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Numerical solution for boundary value problem of fractional order with approximate Integral and derivative

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Abstract Approximating the solution of differential equations of fractional order is necessary because fractional differential equations have extensively been used in physics, chemistry as well as engineering fields.

In this paper with central difference approximation and applying the formula approximate integration, we have found approximate solution for a class of boundary value problems of fractional order. Three numerical examples are presented to describe the fractional usefulness of the suggested method.

Keywords. Boundary value problems of fractional order, Riemann-Liouville fractional derivative, Caputo fractional derivative, central difference.

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1. Introduction

In the past, it was believed that classical fractional calculus can provide a powerful tool that can be used to describe a large group of dynamic processes in various applied sciences. However, it has been proved by more recent studies that fractional calculus can provide more accurate models compared with the classical fractional calculus.

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This is why fractional calculus has received a great degree of interest in recent years. Fractional derivative and fractional integration have many applications in different complex systems such as physics, chemistry, fluid mechanics, viscoelasticity, signal processing, mathematical biology, and bioengineering [1–3, 12, 18]. Moreover, they have various applications in different branches of science and engineering.

Boundary value problems of fractional order are applied in accounting for various physical process of stochastic transport. Also, they have application in investigating the liquid filtration in a strongly porous (fractal) medium [17]. Moreover, boundary value problems with integral boundary conditions form a fascinating and important class of problems. The special cases of these problems include two, three, multipoint and nonlocal boundary value problems.

Integral boundary conditions also play a part in population dynamic and cellular systems [4, 5]. Furthermore, they appear in the mathematical model created for a micro-electro-mechanical system (MEMS) instrument which basically has been developed to measure the viscosity of fluids that we encounter during oil well exploration [7]. It has been argued that solution of fractional differential equations (FDEs) is required in order to analyze and design various systems. The methods in this category include Laplace and Fourier transforms, eigenvector expansion, method based on Laguerre integral formula, direct solution based on Grunewald Letnikov approximation, truncated Taylor series expansion and power series method [6, 10, 11, 13, 14].

For the purpose of solving FDEs numerically, several algorithms have been created. These include fractional Adams-Moulton methods, explicit Adams multistep methods, fractional difference methods, decomposition method, variational iteration method, least squares finite element solution, and extrapolation method. Also, they include the Kansa method which is a convenient, meshless method that has been applied in dealing with a variety of partial differential equation models [8, 9, 15, 16].

In this work, we have present numerical solutions using the central difference approximation for the first and second derivative and with formula approximate integration.

The present research paper is organized in following sections: in section 2 some definitions and theorems, which are necessary for our work, are presented. In section 3, we establish the direction of the proposed method. In section 4, we have included some numerical results in order to illustrate the applications and usefulness of the suggested method. Finally, conclusions are drawn in Section 5.

2. Preliminaries

In this section, we give some definitions and properties of the fractional calculus. Let $f(x)$ be a function defined on $[a,b]$, then

Definition 1. The Riemann-Liouville fractional derivative [14]:

$$^{R}D^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_0^x(x-t)^{m-\alpha-1}f(t)dt,\ \alpha>0,\quad m-1<\alpha<m.$$

Definition 2. Riemann-Liouville fractional integral [14]:

$$D_{a^+}^{-\alpha}f(x)=D_a^{-\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt,\ \alpha>0,$$

where Γ is the gamma function.

Definition 3. The Caputo fractional derivative [14]:

$$D^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x(x-s)^{m-\alpha-1}f^{(m)}(s)ds,\ \alpha>0,\quad m-1<\alpha<m.$$

The relation between the Riemann-Liouville operator and Caputo operator is given by:

$$D^{\alpha}f(x)=^{R}D^{\alpha}\left[f(x)-\sum_{k=0}^{m-1}\frac{1}{k!}(x-a)^k f^{(k)}(a)\right],\ \alpha>0,\quad m-1<\alpha<m.$$

Theorem 1. Let $f(x)\in C^m[0,1]$ and $\alpha\in(m-1,m)$, $m\in N$, and $g(x)\in C^m[0,1]$. Then for $x\in[0,1]$:

1) $D^{\alpha}D^{-\alpha}g(x)=g(x)$
2) $D^{-\alpha}D^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}\frac{x^k}{k!}f^{(k)}(0)$,
3) $D^{m-\alpha}f(x)=D^{-\alpha}f^{(m)}(x)$,
4) $\lim\limits_{x\to0}D^{\alpha}f(x)=\lim\limits_{x\to0}D^{-\alpha}f(x)=0$,
5) if $\alpha\in(0,1]$, $i=1,\dots,n$ with $\alpha=\sum\limits_{i=1}^{n}\alpha_i$ are such that, for each k ($k=1,2,\dots,m-1$), there exist $i_k<n$ with $\sum\limits_{j=1}^{i_k}=k$, then the following composition rule holds [14]:

$$D^{\alpha}f(x)=D^{\alpha_n}\dots D^{\alpha_2}D^{\alpha_1}f(x).$$

3. Analysis of the method

In order to describe the proposed method, we consider the numerical solution of the following fractional boundary value problems (FBVPs):

$$D^{-\alpha}y''(x)+y\ p(x)=g(x),\quad 0\le\alpha<1,\ x\in[a,b],\tag{3.1}$$

subjected to boundary conditions

$$y(a)=y(b)=0,\tag{3.2}$$

in which the function $p(x)$ and $g(x)$ are continuous on the interval $[a,b]$ and the operator D^{α} represents the Caputo fractional derivative. The analytical solution of $(3.1-3.2)$ cannot be taken for arbitrary choices $p(x)$ and $g(x)$. When $\alpha=0$, the problem (3.1) is shortened to the classical second order boundary value problem.
The main goal of this research work is to apply the central difference approximation and approximate integration formula to create a new numerical method for the FBVPs $(3.1-3.2)$. To do so, we firstly convert the FBVPs in (3.1) into the following form

$$y''(x)+D^{\alpha}y\ p(x)=D^{\alpha}g(x),\quad 0\leq\alpha<1,\ x\in[a,b]. \tag{3.3}$$

In the second step, we introduce a finite set of grid points x_i by driving the interval $[a,b]$ into n-equal parts:

$$x_i=a+ih,\ x_0=a,\ x_n=b,\ h=\frac{b-a}{n},\ i=0,1,\ldots,n. \tag{3.4}$$

If we suppose that $y(x)$ is the exact solution of (3.3), then we want to approximate $y(x_i)=y_i$ for $i=1,\ldots,n-1$.
Thus, the equation (3.3) for each x_i will result in

$$y''(x_i)=D^{\alpha}\big(g(x_i)-y\ p(x_i)\big),\quad i=1,\ldots,n-1. \tag{3.5}$$

Otherwise, with definition 3

$$y''(x_i)=\frac{1}{\Gamma(1-\alpha)}\big(\mathcal{R}(x_i)-\mathcal{T}(x_i)-\mathcal{K}(x_i)\big),\ i=1,\ldots,n-1 \tag{3.6}$$

where

$$\begin{aligned}\mathcal{R}(x_i)&=\int_0^{x_i}(x_i-s)^{-\alpha}g'(s)ds,\\ \mathcal{T}(x_i)&=\int_0^{x_i}(x_i-s)^{-\alpha}y(s)p'(s)ds,\\ \mathcal{K}(x_i)&=\int_0^{x_i}(x_i-s)^{-\alpha}p(s)y'(s)ds.\end{aligned} \tag{3.7}$$

Now using the central difference approximation for the first and second derivative in relation (3.6):

$$\begin{aligned}y'(x_i)&\approx\frac{y(x_{i+1})-y(x_{i-1})}{2h},\\ y''(x_i)&\approx\frac{y(x_{i+1})-2y(x_i)+y(x_{i-1})}{h^2},\end{aligned}$$

and applying the formula approximate integration:

$$\int_{x_0}^{x_n} f(x)dx \approx \sum_{i=0}^{n-1} a_i f(x_i),$$

which is accurate for $f(x) = 1, x, \ldots, x^{n-1}$ and with initial condition $y(a) = y(b) = 0$, we will get to the result $y(x_0) = y(x_n) = 0$.
Hence, with approximate equation (3.6) for $i = 1$ we have:

$$y(x_2) - 2y(x_1) + y(x_0) = \frac{h^2}{\Gamma(1-\alpha)}\big(\mathcal{R}(x_1) - \mathcal{T}(x_1) - \mathcal{K}(x_1)\big),$$

where

$$\mathcal{R}(x_1) = \int_0^{x_1} (x_1 - s)^{-\alpha} g'(s)ds,$$
$$\mathcal{T}(x_1) = \int_0^{x_1} (x_1 - s)^{-\alpha} y(s)p'(s)ds,$$
$$\mathcal{K}(x_1) = \frac{1}{h}\int_0^{x_1} (x_1 - s)^{-\alpha} p(s)\left(y(s+h) - y(s)\right)ds,$$

for $i = 2$ we have

$$y(x_3) - 2y(x_2) + y(x_1) = \frac{h^2}{\Gamma(1-\alpha)}\big(\mathcal{R}(x_2) - \mathcal{T}(x_2) - \mathcal{K}(x_2)\big),$$

where

$$\mathcal{R}(x_2) = \int_0^{x_2} (x_2 - s)^{-\alpha} g'(s)ds,$$
$$\mathcal{T}(x_2) = \int_0^{x_2} (x_2 - s)^{-\alpha} y(s)p'(s)ds,$$
$$\mathcal{K}(x_2) = \frac{1}{h}\int_0^{x_2} (x_2 - s)^{-\alpha} p(s)\left(y(s+h) - y(s)\right)ds,$$

and the same can be obtained for $i = 3, \ldots, n-1$. After solving the algebraic equations with Maple, we find $y(i)$ for $i = 1, 2, 3, \ldots, n-1$.

4. Illustrative Example

We now consider some numerical examples, illustrating the solution using the purposed methods. All calculations are implemented with *Maple*.

Example 1. Take the boundary value problem

$$D^{-\alpha}y''(x) + y(x) = \frac{720}{\Gamma(5+\alpha)}x^{4+\alpha} - \frac{40320}{\Gamma(7+\alpha)}x^{6+\alpha} + (1-x^2)x^6, \tag{4.1}$$
$$y(0) = y(1) = 0$$

The analytical solution of (4.1) is

$$y(x) = x^6(1-x^2). \tag{4.2}$$

The numerical solutions for various values of α are represented in Fig.1.

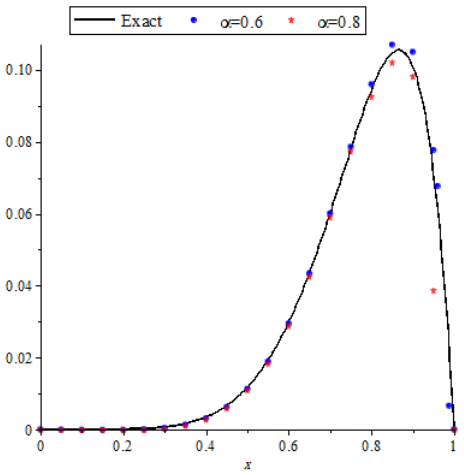

FIGURE 1. Numerical solutions for various values of α of Example 1.

TABLE 1. The absolute error between the exact solution and the approximate solution for $\alpha = 0.9$

x	*Approximate*	*Exact*	$\lvert y_{App} - y_{Exact}\rvert$
0.4	0.0025380001	0.00344064	0.0009026399
0.6	0.060001108	0.02985984	0.030141268
0.8	0.094372102	0.09437184	0.000002621
0.9	0.10097432	0.10097379	0.00000531
1.0	0.00000100	0.0	0.00000100

Example 2. Consider the boundary value problem:

$$D^{-\alpha}y''(x) = -y(x)+ \left(-x^2+(-3+2\alpha)x-\alpha^2\right)\left(x^{\alpha}\Upsilon(x,\alpha)+(x-x^2)\right)e^x \tag{4.3}$$

$$y(0) = y(1) = 0$$

The analytical solution of (4.3) is

$$y(x) = x(1-x)e^x. \tag{4.4}$$

The numerical solutions for various values of α are represented in Fig.2.

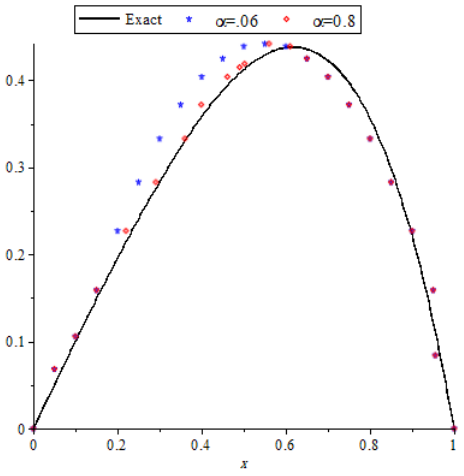

FIGURE 2. Numerical solutions for various values of α of Example 2.

TABLE 2. The absolute error between the exact solution and the approximate solution for $\alpha = 0.9$

x	*Approximate*	*Exact*	$\lvert y_{App} - y_{Exact}\rvert$
0.4	0.3586352541	0.3580379275	0.0005973266
0.6	0.4384018926	0.4373085120	0.0010933806
0.8	0.3578671593	0.3560865485	0.0019586540
0.9	0.2235781442	0.2213642800	0.0022138642
1.0	0.000274546	0.0	0.000274546

Example 3. Assuming the boundary value problem

$$D^{-\alpha}y''(x) + xy(x) = \frac{120}{\Gamma(4+\alpha)}x^{3+\alpha} - \frac{\Gamma(5+\alpha)}{\Gamma(3+2\alpha)}x^{2+2\alpha} + x^6 - x^{5+\alpha} \tag{4.5}$$
$$y(0) = y(1) = 0$$

The analytical solution of (4.5) is

$$y(x) = x^5(x - x^\alpha). \tag{4.6}$$

The numerical solution for $\alpha = 0.7$ and $\alpha = 0.8$ are represented in Fig. 3.

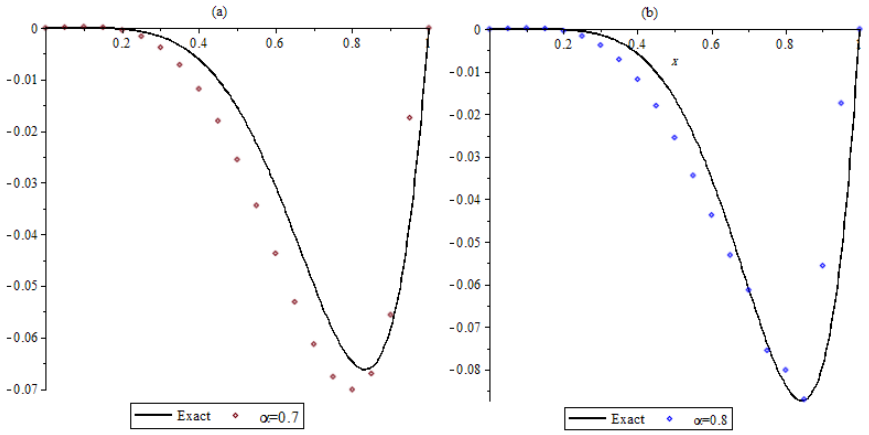

FIGURE 3. Numerical solutions for various values of α of Example 3.

TABLE 3. The absolute error between the exact solution and the approximate solution for $\alpha = 0.9$

x	*Approximate*	*Exact*	$\lvert y_{App} - y_{Exact}\rvert$
0.4	$-0.389459916e-3$	$-0.3930448957e-3$	0.0000035849797
0.6	$-0.2445230657e-2$	$-0.243277168e-2$	0.000012458977
0.8	$-0.590214224e-2$	$-0.5915327234e-2$	0.000013184994
0.9	$-0.562296329e-2$	$-0.5628890861e-2$	0.000005927571
1.0	0.0	0.0	0.0

5. CONCLUSIONS

We have presented a new method to solve fractional boundary value problem. The numerical results obtained in this paper indicate that the suggested method maintains

a considerable degree of high accuracy, which is promising in dealing with the solution of two point boundary value problem of fractional order.

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