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A new total variation diminishing implicit nonstandard finite difference scheme for conservation laws

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Abstract In this paper, a new implicit nonstandard finite difference scheme for conservation laws preserving total variation diminishing (TVD) property, is proposed. This scheme is derived by using nonlocal approximation for nonlinear terms of partial differential equation. Schemes preserving the essential physical property, such as TVD are of great importance in practice. Such schemes are free of spurious oscillations around discontinuities. Numerical results for Burgers' equation are presented. Comparison of numerical results with a classical difference scheme is given.

Keywords. Nonstandard finite difference scheme, Total variation diminishing, Conservation law, Nonlocal approximation.

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1. Introduction

The general setting of this work is conservation laws in the form

$$U_t + (f(U))_x = 0, \quad x \in \mathbb{R}, \quad t \in [0, T], \qquad U(x, 0) = U_0(x), \tag{1.1}$$

where $f(U)$ is the nonlinear flux function. The equation (1.1) describe the behavior of many different physical phenomena. For example, in theory of fluid flow, the equations of motion, continuity and energy can be combined into one conservation equation of the form (1.1). As typical for partial differential equations, problem (1.1) can not be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions. But, it is crucial to design numerical methods which replicate essential physical properties of the solutions [9]. This motivates the following concept of stability [2]:

Definition 1. Assume that the solution of (1.1) satisfies some property (P). A numerical method approximating (1.1) is called qualitatively stable with respect to (P) or P-stable if the numerical solutions satisfy property (P) for all values of the involved step sizes.

For problems with smooth solutions, usually a linear stability analysis is adequate. For problems with discontinuous solutions, however, such as solutions to (1.1), a stronger measure of stability is usually required. Almost all of the standard procedures yield schemes which are convergent with restriction on the step size. One response to this situation was the initiation by Mickens [9] of a research program for the investigation of new methods for constructing finite difference schemes which are convergent for any step size. These new procedures are called nonstandard finite difference methods [2, 3, 4, 9]. A formal definition is as follows:

Definition 2. A finite difference method for (1.1) is called nonstandard if at least one of the following is met
a) In the discrete derivatives the traditional denominator Δt or Δx is replaced by a nonnegative function $\varphi(\Delta t)$ or $\varphi(\Delta x)$ such that

$$\varphi(z) = z + O(z^2) \ \ as \ \ 0 < z \to 0.$$

b) Nonlinear terms are approximated in a nonlocal way, i.e. by a suitable function of several points of the mesh. For instance, the nonlinear terms U^2 and U^3 can be modelled as in Anguelov and Lubuma [2]:

$$U^2 \approx aU_k^2 + bU_kU_{k+1}, \ a + b = 1, \ a, \ b \in \mathbb{R},$$

$$U^3 \approx aU_k^3 + (1-a)U_k^2U_{k+1}, \ \ a \in \mathbb{R}.$$

One of the main advantages of the nonstandard finite difference methods is that in addition to the usual properties of consistency, stability and hence convergence, they produce numerical solutions which also transfer essential qualitative property of the exact solution. Physical properties, namely, monotonicity, positivity and boundedness have received extensive attention in the design of qualitatively stable nonstandard finite difference schemes [2, 3, 5, 10]. The purpose of this paper is to construct an implicit nonstandard finite difference scheme using nonlocal approximation of nonlinear term, to be total variation diminishing (TVD). The computational complexity of TVD implicit methods is significantly higher particularly when nonlinear functions are involved. Our approach is to use the tool of the nonstandard finite difference method in constructing TVD scheme which have the advantages of being computationally simpler. In section 2, we give some preliminary setting and definitions including Roe numerical flux [12], TVD and nonlocal approximations to be used in the rest of the paper. Section 3 is devoted to the construction of new TVD method. In the section 4, some numerical experiments are given to confirm the validity of our new method.

2. Preliminaries

Following a space discretization, Eq. (1.1) is written as a system of ODEs of the form

$$U_t = L(U), \tag{2.1}$$

where $U = (U_j)$ and $U_j(t) \simeq U(x_j, t)$. We consider the case when the operator L in (2.1) is obtained from special discretization using the Roe numerical flux. More precisely, we have

$$\hat{f}_{j+\frac{1}{2}} = \begin{cases} f(U_j), & \alpha_{j+\frac{1}{2}} \geq 0, \\ f(U_{j+1}), & \alpha_{j+\frac{1}{2}} < 0, \end{cases} \tag{2.2}$$

with

$$\alpha_{j+\frac{1}{2}} = \frac{f(U_{j+1}) - f(U_j)}{U_{j+1} - U_j},$$

where we assume that the mesh in the space dimension is uniform with a step-size Δx and

$$(L(U))_j = \frac{1}{\Delta x}(\hat{f}_{j-\frac{1}{2}} - \hat{f}_{j+\frac{1}{2}}). \tag{2.3}$$

Let a mesh $t_n = n\Delta t,\ n = 0, 1, ...,$ in the time direction be given. As usual U^n denotes an approximation of U at $t = t_n$. The total variation of U^n is given by

$$TV(U^n) = \sum_j \mid U^n_{j+1} - U^n_j \mid .$$

A numerical scheme is called TVD if $TV(U^n)$ is decreasing with respect to n [1, 11, 13], i.e.,

$$TV(U^n) \geq TV(U^{n+1}) \quad n = 0, 1, 2, ... \tag{2.4}$$

The TVD property is more generally referred to as strong stability preserving (SSP) or monotonicity when norms other that the total variation norm or even sublinear functionals are considered [7].

3. Construction of the new scheme

We construct our numerical scheme for conservation laws using Roe numerical flux and nonlocal approximation of nonlinear terms. The new implicit method is given by

$$U^{n+1}_j = \begin{cases} U^n_j - \frac{\Delta t}{\Delta x}\left(\frac{f(U^n_j)-f(U^n_{j-1})}{U^n_j - U^n_{j-1}}\right)(U^{n+1}_j - U^{n+1}_{j-1}), & \text{if } \alpha_{j+\frac{1}{2}} \geq 0,\ \ \alpha_{j-\frac{1}{2}} \geq 0, \\ U^n_j - \frac{\Delta t}{\Delta x}\left(\frac{f(U^n_{j+1})-f(U^n_j)}{U^n_{j+1} - U^n_j}\right)(U^{n+1}_{j+1} - U^{n+1}_j), & \text{if } \alpha_{j+\frac{1}{2}} < 0,\ \ \alpha_{j-\frac{1}{2}} < 0. \end{cases} \tag{3.1}$$

The goal of using nonlocal approximation in the nonstandard discretization of $L(U)$ is to obtain a numerical scheme that preserve TVD property. This fact will be reflected in Theorem 1. The TVD property of numerical methods is often proved by using interesting research due to Harten [5]. Here we give a version dealing with the implicit cases.

Lemma 1 (Harten).

If an implicit scheme can be written as

$$U_j^{n+1} = U_j^n + C_{j+\frac{1}{2}}(U_{j+1}^{n+1} - U_j^{n+1}) - D_{j-\frac{1}{2}}(U_j^{n+1} + U_{j-1}^{n+1}), \tag{3.2}$$

with $C_{j+\frac{1}{2}} \geq 0$, $D_{j-\frac{1}{2}} \geq 0$ then it is TVD.

Theorem 1. The scheme (3.1) is qualitatively stable with respect to the TVD property (2.4) without any time step restriction.

Proof.
(i). When $\alpha_{j+\frac{1}{2}} \geq 0$, the scheme can be represented in the form (3.2) with

$$D_{j-\frac{1}{2}} = \frac{\Delta t}{\Delta x}\Big(\frac{f(U_j^n) - f(U_{j-1}^n)}{U_j^n - U_{j-1}^n}\Big),$$

$$C_{j+\frac{1}{2}} = 0.$$

Using the definition of $\alpha_{j+\frac{1}{2}}$, it can be easily seen that $D_{j-\frac{1}{2}} \geq 0$.

(ii). For $\alpha_{j+\frac{1}{2}} < 0$, one can easily obtain that

$$C_{j+\frac{1}{2}} = -\frac{\Delta t}{\Delta x}\Big(\frac{f(U_{j+1}^n) - f(U_j^n)}{U_{j+1}^n - U_j^n}\Big),$$

$$D_{j-\frac{1}{2}} = 0.$$

Hence it follows from Lemma 1 that the scheme (3.1) is TVD. □

Remark 1. *Since the scheme (3.1) is TVD, The convergence follows from [8, Theorem 15.2].*

Remark 2. *We have considered nonlocal approximation of the function L for deriving nonstandard TVD schemes for (2.1). The proposed method is of Euler type:*

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x}(\hat{f}_{j-\frac{1}{2}} - \hat{f}_{j+\frac{1}{2}}),$$

therefore it is of order one.

Remark 3. *We should note that, in this paper, we have discussed a nonstandard finite difference method, which is qualitatively stable with respect to the TVD property. It has been shown that new scheme with such qualitative stability resolve discontinuities in the solution without spurious oscillations. Furthermore, the new scheme require no restriction on the time step-size.*

4. Numerical results

In this section we present some results of numerical computations using nonstandard finite difference scheme proposed in the previous section. Here, we corroborate the properties of new scheme for Burgers' equation:

$$U_t + \left(\frac{1}{2}U^2\right)_x = 0. \tag{4.1}$$

It is well known that the entropy solution of this equation develops discontinuities (shocks) even for smooth initial condition. To simplify the matters we take

$$U(x,0) = \begin{cases} 1.2, & \text{if } -1 \le x < 0, \\ 0, & \text{if } \quad 0 \le x \le 1. \end{cases} \tag{4.2}$$

It was shown in [11, 7] that non-TVD methods typically produce oscillations around the points of discontinuity. Figure 1 shows such oscillations produced by the standard Euler method applied to problem (4.1)-(4.2). Also, Figure 2 shows the numerical solution of the problem (4.1)-(4.2) by the new implicit scheme for $\Delta x = 0.2$ with different value Δt. Our final time is taken as $t_f = 5$. The solid line is the exact solution $U(x,t)$, the points joined by a dashed line are numerical solutions. We can see that our scheme is able to produce an accurate solution by decreasing the step sizes.

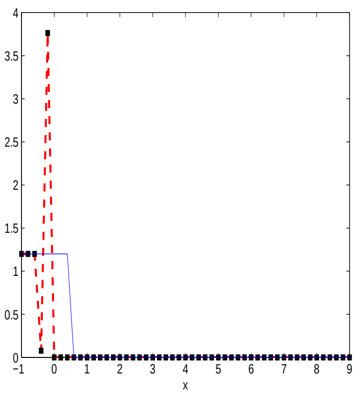

FIGURE 1. Numerical solution of the Burgers' equation with initial condition (4.2) given by the standard Euler method using Roe flux with $\Delta x = \Delta t = 0.2$ and $t_f = 0.8$.

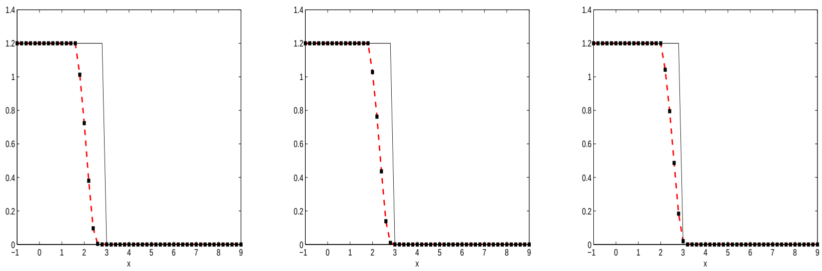

FIGURE 2. Numerical solution of the Burgers' equation with initial condition (4.2) given by the new scheme with $\Delta t = 1$ (left), with $\Delta t = 0.5$ (center), and $\Delta t = 0.2$ (right).

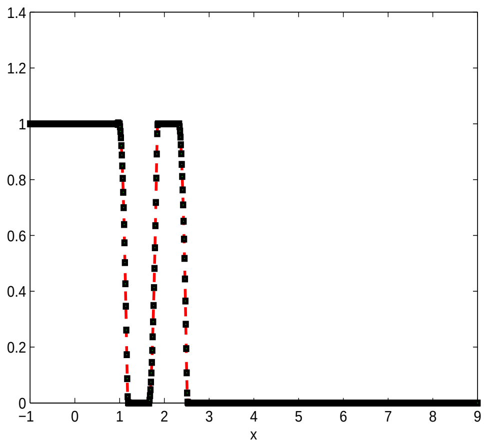

FIGURE 3. Numerical solution of the Burgers' equation with initial condition (4.3) given by the new scheme with $\Delta x = \Delta t = 0.01$ and $t_f = 5$.

The next numerical result of the new scheme is obtained for the Burger's equation with a square wave initial condition:

$$U(x,0) = \begin{cases} 1, & \text{if } -1 \leq \mid x \mid \leq \frac{1}{3}, \\ 0, & \text{if } \quad \frac{1}{3} < \mid x \mid \leq 1. \end{cases} \tag{4.3}$$

Figure 3 shows that our proposed scheme produces smooth and nonoscillatory solution for $\Delta t = \Delta x = 0.01$ and $t_f = 5$. The evolution of the total variation of U^n is shown in Figure 4, for the output times $t_f = T$ with $T = 1, 2, ..., 5$ revealing a decreasing.

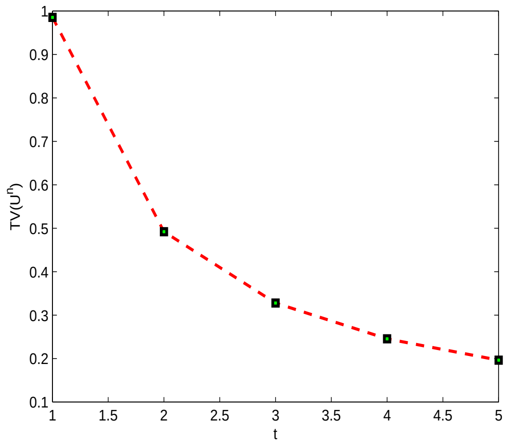

FIGURE 4. Values of $TV(U^n)$ for $T = 1, 2, ..., 5$.

5. Conclusion

In this paper, we constructed a new implicit nonstandard finite difference method based on nonlocal approximation of nonlinear terms. In particular, the proposed method uses Roe numerical flux. The power of our scheme over the standard one is that it is reliable numerical simulation that preserve the linear stability and TVD property of the exact solution. Our interest for future is to establish the positivity property of proposed nonstandard method, since we have numerical evidence that positivity ensured when the new scheme applied to positive systems.

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