Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 1, No. 1, 2013, pp. 71-77

The modified simplest equation method and its application

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Abstract In this paper, the modified simplest equation method is successfully implemented to find travelling wave solutions of the generalized forms $B(n, 1)$ and $B(-n, 1)$ of Burgers equation. This method is direct, effective and easy to calculate, and it is a powerful mathematical tool for obtaining exact travelling wave solutions of the generalized forms $B(n, 1)$ and $B(-n, 1)$ of Burgers equation and can be used to solve other nonlinear partial differential equations in mathematical physics.

Keywords. The modified simplest equation method, traveling wave solutions, homogeneous balance, solitary wave solutions, The generalized forms $B(n, 1)$ and $B(-n, 1)$ of Burgers equation.
2010 Mathematics Subject Classification. 47F05.

1. Introduction

Exact solutions to nonlinear partial differential equations (NLPDEs) play an essential role in the nonlinear science, especially they may provide much physical information and help one to understand the mechanism that governs these physical models. In recent years, new exact solutions may help to find new phenomena. A variety of powerful methods, such as the $(\frac{G'}{G})$-expansion method [10, 14], the Exp-function method [4], extended tanh method [1, 2], Jacobi elliptic function method [6] and F-expansion method [11], the homogeneous balance method [3] ,and so on.
One of the most powerful and direct methods for constructing solutions of nonlinear partial differential equations is the modified simplest equation method [5, 7, 8, 9, 12, 13].
The modified simplest equation method is based on the assumptions that the exact solutions can be expressed by a polynomial in $\frac{F'}{F}$, such that $F = F(\xi)$ is a solution of an unknown linear ordinary equation to be determined later.
The generalized forms $B(n, 1)$ and $B(-n, 1)$ of Burgers equation appear in various areas of mathematics, such as the modeling of fluid dynamics, the

propagation of waves, and traffic flow. Here, our goal is finding exact solutions of the generalized forms $B(n,1)$ and $B(-n,1)$ of Burgers equation by using the modified simplest equation method.

2. Modified simplest equation method

This method consists of the following steps:

Step 1. Consider a general form of nonlinear partial differential equation (PDE)

$$P(u, u_x, u_t, u_{xx}, u_{xt}, \cdots) = 0. \tag{2.1}$$

Assume that the solution is given by $u(x,t) = U(\xi)$ where $\xi = x - ct$. Hence, we use the following changes:

$$\begin{aligned} \frac{\partial}{\partial t}(.) &= -c\frac{\partial}{\partial \xi}(.), \\ \frac{\partial}{\partial x}(.) &= \frac{\partial}{\partial \xi}(.), \\ \frac{\partial^2}{\partial x^2}(.) &= \frac{\partial^2}{\partial \xi^2}(.). \end{aligned} \tag{2.2}$$

and so on for other derivatives. Using (2.2) changes the PDE (2.1) to an ODE

$$Q(U, U', U'', \cdots) = 0. \tag{2.3}$$

where $U = U(\xi)$ is an unknown function, Q is a polynomial in the variable U and its derivatives.

Step 2. We suppose that Eq. (2.3) has the following formal solution:

$$U(\xi) = \sum_{i=0}^{N} A_i (\frac{F'}{F})^i, \tag{2.4}$$

where A_i are arbitrary constants to be determined such that $A_N \neq 0$, while $F(\xi)$ is an unknown function to be determined later.

Step 3. We determine the positive integer N in (2.4) by balancing the highest order derivatives and the nonlinear terms in Eq.(2.3).

Step 4. We substitute (2.4) into (2.3), we calculate all the necessary derivatives $U', U'', \cdots$ and then we account the function $F(\xi)$. As a result of this substitution, we get a polynomial of $\frac{F'(\xi)}{F(\xi)}$ and its derivatives. In this polynomial, we equate with zero all the coefficients of it. This operation yields a system of equations which can be solved to find A_i and $F(\xi)$. Consequently, we can get the exact solution of Eq. (2.1).

3. Applications

In this section, we apply the modified simplest equation method to construct the travelling wave solution of generalized forms of Burgers equation.

3.1. **The $B(n,1)$ Burgers equation.** This $B(n,1)$ Burgers equation is given as

$$u_t + a(u^n)_x + bu_{xx} = 0,\ n > 1,\ a, b \neq 0. \tag{3.1}$$

Using the transformation $u(x,t) = U(\xi)$, where $\xi = x - ct$, the PDE is reduced to an ODE

$$-cU' + a(U^n)' + b(U)'' = 0, \tag{3.2}$$

where primes denote the derivative with respect to ξ. Integrating once with respect to ξ and taking constant of integration to be zero, (3.2) reduces to

$$-cU + a(U^n) + b(U)' = 0. \tag{3.3}$$

Now balancing U^n and U', we obtain

$$m = \frac{1}{n-1},\ n > 1$$

A necessary condition for obtaining a closed form analytic solution is that m must a positive integer. Using the transformation

$$U = V^{\frac{1}{n-1}}, \tag{3.4}$$

(3.3) converts to

$$-c(n-1)V + a(n-1)V^2 + bV' = 0. \tag{3.5}$$

Balancing V^2 with V' gives $N = 1$. Therefore, we have

$$V(\xi) = A_0 + A_1\frac{F'}{F},\ A_1 \neq 0. \tag{3.6}$$

Using (3.6), we obtain

$$V^2 = {A_0}^2 + 2A_0A_1(\frac{F'}{F}) + {A_1}^2(\frac{F'}{F}), \tag{3.7}$$

$$V' = A_1(\frac{F''}{F} - (\frac{F'}{F})^2). \tag{3.8}$$

Substituting (3.6) to (3.8) into Eq. (3.5) and setting the coefficients of $F^j(j = 0, -1, -2)$ to zero, we obtain

$$-c(n-1)A_0 + a(n-1)A_0^2 = 0, \tag{3.9}$$

$$-c(n-1)A_1F' + 2a(n-1)A_0A_1F' + bA_1F'' = 0, \tag{3.10}$$

$$a(n-1)A_1^2F'^2 - bA_1F'^2 = 0. \tag{3.11}$$

Eqs. (3.9) and (3.11) directly imply following solutions:

$$A_0 = \frac{c}{a},\ A_1 = \frac{b}{a(n-1)}.$$

Thus, Eq. (3.10) becomes

$$c(n-1)F' + bF'' = 0. \tag{3.12}$$

The general solution of Eq. (3.12) is

$$F(\xi) = a_0 + a_1e^{-\frac{c(n-1)}{b}\xi}. \tag{3.13}$$

where $a_i(i = 0, 1)$ are arbitrary constants.
Thus, we have

$$V(\xi) = \frac{c}{a} + \frac{b}{a(n-1)}\left(\frac{-a_1\frac{c(n-1)}{b}e^{-\frac{c(n-1)}{b}\xi}}{a_0 + a_1e^{-\frac{c(n-1)}{b}\xi}}\right)$$

Now using of $U = V^{\frac{1}{n-1}}$ have:

$$U(\xi) = \left[\frac{c}{a} + \frac{b}{a(n-1)}\left(\frac{-a_1\frac{c(n-1)}{b}e^{-\frac{c(n-1)}{b}\xi}}{a_0 + a_1e^{-\frac{c(n-1)}{b}\xi}}\right)\right]^{\frac{1}{n-1}} \tag{3.14}$$

If we set $a_0 = a_1 = 1$, $n = 2$ in (3.14), then solution of (3.1) is obtained as

$$u(x,t) = \frac{c}{2a}[1 + \tanh(\frac{c}{2b})(x - ct)]$$

3.2. **The $B(-n,1)$ Burgers equation.** This $B(-n,1)$ Burgers equation is given as

$$u_t + a(u^{-n})_x + bu_{xx} = 0,\ n > 1,\ a, b \neq 0. \tag{3.15}$$

Using the transformation $u(x,t) = U(\xi)$, where $\xi = x - ct$, the PDE is reduced to an ODE

$$-cU' + a(U^{-n})' + b(U)'' = 0, \tag{3.16}$$

where primes denote the derivative with respect to ξ. Integrating once with respect to ξ and taking constant of integration to be zero, (3.16) reduces to

$$-cU + a(U^{-n}) + b(U)' = 0. \tag{3.17}$$

Now balancing U^{-n} and U', we obtain

$$m = -\frac{1}{n+1}, \ n > 1$$

By using of the transformation

$$U = V^{-\frac{1}{n+1}}, \tag{3.18}$$

(3.17) converts to

$$-c(n+1)V + a(n+1)V^2 - bV' = 0. \tag{3.19}$$

Balancing V^2 with V' gives $N = 1$. Therefore, we have

$$V(\xi) = A_0 + A_1\frac{F'}{F}, \ A_1 \neq 0. \tag{3.20}$$

Using (3.20), we obtain

$$V^2 = {A_0}^2 + 2A_0A_1(\frac{F'}{F}) + {A_1}^2(\frac{F'}{F}), \tag{3.21}$$

$$V' = A_1(\frac{F''}{F} - (\frac{F'}{F})^2). \tag{3.22}$$

Substituting (3.20)-(3.22) into Eq. (3.19) and setting the coefficients of $F^j(j = 0, -1, -2)$ to zero, we obtain

$$-c(n+1)A_0 + a(n+1)A_0^2 = 0, \tag{3.23}$$

$$-c(n+1)A_1F' + 2a(n+1)A_0A_1F' - bA_1F'' = 0, \tag{3.24}$$

$$a(n+1)A_1^2F'^2 + bA_1F'^2 = 0. \tag{3.25}$$

Eqs. (3.23) and (3.25) directly imply following solutions:

$$A_0 = \frac{c}{a}, \ A_1 = -\frac{b}{a(n+1)}.$$

Thus, Eq. (3.24) becomes

$$c(n+1)F' - bF'' = 0. \tag{3.26}$$

The general solution of Eq. (3.26) is

$$F(\xi) = a_0 + a_1e^{\frac{c(n+1)}{b}\xi}. \tag{3.27}$$

where $a_i(i = 0, 1)$ are arbitrary constants.
Thus, we have

$$V(\xi) = \frac{c}{a} - \frac{b}{a(n+1)}\left(\frac{a_1\frac{c(n+1)}{b}e^{\frac{c(n+1)}{b}\xi}}{a_0 + a_1e^{\frac{c(n+1)}{b}\xi}}\right)$$

. Now using $U = V^{-\frac{1}{n+1}}$ we have:

$$U(\xi) = [\frac{c}{a} - \frac{b}{a(n+1)}(\frac{a_1 \frac{c(n+1)}{b} e^{\frac{c(n+1)}{b}\xi}}{a_0 + a_1 e^{\frac{c(n+1)}{b}\xi}})]^{-\frac{1}{n+1}} \tag{3.28}$$

If we set $a_0 = a_1 = 1$, $n = 2$ in (3.28), then solution of (3.15) is obtained as

$$u(x,t) = \frac{1}{[\frac{c}{2a}(1 - \tanh(\frac{3c}{2b})(x - ct))]^{\frac{1}{3}}}$$

4. Conclusion

In this paper, the modified simplest equation method is applied successfully for solving the $B(n,1)$ burgers equation and the $B(-n,1)$ burgers equation. The results show that this method is efficient in finding the exact solutions of nonlinear differential equations.

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