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New Solutions for Singular Lane-Emden Equations Arising in Astrophysics Based on Shifted Ultraspherical Operational Matrices of Derivatives

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Abstract In this paper, the ultraspherical operational matrices of derivatives are constructed. Based on these operational matrices, two numerical algorithms are presented and analyzed for obtaining new approximate spectral solutions of a class of linear and nonlinear Lane-Emden type singular initial value problems. The basic idea behind the suggested algorithms is built on transforming the equations with their initial conditions into systems of linear or nonlinear algebraic equations which can be solved by using suitable numerical solvers. The Legendre and first and second kind Chebyshev operational matrices of derivatives can be deduced as special cases of the constructed operational matrices. For the sake of testing the validity and applicability of the suggested numerical algorithms, three illustrative examples are presented.

Keywords. Ultraspherical polynomials, operational matrix of derivatives, Lane-Emden equations, isothermal gas spheres equation.

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1. Introduction

Spectral methods play significant parts in several disciplines such as fluid dynamics and engineering. The main idea behind spectral methods is basically based on approximating solutions of differential equations in terms of expansions of various orthogonal polynomials. The most commonly used versions of spectral methods are the collocation, tau and Galerkin methods (see, for instance [1,2]). The Galerkin method

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is effective in handling multidimensional linear boundary value problems (see, for example [3–5]). The collocation methods are powerful in handling nonlinear equations (see, for example [6]).

Scientific literature embrace different techniques to model and formulate physical structures. The singular phenomenon and studies of singular intial value problems in second order ordinary differential equations are of considerable importance in mathematical physics and have attracted the attention of many mathematicians and physicists. One of the equations describing this type of phenomenon is the Lane-Emden type equation formulated as

$$y'' + \frac{\alpha}{x} y' + f(x, y) = g(x), \quad 0 < x < 1, \quad \alpha \geq 0, \tag{1.1}$$

subject to the following initial conditions:

$$y(0) = A, \; y'(0) = B, \tag{1.2}$$

where A and B are known constants, $f(x, y)$ is a continuous real valued function, and $g(x) \in C[0, 1]$.

Lane-Emden type equation models many phenomena in mathematical physics and astrophysics. It is categorized as singular nonlinear initial value problem. This equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. This equation is one of the basic equations in the theory of stellar structure [7] and has been the focus of many studies.

Historically, the first equation of this type is the standard Lane-Emden equation, obtained from (1.1)-(1.2) by taking $\alpha = 2$, $f(x, y) = y^n$, $g(x) = 0$ and $A = 1, B = 0$, i.e.

$$y'' + \frac{2}{x} y' + y^n = 0, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{1.3}$$

Equation (1.3) represents a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-graviting, spherically symmetric, polytropic fluid. For this reason equation (1.3) is also called the polytropic differential equation and it gives a useful approximation for self-graviting gaseous spheres such as stars (see, [8]). The three cases correspond to $n = 0, 1, 5$ can be solved analytically, while the other must be treated numerically.

Due to their multiple applicability, the Lane-Emden type equations were extensively studied by a large number of authors (see for example [9–16]). There are several numerical techniques employed for solving such equations. We list here some of these numerical methods:

- Homotopy perturbation method (Ramos, 2008 [17]).
- Sinc-collocation method (Parand and Pirkhedri, 2010 [18]).
- Lagrangian interpolation method (Parand et al., 2010 [19]).
- Optimal homotopy asymptotic method (Iqbal and Javed, 2011 [15]).
- Jacobi matrix method (Eslachi et al. 2012 [20]).
- Bernstein operational matrix of differentiation (Pandey and Kumar, 2012 [21]).
- Boubaker polynomials expansion scheme (Boubaker and Van Gorder, 2012 [22]).
- Modified Legendre-spectral method (Rismani and Monfared, 2012 [23]).
- Legendre operational matrix of differentiation (Pandey et al., 2012 [24]).
- Birkhoff interpolation method (Dehghan et al., 2013 [25]).
- Cubic Hermite spline functions collocation method (Mohammad zadeh et al., 2014 [6]).
- Rational approximation method (Iacono and Felice, 2014 [26]).
- Squared remainder minimization method (Cauntu and Bota, 2013 [27]).
- Second kind Chebyshev operational matrix method (Doha et al., 2013 [28]).

The approach of utilizing the operational matrices of derivatives is followed by a large number of authors due to its efficiency and applicability on various types of boundary value problems, and in particular nonlinear BVPs. For example, a novel operational matrix of derivatives based on harmonic numbers is employed for solving linear and nonlinear sixth-order two point boundary value problems in [29]. For some other articles in this direction, see for example, [21, 30].

The main purpose of this article can be summarized in the following three items:

i: Constructing ultraspherical operational matrices of derivatives.
ii: Employing shifted ultraspherical tau method (SUTM) for solving a linear singular IVPs of Lane-Emden type.

iii: Employing shifted ultraspherical collocation method (SUCM) for solving a class of nonlinear singular IVPs of Lane-Emden type.

The outlines of this paper is as follows. In Section 2, some relevant properties of ultraspherical polynomials and their shifted ones are given. Section 3 is dedicated to introducing a new shifted ultraspherical operational matrices of derivatives (SUOMD). In Section 4, two numerical algorithms based on utilizing SUOMD are presented and analyzed for the sake of handling singular IVPs of Lane-Emden type. Three illustrative examples are discussed in Section 5 aiming to illustrate the efficiency and the applicability of our proposed numerical algorithms. Some concluding remarks are reported in Section 6.

2. Ultraspherical polynomials and their shifted ones

In this section, we give some relevant properties of ultraspherical polynomials and their shifted ones.

2.1. **Ultraspherical polynomials.** The ultraspherical polynomials are a special class of Jacobi polynomials associated with the real parameter ($\lambda > -\frac{1}{2}$). They are orthogonal on the interval [-1,1], with respect to the weight function $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$, in the sense that

$$\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\ C_m^{(\lambda)}(x)C_n^{(\lambda)}(x)\ dx = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \tag{2.1}$$

where

$$h_n = \frac{\sqrt{\pi}\ n!\ \Gamma(\lambda+\frac{1}{2})}{(2\lambda)_n\ (n+\lambda)\ \Gamma(\lambda)}, \qquad (2\lambda)_n = \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda)}.$$

Here, the ultraspherical polynomials are standardized such that $C_n^{(\lambda)}(1) = 1$. The main advantage of such standardization is that the Legendre polynomials $L_n(x)$, and the Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)$ can be obtained as direct special cases of $C_n^{(\lambda)}(x)$. Explicitly, the following relations are valid.

$$L_n(x) = C_n^{(\frac{1}{2})}(x), \qquad T_n(x) = C_n^{(0)}(x), \qquad U_n(x) = (n+1)C_n^{(1)}(x).$$

The following properties of ultraspherical polynomials (see, for instance, [31]) are of fundamental importance. They are eigenfunctions of the following singular Sturm-Liouville equation

$$(1-x^2)\, D^2\, \phi_k(x) - (2\lambda+1)\, x\, D\, \phi_k(x) + k(k+2\lambda)\, \phi_k(x) = 0, \qquad D \equiv \frac{d}{dx},$$

and may be generated using the recurrence relation

$$(k+2\lambda) C_{k+1}^{(\lambda)}(x) = 2(k+\lambda)\, x\, C_k^{(\lambda)}(x) - k\, C_{k-1}^{(\lambda)}(x), \quad k = 1, 2, 3, \ldots,$$

starting from $C_0^{(\lambda)}(x) = 1$ and $C_1^{(\lambda)}(x) = x$, or obtained from the Rodrigues' formula

$$C_n^{(\lambda)}(x) = \left(-\tfrac{1}{2}\right)^n \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(n+\lambda+\frac{1}{2})} (1-x^2)^{\frac{1}{2}-\lambda} \frac{d^n}{dx^n}[(1-x^2)^{n+\lambda-\frac{1}{2}}].$$

Now, the derivatives of ultraspherical polynomials are given in the following theorem.

Theorem 2.1. *For all $q \geq 1,\ k \geq q$, the qth derivative of the ultraspherical polynomials $C_k^{(\lambda)}(x)$ can be expressed in terms of their corresponding polynomials by the formula:*

$$D^q C_k^{(\lambda)}(x) = \frac{2^q\ k!}{(q-1)!\ \Gamma(k+2\lambda)} \sum_{\substack{m=0 \\ (k+m-q)even}}^{k-q} \xi_{q,m,k,\lambda}\ C_m^{(\lambda)}(x), \tag{2.2}$$

where

$$\xi_{q,m,k,\lambda} = \frac{(m+\lambda)\Gamma(m+2\lambda)\left(\frac{k-m+q-2}{2}\right)!\ \Gamma\left(\frac{k+m+q+2\lambda}{2}\right)}{m!\ \left(\frac{k-q-m}{2}\right)!\ \Gamma\left(\frac{k+m-q+2\lambda+2}{2}\right)}. \tag{2.3}$$

(For a proof of Theorem 2.1, see, Doha [32]).

2.2. **Shifted ultraspherical polynomials.** Shifted ultraspherical polynomials are defined on $[0,1]$ by $\tilde{C}_n^{(\lambda)}(x) = C_n^{(\lambda)}(2x-1)$. All results of ultraspherical polynomials can be easily transformed to give the corresponding results for their shifted ones. The orthogonality relation for $\tilde{C}_n^{(\lambda)}(x)$ with respect to the weight function $(x-x^2)^{\lambda-\frac{1}{2}}$ is given by

$$\int_0^1 (x-x^2)^{\lambda-\frac{1}{2}}\, \tilde{C}_n^{(\lambda)}(x)\, \tilde{C}_m^{(\lambda)}(x)\, dx = \begin{cases} 0, & m \neq n, \\ \dfrac{4^{-\lambda}\sqrt{\pi}\ n!\ \Gamma(\lambda+\frac{1}{2})}{(2\lambda)_n\ (n+\lambda)\ \Gamma(\lambda)}, & m = n. \end{cases}$$

As a direct consequence of Theorem 2.2, the qth derivative of $\tilde{C}_n^{(\lambda)}(x)$ can be easily obtained in the following corollary.

Corollary 2.2. *For all $q \geq 1$, $k \geq q$, the qth derivative of the shifted ultraspherical polynomial $\tilde{C}_k^{(\lambda)}(x)$ is given explicitly by*

$$D^q \tilde{C}_k^{(\lambda)}(x) = \frac{2^{2q}\ k!}{(q-1)!\ \Gamma(k+2\lambda)} \sum_{\substack{m=0 \\ (k+m-q) even}}^{k-q} \xi_{q,m,k,\lambda}\ \tilde{C}_m^{(\lambda)}(x), \tag{2.4}$$

where $\xi_{q,m,k,\lambda}$ is given in (2.3).

3. Ultraspherical operational matrices of derivatives

If we assume that a function, $y(x) \in L_w^2[0,1]$, $w = (x-x^2)^{\lambda-\frac{1}{2}}$, then it can be expanded in terms of shifted ultraspherical polynomials as:

$$y(x) = \sum_{i=0}^{\infty} a_i\, \tilde{C}_i^{(\lambda)}(x), \tag{3.1}$$

where

$$a_i = \frac{4^{\lambda}\,(2\lambda)_i\ (i+\lambda)\ \Gamma(\lambda)}{\sqrt{\pi}\ i!\ \Gamma(\lambda+\frac{1}{2})} \int_0^1 (x-x^2)^{\lambda-\frac{1}{2}}\, y(x)\, \tilde{C}_i^{(\lambda)}(x)\, dx.$$

Assume that the series in Eq. (3.1) is approximated by the first $(N+1)$ shifted ultraspherical polynomials as:

$$y(x) = \sum_{i=0}^{N} a_i\, \tilde{C}_i^{(\lambda)}(x) = \boldsymbol{A}^T\, \boldsymbol{\Phi}(x), \tag{3.2}$$

where

$$\boldsymbol{A}^T = [a_0, a_1, \ldots, a_N], \quad \boldsymbol{\Phi}(x) = [\tilde{C}_0^{(\lambda)}(x), \tilde{C}_1^{(\lambda)}(x), \ldots, \tilde{C}_N^{(\lambda)}(x)]^T, \tag{3.3}$$

then the operational matrices of derivatives of the shifted ultraspherical polynomials are given by:

$$\frac{d^q \boldsymbol{\Phi}(x)}{dx^q} = D^{(q)} \boldsymbol{\Phi}(x), \quad q = 1, 2, \ldots, \tag{3.4}$$

where for every fixed q, $D^{(q)}$ is the $(N+1)\times(N+1)$ operational matrix of derivative, which is given explicitly in the following theorem.

Theorem 3.1. *For every $q \geq 1$, the matrix $D^{(q)}$ is given explicitly by:*

$$D^{(q)} = (d_{ij}^{(q)})_{0 \leq k,j \leq N} = \begin{cases} \dfrac{2^{2q}\, i!\,(j+\lambda)\;\Gamma(j+2\lambda)\;\left(\frac{1}{2}(i-j+q-2)\right)!\;\Gamma\left(\frac{1}{2}(i+j+q+2\lambda)\right)}{j!\,(q-1)!\,\Gamma(i+2\lambda)\;\left(\frac{1}{2}(i-j-q)\right)!\;\Gamma\left(\frac{1}{2}(i+j-q+2\lambda+2)\right)}, \\ \qquad i \geq j+q \text{ and } (i+j+q) \text{ even}, \\ \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.5}$$

and in particular, for $q = 1$, we have

$$D^{(1)} = (d_{ij}^{(1)})_{0 \leq i,j \leq N} = \begin{cases} \dfrac{4(j+\lambda)\, i!\, \Gamma(j+2\lambda)}{j!\, \Gamma(i+2\lambda)}, & i > j, \quad (i+j) \text{ odd}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

Proof. The proof of Theorem 3.1 follows immediately from formula (2.4). □

Remark 3.2. The operational matrices $D^{(q)}$ defined explicitly in (3.5) can be expressed in terms of the operational matrix $D^{(1)}$ as:

$$D^{(q)} = \left(D^{(1)}\right)^q, \tag{3.7}$$

where q in the right hand side of (3.7) denotes the matrix power.

Remark 3.3. The operational matrices given in Pandey and Kumar [24] can be obtained as a direct special case for $\lambda = \frac{1}{2}$. This shows clearly that our result in Theorem 2 is more general than that given in [24].

4. Solutions of Lane-Emden type equations

In this section, we explain and demonstrate how the approximate spectral solutions of singular IVPs of Lane-Emden type can be obtained by applying shifted ultraspherical tau method (SUTM) for linear problems, and shifted ultraspherical collocation method (SUCM) for nonlinear ones. The two methods are essentially based on our previous constructed operational matrices of derivatives.

Now, consider the Lane-Emden equation of the form

$$y'' + \frac{\alpha}{x}\, y' + f(x,y) = g(x), \quad 0 < x \leq 1, \quad \alpha \geq 0, \tag{4.1}$$

subject to the following initial conditions:

$$y(0) = a,\; y'(0) = 0. \tag{4.2}$$

If the functions $y(x), f(x,y)$ and $g(x)$ are approximated in terms of the shifted ultraspherical polynomials as:

$$y(x) \approx y_N(x) = \sum_{i=0}^{N} a_i\, \tilde{C}_i^{(\lambda)}(x) = \boldsymbol{A}^T\, \boldsymbol{\Phi}(x), \tag{4.3}$$

$$f(x,y) \approx f(x, \boldsymbol{A}^T\, \boldsymbol{\Phi}(x)), \tag{4.4}$$

$$g(x) \approx \sum_{i=0}^{N} g_i\, \tilde{C}_i^{(\lambda)}(x) = \boldsymbol{G}^T\, \boldsymbol{\Phi}(x), \tag{4.5}$$

then, after making use of the ultraspherical operational matrices of derivatives, Eq. (4.1) can be written in the form:

$$\boldsymbol{A}^T D^{(2)} \boldsymbol{\Phi}(x) + \frac{\alpha}{x}\, \boldsymbol{A}^T D^{(1)} \boldsymbol{\Phi}(x) + f(x, \boldsymbol{A}^T \boldsymbol{\Phi}(x)) \approx \boldsymbol{G}^T \boldsymbol{\Phi}(x). \tag{4.6}$$

Now, the residual $R_N(x)$ of Eq. (4.1) is given by

$$R_N(x) = \boldsymbol{A}^T D^{(2)} \boldsymbol{\Phi}(x) + \frac{\alpha}{x}\, \boldsymbol{A}^T D^{(1)} \boldsymbol{\Phi}(x) + f(x, \boldsymbol{A}^T \boldsymbol{\Phi}(x)) - \boldsymbol{G}^T \boldsymbol{\Phi}(x). \tag{4.7}$$

For linear Lane-Emden equation, a typical tau method (see, [19,24,33]) is applied. In such case, and as a direct consequence, Eq. (4.7) immediately yiels

$$\begin{gathered}(R_N(x), \tilde{C}_i^{(\lambda)}(x))_w = \int_0^1 (x-x^2)^{\lambda-\frac{1}{2}}\ R_N(x)\, \tilde{C}_i^{(\lambda)}(x)\, dx = 0, \\ i = 0, 1, \ldots, N-2,\end{gathered} \tag{4.8}$$

while for nonlinear Lane-Emden equation, the collocation method is applied to give

$$R_N(x_i) = 0, \qquad i = 0, 1, 2, \ldots N-2, \tag{4.9}$$

where x_i are the first $(N-1)$ distinct roots of $\tilde{C}_{N+1}^{(\lambda)}(x)$. Now, the two initial conditions (4.2) lead to the following two equations:

$$y(0) = \boldsymbol{A}^T \boldsymbol{\Phi}(0) = a, \quad y'(0) = \boldsymbol{A}^T D^{(1)} \boldsymbol{\Phi}(0) = 0, \tag{4.10}$$

and therefore Eqs. (4.8) or (4.9) together with Eqs. (4.10) generate a set of $(N+1)$ linear or nonlinear equations. The resulting linear system can be solved by using any suitable solver, while the nonlinear ones can be solved with the aid of the well-known Newton's iterative method for the unknown components of vector $\boldsymbol{A}$, and hence the approximate solution $y_N(x)$ in (4.3) can be obtained.

5. Numerical results and discussions

In this section, we test the two suggested numerical algorithms from the point of view of their applicability and efficiency. Three illustrative examples are considered in this respect.

Example 1. *We consider the following Lane-Emden equation for the three cases,* $n = 0, 1$ *and* $n = 5$.

$$y''(x) + \frac{2}{x}\, y'(x) + (y(x))^n = 0, \quad 0 < x \leq 1, \qquad y(0) = 1,\, y'(0) = 0. \tag{5.1}$$

Case 1 *(n=0): In such case, the exact solution of Eq. (5.1) is given by* $y(x) = 1 - \frac{x^2}{6}$. *If we apply SUTM with* $N = 2$, *then we have*

$$y_N(x) = \boldsymbol{A}^T\, \boldsymbol{\Phi}(x) = a_0\, \tilde{C}_0^{(\lambda)}(x) + a_1\, \tilde{C}_1^{(\lambda)}(x) + a_2\, \tilde{C}_2^{(\lambda)}(x).$$

From Eqs. (3.5), *the two operational matrices* $D^{(1)}$ *and* $D^{(2)}$ *are given by*

$$D^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & \frac{8(\lambda+1)}{2\lambda+1} & 0 \end{pmatrix}, \quad D^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{16(\lambda+1)}{2\lambda+1} & 0 & 0 \end{pmatrix}.$$

Therefore Eq. (4.8), yields

$$\int_0^1 (x - x^2)^{\lambda - \frac{1}{2}} \left(\boldsymbol{A}^T\, D^{(2)}\, \boldsymbol{\Phi}(x) + \frac{2}{x}\, \boldsymbol{A}^T\, D\, \boldsymbol{\Phi}(x) + 1 \right) \tilde{C}_i^{(\lambda)}(x) dx = 0, i = 0. \tag{5.2}$$

Moreover, the two initial conditions in (4.2) *lead to the following two equations:*

$$\boldsymbol{A}^T\, \boldsymbol{\Phi}(0) = 1, \tag{5.3}$$

and

$$\boldsymbol{A}^T\, D\, \boldsymbol{\Phi}(0) = 0. \tag{5.4}$$

If the linear system of Eqs. (5.2)-(5.4) is solved, then we get

$$a_0 = \frac{46\lambda + 45}{48(\lambda + 1)}, \qquad a_1 = \frac{-1}{12}, \qquad a_2 = \frac{-(2\lambda + 1)}{48(\lambda + 1)},$$

and accordingly

$$\begin{aligned} y(x) &= \left(\frac{46\lambda + 45}{48(\lambda + 1)},\ \frac{-1}{12},\ \frac{-(2\lambda + 1)}{48(\lambda + 1)} \right) \begin{pmatrix} 1 \\ (2x - 1) \\ \dfrac{2(1 + \lambda)(2x - 1)^2 - 1}{1 + 2\lambda} \end{pmatrix} \\ &= 1 - \frac{x^2}{6}, \end{aligned}$$

TABLE 1. The error E for Example 1, Case 2

λ	$-\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{3}{4}$	1
E	$7.49.10^{-13}$	$2.00.10^{-13}$	$3.93.10^{-14}$	$3.88.10^{-15}$	$3.86.10^{-14}$

TABLE 2. Comparison between best errors for Example 1, Case 2

Method	*BOM [21]*	*LOM [24]*	*SUTM*
N	*8*	*10*	*8*
Best error	$5.00.10^{-10}$	$2.00.10^{-6}$	$7.48.10^{-11}$

which is the exact solution.
Case 2 *(n=1): In such case, the exact solution of Eq. (5.1) is given by* $y(x) = \dfrac{\sin x}{x}$. *We apply SUTM with* $N = 11$. *In Table 1, we list the maximum error E for various values of* λ, *while in Table 2 we give a comparison between the best error resulted from the application of SUTM with the best errors obtained by using Bernstein operational matrix of derivatives (BOM [21]) and Legendre operational matrix of derivatives (LOM [24]).*

Case 3 *(n=5): In such case, the exact solution of Eq. (5.1) is given by* $y(x) = \dfrac{1}{\sqrt{1+\frac{x^2}{3}}}$. *We apply SUCM with* $N = 10, \lambda = 0$. *In Table 3, we present a comparison between our proposed numerical solution with those obtained by using the following methods: Homotopy perturbation method HPM [34], optimal homotopy asymptotic method OHAM [15], Boubaker polynomials expansion scheme BPES [22] and squared remainder minimization method SRMM [27]. This comparison is performed between the best error resulted from the application of our method with the best errors resulted from the applications of the all previously mentioned methods computed at a set of values of* x *on* $[0, 1]$. *Moreover, in Figure 1 we illustrate the exact and approximate solutions in case of* $N = 3$ *for various values of* λ. *The figure shows that the case corresponds to* $\lambda = 0$ *(Chebyshev expansion) is the best.*

Remark 5.1. Numerical results of Example 1, show that the numerical spectral approximations based on using Chebyshev polynomials of the first kind are not always better than those resulted from using other ultraspherical polynomials.

Example 2. *Consider the isothermal gas spheres which are modeled by Davis [35]*

$$y''(x) + \frac{2}{x}\, y'(x) + e^{y(x)} = 0, \quad 0 < x \leq 1, \qquad y(0) = 0,\, y'(0) = 0. \tag{5.5}$$

TABLE 3. Comparison of absolute errors for Example 1, Case 3

Method	*x=0*	*x=0.1*	*x=0.5*	*x=1*
HPM [34]	*0*	$3.36.\,10^{-11}$	$1.22.\,10^{-5}$	$2.59.\,10^{-3}$
OHAM [15]	*0*	$4.01.\,10^{-5}$	$3.56.\,10^{-4}$	$4.49.\,10^{-4}$
BPES [22]	*0*	$5.22.\,10^{-4}$	$1.31.\,10^{-2}$	$8.26.\,10^{-2}$
SRMM [27]	*0*	$1.46.\,10^{-7}$	$3.56.\,10^{-6}$	$5.07.\,10^{-7}$
SUCM	*0*	*0*	$2.22.\,10^{-16}$	$7.04.\,10^{-13}$

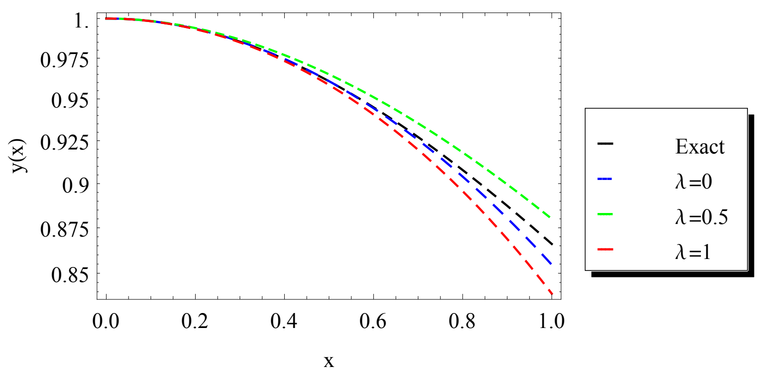

FIGURE 1. Exact and approximate solutions of Example 1, Case 3.

Here, we use the following approximation

$$y(x) \approx \boldsymbol{A}^T\,\boldsymbol{\Phi}(x),$$

$$e^{y(x)} \approx 1 + \boldsymbol{A}^T\,\boldsymbol{\Phi}(x) + \frac{1}{2}\left(\boldsymbol{A}^T\,\boldsymbol{\Phi}(x)\right)^2 + \frac{1}{6}\left(\boldsymbol{A}^T\,\boldsymbol{\Phi}(x)\right)^3 + \frac{1}{24}\left(\boldsymbol{A}^T\,\boldsymbol{\Phi}(x)\right)^4.$$

We apply SUCM with $N = 8, \lambda = 1$. *Since, the exact solution of* (5.5) *is not available, then the approximate solutions are compared with the numerical solution obtained by using a fourth-order Runge-Kutta method. In this respect, we present a comparison in Table 4 between our proposed numerical solution with the approximate solutions obtained by using the following methods: Bernstein operational matrix of differentiation (BOM [21]), Legendre operational matrix of differentiation (LOM [24]), Hermite function collocation method HCM [12] and squared remainder minimization method SRMM [27]. In Figure 2, we illustrate the absolute error of Example 2 for* $N = 4, 5$. *As expected the error decreases as* N *increases.*

TABLE 4. Comparison of absolute errors for Example 2

Method	*x=0*	*x=0.1*	*x=0.5*	*x=1*
BOM [21]	0	$6.10.\,10^{-12}$	$1.43.\,10^{-7}$	$3.47.\,10^{-5}$
LOM [24]	$9.24.\,10^{-18}$	$5.61.\,10^{-10}$	$8.12.\,10^{-6}$	$4.49.\,10^{-4}$
HCM [12]	0	$5.84.\,10^{-7}$	$5.57.\,10^{-7}$	$4.96.\,10^{-7}$
SRMM [27]	0	$1.08.\,10^{-9}$	$3.17.\,10^{-8}$	$3.30.\,10^{-10}$
SUCM	0	$2.68.\,10^{-14}$	$8.21.\,10^{-14}$	$2.50.\,10^{-10}$

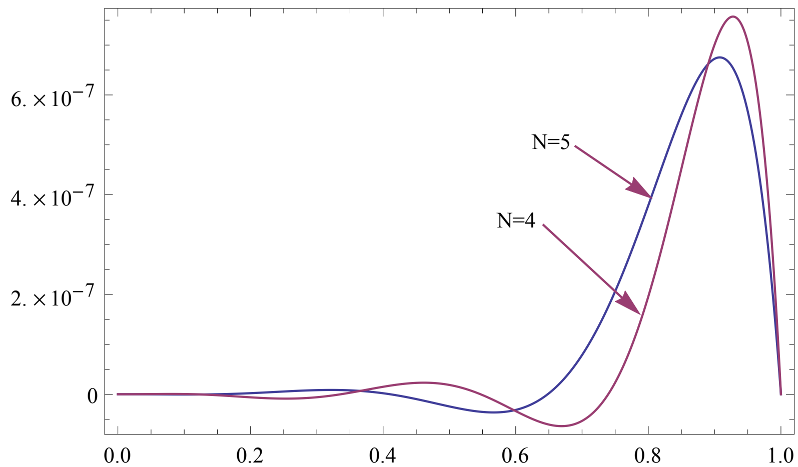

FIGURE 2. Absolute error of Example 2.

Example 3. *Consider the following nonhomogeneous Lane-Emden equation [12]*

$$\begin{aligned} &y''(x) + \frac{8}{x}\, y'(x) + x\, y(x) = x^5 - x^4 + 44\, x^2 - 30\, x; \quad 0 < x \le 1, \\ &y(0) = 0,\ y'(0) = 0, \end{aligned} \tag{5.6}$$

with the exact solution $y(x) = x^4 - x^3$. *If we apply SUTM with* $N = 4$, *then after some manipulations, the vector* $\boldsymbol{A}$ *is given by*

$$\boldsymbol{A} = \Bigg(-\frac{(1+2\lambda)(5+2\lambda)}{64(1+\lambda)(2+\lambda)},\ -\frac{1}{8} + \frac{3}{16(2+\lambda)}, \\ \frac{3(1+2\lambda)}{32(1+\lambda)(3+\lambda)},\ -\frac{(1+2\lambda)}{16(2+\lambda)},\ -\frac{(1+2\lambda)(3+2\lambda)}{64(2+\lambda)(3+\lambda)} \Bigg)^T,$$

and therefore

$$y(x) \approx \boldsymbol{A}^T\, \boldsymbol{\Phi}(x) = x^4 - x^3,$$

which is the exact solution.

Remark 5.2. It is worthy noting here that the obtained numerical results in the previous solved three examples are very accurate, although the number of retained modes in the spectral expansions are very few, and the numerical results are comparing favorably with the known analytical solutions.

6. Concluding remarks

In this paper, we give two numerical algorithms for obtaining solutions of linear and nonlinear singular IVPs of Lane-Emden type. The shifted ultraspherical operational matrices of derivatives are constructed for this purpose. The algorithms based on using shifted Legendre polynomials developed by Pandey [24] are obtained from our ultraspherical algorithms as direct special cases. Moreover, the algorithms based on using the first kind of Chebyshev polynomials can be obtained as another special case. The main advantage of the two proposed algorithms is their simplicity and their availability for application on both liner and nonlinear equations. Another advantage of the proposed algorithms, is that their applications enables one to achieve high accurate approximate solutions using a few number of terms of the approximate expansion.

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