Example 4. The current example is the nonlinear forced Van-DerPol oscillator given in [7, 22]. The cost, dynamics, and constraints are the following:

$$\begin{aligned}
 \min J &= \frac{1}{2} \int_0^5 (x_1^2 + x_2^2 + u^2) dt \\
 \text{s.t.} \quad & \dot{x}_1(t) = x_2(t), \\
 & \dot{x}_2(t) = -x_1(t) + (1 - x_1(t))x_2(t) + u(t), \\
 & x_1(0) = 1, \quad x_2(0) = 0, \quad x_2(5) - x_1(5) - 1 = 0.
 \end{aligned}$$

The analytic solution to this problem is unavailable. However, the convergence of the computed solution can be numerically verified. In Table 4, the computational results of J using the present method for various values of M with methods of [7] and B-spline function [6] are given. This results show the accuracy of our method in comparison with others.



TABLE 4. Results for Example 4

Methods	Parameters	J
Result of [7]	$N = 7$	0.3807970
	$N = 9$	0.3807971
	$N = 11$	0.3807971
B-spline method [6]	$M = 5$	0.3807999
	$M = 6$	0.3807981
	$M = 7$	0.3807977
Present method	$M = 5$	0.3807970
	$M = 6$	0.3807969
	$M = 7$	0.3807968

6. CONCLUSION

In this paper we presented a numerical scheme for solving nonlinear constrained quadratic optimal control problems. The method of B-spline functions was employed. Also several test problems were used to see the applicability and efficiency of the method. The obtained results show that the new approach can solve the problem effectively.

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