Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 2, No. 4, 2014, pp. 216-226

The Bäcklund transformation method of Riccati equation applied to coupled Higgs field and Hamiltonian amplitude equations

A. H. Arnous
Department of Engineering Mathematics and Physics,
Higher Institute of Engineering El Shorouk, Egypt
E-mail: ahmed.h.arnous@gmail.com

M. Mirzazadeh*
Department of Engineering Sciences,
Faculty of Technology and Engineering,
East of Guilan, University of Guilan,
P.C. 44891-63157, Rudsar-Vajargah, Iran
E-mail: mirzazadehs2@guilan.ac.ir

M. Eslami
Department of Mathematics,
Faculty of Mathematical Sciences,
University of Mazandaran, Babolsar, Iran
E-mail: mostafa.eslami@umz.ac.ir

Abstract In this paper, we establish new exact solutions for some complex nonlinear wave equations. The Bäcklund transformation method of Riccati equation is used to construct exact solutions of the Hamiltonian amplitude equation and the coupled Higgs field equation. This method presents a wide applicability to handling nonlinear wave equations. These equations play a very important role in mathematical physics and engineering sciences. Obtained solutions may also be important of significance for the explanation of some practical physical problems.

Keywords. Complex nonlinear wave equations, Exact solutions, Bäcklund transformation method of Riccati equation.

2010 Mathematics Subject Classification. 35Q51, 37K40.

1. Introduction

Nonlinear evolution equations (NLEEs) are widely used as models to describe complex physical phenomena in various fields of science, especially in fluid mechanics, solid-state physics, plasma physics, plasma wave and chemical physics [1–32]. Solving NLEEs is one of the absolute necessities in the area of nonlinear waves. It is difficult to make progress without a closed form analytical solution to these NLEEs. Thus, solving NLEEs is an important task in this area of research. Various methods

Received: 25 December 2014 ; Accepted: 8 March 2015.
* Corresponding.

have been utilized to explore different kinds of solutions of physical models described by NLEEs. These methods include such as trial equation method [11], multiple exp-function method [18], tanh-sech method [15,16,19], extended tanh method [30], ansatz method [4], transformed rational function method [17] and so on. Lu [13] has introduced a reliable and effective method called the Bäcklund transformation method of Riccati equation to look for new exact solutions of nonlinear fractional PDEs. The Bäcklund transformation method of Riccati equation [13] is based on the assumptions that the exact solutions of NLEEs can be expressed by a polynomial in ψ, such that $\psi = \psi(\xi)$ satisfies the Bäcklund transformation of Riccati equation. The aim of this paper is to extract the new exact 1-soliton solutions of the Hamiltonian amplitude equation and the coupled Higgs field equation by the Bäcklund transformation method of Riccati equation [13].

2. The Bäcklund transformation method of Riccati equation

Recall that the Riccati equation:

$$\phi'(\xi) = \sigma + \phi^2(\xi), \tag{2.1}$$

which has the following exact solutions

$$\phi = \begin{cases} -\sqrt{-\sigma}\,\tanh\left(\sqrt{-\sigma}\,\xi\right), & \sigma < 0, \\ -\sqrt{-\sigma}\,\coth\left(\sqrt{-\sigma}\,\xi\right), & \sigma < 0, \\ \sqrt{\sigma}\,\tan\left(\sqrt{\sigma}\,\xi\right), & \sigma > 0, \\ -\sqrt{\sigma}\,\cot\left(\sqrt{\sigma}\,\xi\right), & \sigma > 0, \\ -\frac{1}{\xi+\varpi},\ \varpi = const. & \sigma = 0. \end{cases} \tag{2.2}$$

Next, let us consider the nonlinear evolution equation (NLEE):

$$F\left(u, u_t, u_x, u_{xx}, u_{xt}, \ldots\right) = 0, \tag{2.3}$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in u and its various partial derivatives u_t, u_x with respect to t, x respectively, in which the highest order derivatives and nonlinear terms are involved.

Using the traveling wave transformation

$$u(x,t) = U(\xi),\ \ \xi = k(x - ct), \tag{2.4}$$

where k, c are constant to be determined later, to reduce Eq. (2.3) to a nonlinear ordinary differential equation (NLODE) of the form

$$P\left(U, U', U'', ...\right) = 0. \tag{2.5}$$

Step 1: Suppose that Eq. (2.5) has the following solution

$$U(\xi) = \sum_{l=0}^{N} a_l \psi^l(\xi), \tag{2.6}$$

where $a_l(l = 0, ..., N)$ are constants to be determined and $\psi(\xi)$ comes from the following Bäcklund transformation for the Riccati equation:

$$\psi(\xi) = \frac{-\sigma B + D\,\phi(\xi)}{D + B\,\phi(\xi)}, \tag{2.7}$$

that is $\psi(\xi)$ satisfies the Riccati equation

$$\psi'(\xi) = \sigma + \psi^2(\xi), \tag{2.8}$$

where B, D are arbitrary parameters, σ is a constant to be determined and $B \neq 0$, $\phi(\xi)$ are the well-known solutions (2.2).

Step 2 : Balancing the highest order derivatives and nonlinear term in (2.5) to determine the positive integer N in (2.6).

Step 3 : Substituting the explicit formal solution (2.6) with (2.7) into Eq. (2.5) and setting the coefficients of the powers of $\phi(\xi)$ to be zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica to get the unknown constants $a_l(l = 0, ..., N)$, σ, k and c. Consequently, we obtain the exact solutions of Eq. (2.3).

3. Applications

In this section, we apply the Bäcklund transformation of Riccati equation to seek the exact solutions of the following NLEEs:

3.1. **New Hamiltonian amplitude equation.**

A new Hamiltonian amplitude equation

$$iu_x + u_{tt} + 2\rho|u|^2 u - \delta u_{xt} = 0, \tag{3.1}$$

where $\rho = \pm 1$, $\delta << 1$, was recently introduced by Wadati et al. [26]. This equation governs certain instabilities of modulated wave trains with the additional term $-\delta u_{xt}$ overcoming the ill-posedness of the unstable nonlinear Schrodinger equation. The equation is apparently not integrable but a Hamiltonian analogue of the Kuramoto-Sivashinsky equation, which arises in dissipative system. By using the transformation

$$u(x,t) = e^{i\eta}U(\xi), \ \ \eta = \alpha x + \beta t + \theta, \ \ \xi = k(x - ct), \tag{3.2}$$

we can reduce Eq. (3.1) to the following ordinary differential equation (ODE):

$$k^2(c^2 + c\delta)U'' - (\alpha + \beta^2 - \alpha\beta\delta)U + 2\rho U^3 = 0, \tag{3.3}$$

with the condition

$$c = \frac{\delta\beta - 1}{\alpha\delta - 2\beta}. \tag{3.4}$$

Balancing U'' with U^3 in Eq. (3.3), we obtain $N = 1$. So we look for solution of Eq. (3.3) in the following form

$$U(\xi) = a_0 + a_1\left(\frac{-\sigma B + D\phi(\xi)}{D + B\phi(\xi)}\right). \tag{3.5}$$

Substituting (3.5) along with (2.1) into (3.3) and then setting the coefficients of $\phi(\xi)$ to be zero, we can obtain a set of algebraic equations, which can be solved by Mathematica to get the following solution

$$a_0 = 0, \ \ a_1 = \pm\sqrt{-\frac{\alpha + \beta^2 - \alpha\beta\delta}{2\rho\sigma}}, \ \ \sigma = \frac{\alpha + \beta^2 - \alpha\beta\delta}{2k^2(c^2 + c\delta)}, \tag{3.6}$$

where α, β, k are arbitrary real constants. The solution of Eq. (3.1) corresponding to Eq. (3.6) has the following cases:
If $\frac{\alpha+\beta^2-\alpha\beta\delta}{c^2+c\delta} < 0$, we obtain

$$\begin{aligned} u_1(x,t) = &\sqrt{\frac{\alpha + \beta^2 - \alpha\beta\delta}{2\rho}} \\ &\times\left(\frac{\sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,B - D\,\tanh\left(\sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,\xi\right)}{D - \sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}B\,\tanh\left(\sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,\xi\right)}\right) \\ &\times e^{i(\alpha x + \beta t + \theta)}, \end{aligned} \tag{3.7}$$

and

$$u_2(x,t) = \sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2\rho}}$$
$$\times\left(\frac{\sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,B - D\,\coth\left(\sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,\xi\right)}{D - \sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}B\,\coth\left(\sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,\xi\right)}\right)$$
$$\times e^{i(\alpha x+\beta t+\theta)}. \tag{3.8}$$

Eqs. (3.7) and (3.8), are valid when $(\alpha+\beta^2-\alpha\beta\delta)\rho > 0$.

If $\frac{\alpha+\beta^2-\alpha\beta\delta}{c^2+c\delta} > 0$, we obtain

$$u_3(x,t) = \sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2\rho}}$$
$$\times\left(\frac{-\sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,B + D\,\tan\left(\sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,\xi\right)}{D + \sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}B\,\tan\left(\sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,\xi\right)}\right)$$
$$\times e^{i(\alpha x+\beta t+\theta)}, \tag{3.9}$$

and

$$u_4(x,t) = \sqrt{-\frac{\alpha+\beta^2-\alpha\beta\delta}{2\rho}}$$
$$\times\left(\frac{\sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,B + D\,\cot\left(\sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,\xi\right)}{-D + \sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}B\,\cot\left(\sqrt{\frac{\alpha+\beta^2-\alpha\beta\delta}{2k^2(c^2+c\delta)}}\,\xi\right)}\right)$$
$$\times e^{i(\alpha x+\beta t+\theta)}. \tag{3.10}$$

Eqs. (3.9) and (3.10), are valid when $(\alpha+\beta^2-\alpha\beta\delta)\rho < 0$.

If $\alpha = \frac{\beta^2}{\beta\delta-1}$, we obtain

$$u_5(x,t) = \pm k\sqrt{-\frac{c^2+c\delta}{\rho}}\left(\frac{1}{k\left(x-\frac{\delta\beta-1}{\alpha\delta-2\beta}t\right)+\varpi}\right)$$
$$\times e^{i\left(\frac{\beta^2}{\beta\delta-1}x+\beta t+\theta\right)}, \tag{3.11}$$

where $\frac{c^2+c\delta}{\rho} < 0$.

3.2. **The Coupled Higgs field equation.** The Higgs field equation

$$\begin{aligned} &u_{tt} - u_{xx} - \rho u + \delta |u|^2 u - 2uv = 0, \\ &v_{tt} + v_{xx} - \delta \left(|u|^2\right)_{xx} = 0, \end{aligned} \tag{3.12}$$

describes a system of conserved scalar nucleons interacting with neutral scalar mesons. Here, real constant v represents a complex scalar nucleon field and u a real scalar meson field. Eq. (3.12) is the coupled nonlinear Klein-Gordon equation for $\rho < 0$, $\delta < 0$ and the coupled Higgs field equation $\rho > 0$, $\delta < 0$. Tajiri [25] obtained N-soliton solutions to system (3.12). By using the transformation

$$\begin{aligned} &u(x,t) = e^{i\eta} U(\xi), \;\; v(x,t) = V(\xi), \\ &\eta = \alpha x + \beta t + \theta, \;\; \xi = k(x - ct), \end{aligned} \tag{3.13}$$

we can reduce system (3.12) to the following ODE:

$$k^2(c^2-1)U'' + \left(\beta^2(c^2-1) - \rho\right) U + \delta \left(1 - \frac{2}{c^2+1}\right) U^3 = 0, \tag{3.14}$$

and

$$V = \frac{\delta}{c^2+1} U^2. \tag{3.15}$$

with the condition

$$c = -\frac{\alpha}{\beta}. \tag{3.16}$$

Balancing U'' with U^3 in Eq. (3.14), we obtain $N = 1$. So we look for solution of Eq. (3.14) in the following form

$$U(\xi) = a_0 + a_1 \left(\frac{-\sigma B + D\phi(\xi)}{D + B\phi(\xi)}\right). \tag{3.17}$$

Substituting (3.17) along with (2.1) into (3.14) and then setting the coefficients of $\phi(\xi)$ to be zero, we can obtain a set of algebraic equations, which can be solved by Mathematica to get the following solution

$$a_0 = 0, \;\; a_1 = \pm\sqrt{\frac{(c^2+1)\left(\beta^2(c^2-1)-\rho\right)}{(c^2-1)\delta\sigma}}, \;\; \sigma = \frac{\rho - \beta^2(c^2-1)}{2k^2(c^2-1)}, \tag{3.18}$$

where β, k are arbitrary real constants. The solution of Eq. (3.12) corresponding to Eq. (3.18) has the following cases

If $\frac{\rho}{c^2-1} < \beta^2$, we obtain

$$
\begin{aligned}
u_1(x,t) &= \sqrt{\frac{(c^2+1)\left(\rho-\beta^2(c^2-1)\right)}{(c^2-1)\delta}} \\
&\times \left(\frac{\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,B - D\tanh\left(\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,\xi\right)}{D-\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}B\tanh\left(\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,\xi\right)} \right) \\
&\times e^{i(\alpha x+\beta t+\theta)},
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
v_1(x,t) &= \frac{\rho-\beta^2(c^2-1)}{(c^2-1)} \\
&\times \left(\frac{\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,B - D\tanh\left(\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,\xi\right)}{D-\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}B\tanh\left(\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,\xi\right)} \right)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
u_2(x,t) &= \sqrt{\frac{(c^2+1)\left(\rho-\beta^2(c^2-1)\right)}{(c^2-1)\delta}} \\
&\times \left(\frac{\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,B - D\coth\left(\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,\xi\right)}{D-\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}B\coth\left(\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,\xi\right)} \right) \\
&\times e^{i(\alpha x+\beta t+\theta)},
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
v_2(x,t) &= \frac{\rho-\beta^2(c^2-1)}{(c^2-1)} \\
&\times \left(\frac{\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,B - D\coth\left(\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,\xi\right)}{D-\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}B\coth\left(\sqrt{\frac{\beta^2(c^2-1)-\rho}{2k^2(c^2-1)}}\,\xi\right)} \right)^2.
\end{aligned}
$$

Eqs. (3.19) and (3.20), are valid when $\left(\rho-\beta^2(c^2-1)\right)(c^4-1)\delta > 0$.

If $\frac{\rho}{c^2-1} > \beta^2$, we obtain

$$u_3(x,t) = \sqrt{\frac{(c^2+1)\left(\beta^2(c^2-1)-\rho\right)}{(c^2-1)\delta}} \times \left(\frac{-\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\, B + D \tan\left(\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\,\xi\right)}{D + \sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}} B \tan\left(\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\,\xi\right)} \right) \times e^{i(\alpha x+\beta t+\theta)}, \tag{3.21}$$

$$v_3(x,t) = \frac{\beta^2(c^2-1)-\rho}{(c^2-1)} \times \left(\frac{-\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\, B + D \tan\left(\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\,\xi\right)}{D + \sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}} B \tan\left(\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\,\xi\right)} \right)^2,$$

and

$$u_4(x,t) = \sqrt{\frac{(c^2+1)\left(\beta^2(c^2-1)-\rho\right)}{(c^2-1)\delta}} \times \left(\frac{\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\, B + D \cot\left(\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\,\xi\right)}{-D + \sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}} B \cot\left(\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\,\xi\right)} \right) \times e^{i(\alpha x+\beta t+\theta)}, \tag{3.22}$$

$$v_4(x,t) = \frac{\beta^2(c^2-1)-\rho}{(c^2-1)} \times \left(\frac{\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\, B + D \cot\left(\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\,\xi\right)}{-D + \sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}} B \cot\left(\sqrt{\frac{\rho-\beta^2(c^2-1)}{2k^2(c^2-1)}}\,\xi\right)} \right)^2.$$

Eqs. (3.21) and (3.22), are valid when $\left(\rho-\beta^2(c^2-1)\right)(c^4-1)\delta < 0$.

If $\rho = \beta^2(c^2-1)$, we obtain

$$u_5(x,t) = \pm k\sqrt{-\frac{2(c^2+1)}{\delta}} \left(\frac{1}{k\left(x-\frac{\alpha}{\beta}t\right)+\varpi} \right) \times e^{i\left(\pm\sqrt{\rho+\beta^2}x+\beta t+\theta\right)}, \tag{3.23}$$

$$v_5(x,t) = -2k^2 \left(\frac{1}{k\left(x + \frac{\alpha}{\beta}t\right) + \varpi} \right)^2,$$

where $\delta < 0$.

4. Conclusions

The Bäcklund transformation method of Riccati equation has been successfully applied here for solving the Hamiltonian amplitude equation and the coupled Higgs field equation. This method is an efficient method for obtaining new exact solutions of complex nonlinear wave equations. It may also be applied to nonintegrable equations as well as integrable ones. The solutions are new and the method can be extended to solve problems of nonlinear fractional systems arising in the theory of solitons and other areas.

References

[1] A. Bekir, O. Guner, Exact solutions of distinct physical structures to the fractional potential Kadomtsev-Petviashvili equation, *Computational Methods for Differential Equations,* 2 (1) (2014) 26–36.

[2] A. Bekir, Abdelfattah El Achab, Exact solutions of the 2D Ginzburg-Landau equation by the first integral method, *Computational Methods for Differential Equations,* 2 (2) (2014) 69–76.

[3] A.H. Bhrawy, A.A. Alshaery, E.M. Hilal, Michelle Savescu, Daniela Milovic, Kaisar R. Khan, Mohammad F. Mahmood, Zlatko Jovanoski, A. Biswas, Optical solitons in birefringent fibers with spatio-temporal dispersion, *Optik*. 125, 20 (2014) 4935–4944.

[4] A. Biswas, Solitary wave solution for the generalized Kawahara equation, *Appl. Math. Lett.* 22 (2009) 208–210.

[5] A. Biswas, A.H. Bhrawy, A.A. Alshaeryd , E.M. Hilal, Bright and dark solitons in a cascaded system, *Optik*. 125(9)(2014) 6162–6165.

[6] A. Biswas, K.R. Khan, M. F. Mahmood. Bright and dark solitons in optical metamaterials, *Optik*. 125 (13) (2014) 3299–3302.

[7] A. Biswas, K. Khan, A. Rahman, A. Yildirim, T. Hayat, O. M. Aldossary, Bright and dark optical solitons in birefringent fibers with Hamiltonian perturbations and Kerr law nonlinearity, *Journal of Optoelectronics and Advanced Materials*. 14 (7-8) (2012) 571–576.

[8] M. Bache F. W. Wise, Type-I cascaded quadratic soliton compression in lithium niobate: compressing femtosecond pulses, *Physical Review A*. 81 (2010) 053815.

[9] H. Bulut, Y. Pandir, S. T. Demiray. Exact Solutions of time-fractional KdV equations by using generalized Kudryashov method, *Inter. J. Model. Optim.* 4 (2014) 315–320.

[10] M. Eslami, Solitary Wave solutions to the (3+1)-dimensional Jimbo Miwa equation, *Computational Methods for Differential Equations,* 2 (2) (2014) 115–122.

[11] Y. Gurefe, A. Sonmezoglu, E. Misirli, Application of the trial equation method for solving some nonlinear evolution equations arising in mathematical physics, *Pramana J. Phys.* 77 (2011) 1023–1029.

[12] H. Leblond, H. Triki, D. Mihalache, Theoretical studies of ultrashort soliton propagation in nonlinear optical media from a general quantum model, *Romanian Reports in Physics.* 65(3) (2013) 925–942.

[13] B. Lu, Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations, *Phys. Lett. A.* 376 (2012) 2045–2048.

[14] W.X. Ma, M. Chen, Direct search for exact solutions to the nonlinear Schrödinger equation, *Appl. Math. Comput.* 215 (2009) 2835–2842.

[15] W.X. Ma, Travelling wave solutions to a seventh order generalized KdV equation, *Phys. Lett. A.* 180 (1993) 221–224.

[16] W.X. Ma, B. Fuchssteiner, Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation, *Int. J. Non-Linear Mech.* 31 (1996) 329–338.

[17] W.X. Ma, J.-H. Lee, A transformed rational function method and exact solutions to the (3 + 1)-dimensional Jimbo-Miwa equation, *Chaos Solitons Fract.* 42 (2009) 1356–1363.

[18] W.X. Ma, T.W. Huang, Y. Zhang, A multiple exp-function method for nonlinear differential equations and its application, *Phys. Scr.* 82 (2010) 065003.

[19] W. Malfliet, Solitary wave solutions of nonlinear wave equations, *Amer. J. Phys.* 60 (7) (1992) 650–654.

[20] D. Mihalache, Multidiemnsional localized structures in optics and Bose-Einstein condensates: A selection of recent studies, *Romanian Journal of Physics.* 59 (3-4)(2014) 295–312.

[21] A. F. Mohammed, C. H. Telk, S. P. Majumdar. Limitations of soliton transmission in a cascaded optical amplifier system, *Journal of Optical Communications.* 21 (5) (2000) 165–170.

[22] M. Savescu, A. H. Bhrawy, E. M. Hilal, A. A. Alshaery, A. Biswas, Optical solitons in birefringent fibers with four-wave mixing for Kerr law nonlinearity, *Romanian Journal of Physics.* 59 (5-6)(2014) 582–589.

[23] A. K. Sarma, A. Biswas, Solitary wave solution to the nonlinear evolution equation in caccaded quadratic media beyond the slowly barying envelope approximation, *Optica Applicata.* 41 (2)(2014) 205–212.

[24] N. Taghizadeh, M. Mirzazadeh, F. Farahrooz, Exact solutions of the nonlinear Schrödinger equation by the first integral method, *J. Math. Anal. Appl.* 374 (2011) 549–553.

[25] M. Tajiri, On N-soliton solutions of coupled Higgs field equations, *J. Phys. Soc. Japan.* 52 (1983) 2277.

[26] M. Wadati, H. Segur, M.J. Ablowitz, A new Hamiltonian amplitude equation governing modulated wave instabilities, *J. Phys. Soc. Japan.* 61 (1992) 1187–1193.

[27] Q. Zhou, Analytic study on solitons in the nonlinear fibers with time-modulated parabolic law nonlinearity and Raman effect, *Optik.* 125 (13) (2014) 3142–3144.

[28] Q. Zhou, Q. Zhu, Optical solitons in medium with parabolic law nonlinearity and higher order dispersion, *Waves in Random and Complex Media.* 1 (2014) 1–8.

[29] Q. Zhou, Q. Zhou, Q. Zhu, L. Moraru, A. Biswas, Optical solitons with spatially dependent coefficients by Lie symmetry, *Optoelectronics and Advanced Materials: Rapid Communications.* 8 (7-8)(2014) 800–803.

[30] A.M. Wazwaz, Two reliable methods for solving variants of the KdV equation with compact and noncompact structures, *Chaos Solitons Fract.* 28 (2) (2006) 454–462.

[31] H. Zhang, W. X. Ma, Extended transformed rational function method and applications to complexiton solutions, *Appl. Math. Comput.* 230 (2014) 509–515

[32] Y. Zhou, M. Wang, T. Miao, The periodic wave solutions and solitary for a class of nonlinear partial differential equations, *Phys. Lett. A.* 323 (2004) 77–88.