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A High Order Approximation of the Two Dimensional Acoustic Wave Equation with Discontinuous Coefficients

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Abstract This paper concerns with the modeling and construction of a fifth order method for two dimensional acoustic wave equation in heterogenous media. The method is based on a standard discretization of the problem on smooth regions and a nonstandard method for nonsmooth regions. The construction of the nonstandard method is based on the special treatment of the interface using suitable jump conditions. We derive the required linear systems for evaluation of the coefficients of such a nonstandard method. The given novel modeling provides an overall fifth order numerical model for two dimensional acoustic wave equation with discontinuous coefficients.

Keywords. Interface methods, two dimensional acoustic wave equation, high order methods, Lax-Wendroff method, WENO, discontinuous coefficients, jump conditions.

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1. Introduction

Wave propagation in heterogenous media has many useful applications. This problem arises in the case that some physical quantities confront abrupt changes in the solution domain. Maxwell equations in a media that the conductivity is a piecewise constant function, propagation of acoustic waves in a media (water and weather) with different densities are examples of applications of this problem. The same author have developed a similar method for one way wave equation[2]. In this paper two dimensional acoustic wave equation with discontinuous coefficients is considered. There are many papers in the literature that deal with this problem with various techniques [3, 4, 5, 9]. High order methods for such problems are very important to provide some more accurate solutions with less evaluation costs. So, here the Lax-Wendroff method for time stepping to obtain a high order method in both space and time is used. Along with formulating a fifth order method for smooth regions, based on any standard fifth order method, for nonsmooth regions, i.e. irregular points, a special formula of the same order is obtained to have a fifth order method over the entire domain of solution. For smooth regions the Lax-Wendroff time evolution with WENO reconstruction for space derivatives [7] is used. The paper is organized as follows: In Section 2, the general model problem and related jump conditions are introduced. The approximating formulas at the interface are explained in Section 3, and finally the high order jump conditions are given in Section 4.

2. 2D Acoustic Wave Equation

In numerical treatment of two dimensional wave equation it is more appropriate to consider the following two dimensional form

$$U_t + AU_x + BU_y = 0, \tag{2.1}$$

where

$$U(x,y,t) = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho} \\ 0 & \kappa & 0 \end{pmatrix},$$

here p is the acoustic pressure, u and v are the components of acoustic velocity in x and y directions, respectively. The coefficient matrices $A(\mathbf{x})$ and $B(\mathbf{x})$ are functions of position consisting of physical quantities such as density $\rho(\mathbf{x})$, sound speed $c(\mathbf{x})$ and bulk of modulus $\kappa = \rho c^2$. It is supposed that the density and sound speeds are piecewise constant functions and have a jump discontinuity at the point of $F_{int}(x,y) = 0$, where we call it interface,

$$(\rho, c) = \begin{cases} (\rho^-, c^-), & F_{int}(x,y) < 0, \\ (\rho^+, c^+), & F_{int}(x,y) > 0. \end{cases} \tag{2.2}$$

The interface $F_{int}(x,y) = 0$ is a curve in $x - y$ plane and for the sake of simplicity we consider it to be linear. We use the following jump conditions to find unique solution of the problem (2.1).

$$[\vec{n}.\vec{u}] = [\vec{p}] = 0, \tag{2.3}$$

where $\vec{n}$ is the unit vector normal to the interface. We assume that $u_y - v_x = 0$, i.e., deformation is irrotational.

A standard method for updating numerical solution at the grid ij takes the following form:

$$U_{ij}^{n+1} = U_{ij} + \text{updating terms}. \tag{2.4}$$

We note that all terms on the right hand side are evaluated at time level n, and we drop it for simplicity in the subsequent discussion. For a fifth order method we will use a 21-point grid method

$$U_{ij}^{n+1} = U_{ij} + \sum_{l=1}^{21} \Gamma_{ij,l} U_{ij,l} \tag{2.5}$$

the $\Gamma_{ij,l}$, $l = 1, 2, \ldots, 21$ matrices can be determined with the Lax-Wendroff method. However, we can construct a fifth order WENO-Lax-Wendroff method, where in this case we have to start with a simple Taylor series in time,

$$U_{ij}^{n+1} = U_{ij} + kU'_{ij} + \frac{k^2}{2} U''_{ij} + \cdots + \frac{k^5}{5} U_{ij}^{(5)}. \tag{2.6}$$

Following[7] we approximate the first derivative $U' = -AU_x - BU_y$ using WENO5 (WENO order 5) and then for $U'', \ldots, U^{(5)}$ we use appropriate finite differences of orders $4, 4, 2, 2$ respectively. We concentrate on the point (x_i, y_j) and consider the point (x_0, y_0) on the interface to be near the point ij and define the following transformation which specifies a new coordinates system $\xi - \eta$ where ξ and η are, respectively, the new variables in the tangential and normal directions to the interface.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{2.7}$$

where β is the rotation angle. Now we can write the equation(2.1) in the new coordinate system

$$\bar{U}(\xi, \eta, t) = \begin{pmatrix} \bar{u}(\xi, \eta, t) \\ \bar{v}(\xi, \eta, t) \\ p(\xi, \eta, t) \end{pmatrix}, \; Q_0 = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.8}$$

$$U(x, y, t) = Q_0 \bar{U}(\xi, \eta, t). \tag{2.9}$$

Substituting in (2.1) we obtain

$$\bar{U}_t + A\bar{U}_\xi + B\bar{U}_\eta = 0. \tag{2.10}$$

The jump conditions (2.3) now become [6],

$$[\bar{u}] = 0, \quad [\rho\bar{v}] = 0, \quad [p] = 0. \tag{2.11}$$

3. Approximation at the interface

At irregular points, where some of the grid points in the stencil are on opposite sides of the interface, the ordinary classical fifth order method is no longer valid. In order to obtain a fifth order method, we have to force the coefficients of the following derivatives to zero:

$$U, U_x, U_y, U_{xx}, U_{xy}, U_{yy}, U_{xxx}, U_{xxy}, U_{xyy}, U_{yyy}, U_{xxxx}, U_{xxxy}, U_{xxyy}, U_{xyyy}, U_{yyyy}, U_{xxxxx}, U_{xxxxy}, U_{xxxyy}, U_{xxyyy}, U_{xyyyy}, U_{yyyyy}. \tag{3.1}$$

This means that at least a 21-point stencil is required. It is possible to choose different 21-point stencils but we prefer a symmetric stencil in computations. One of the choices of these 21 points can be the dots illustrated in Figure 1. At these points a 21-point difference scheme of the form

$$U_{ij}^{n+1} = U_{ij} + \sum_{l=1}^{21} \Gamma_{ij,l} U_{ij,l} \tag{3.2}$$

is used. Now the Γ's are 3×3 unknown matrices, and the $U_{ij,l}, l = 1, 2 \ldots, 21$ are the solution values at grid points in the stencil containing the point ij (see Figure 1).

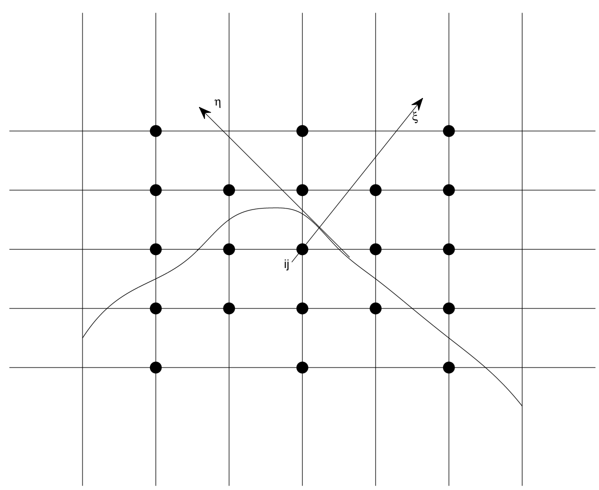

FIGURE 1. 2D interface, new coordinates and a typical 21-point stencil.

In order to obtain the Γ matrices so that the method becomes fifth order we consider the local truncation error at the point ij:

$$\begin{aligned} L =&\frac{1}{k}\sum_{l=1}^{21}\Gamma_{ij,l}U_l - (U_t + \frac{1}{2}kU_{tt} + \frac{1}{6}k^2U_{ttt} + \frac{1}{24}k^3U_{tttt} \\ &+ \frac{1}{120}k^4U_{ttttt})_{ij} + O(k^5), \end{aligned} \tag{3.3}$$

which will be formulated in the new coordinates system. The time derivatives in the new coordinates systems are

$$\begin{aligned} U_t =&Q_0\bar{U}_t = -Q_0(A\bar{U}_\xi + B\bar{U}_\eta) \\ U_{tt} =&Q_0\bar{U}_{tt} = c^2Q_0(\bar{U}_{\xi\xi} + \bar{U}_{\eta\eta}) \\ U_{ttt} =&Q_0\bar{U}_{ttt} = -c^2Q_0((A\bar{U}_\xi + B\bar{U}_\eta)_{\xi\xi} + (A\bar{U}_\xi + B\bar{U}_\eta)_{\eta\eta}) \\ U_{tttt} =&Q_0\bar{U}_{tttt} = c^4Q_0(\bar{U}_{\xi\xi\xi\xi} + 2\bar{U}_{\xi\xi\eta\eta} + \bar{U}_{\eta\eta\eta\eta}) \\ U_{ttttt} =&Q_0\bar{U}_{ttttt} = -c^4Q_0(A\bar{U}_{\xi\xi\xi\xi\xi} + B\bar{U}_{\xi\xi\xi\xi\eta} + 2A\bar{U}_{\xi\xi\xi\eta\eta} + 2B\bar{U}_{\xi\xi\eta\eta\eta} \\ &\qquad + A\bar{U}_{\xi\eta\eta\eta\eta} + B\bar{U}_{\eta\eta\eta\eta\eta}), \end{aligned} \tag{3.4}$$

and so the summation in (3.3) reads,

$$\sum_{l=1}^{21}\Gamma_{ij,l}U_l = \sum_{l=1}^{21}\Gamma_{ij,l}Q_0\bar{U}_{ij,l}. \tag{3.5}$$

It is now convenient to use the following new notations

$$\Gamma_l = Q_0^{-1}\Gamma_{ij,l}Q_0, \quad U_l = \bar{U}_{ij,l}. \tag{3.6}$$

Now from (3.6) and (3.5), the local truncation error (3.3) in the new coordinates system reads,

$$\begin{aligned} L =&Q_0\{\frac{1}{k}\sum_{l=1}^{20}\Gamma_l U_l - (\bar{U}_t + \frac{1}{2}k\bar{U}_{tt} + \frac{1}{6}k^2\bar{U}_{ttt} + \frac{1}{24}k^3\bar{U}_{tttt} \\ &+ \frac{1}{120}k^4\bar{U}_{ttttt})_{ij}\} + O(k^5). \end{aligned} \tag{3.7}$$

We now expand U_l's about the point (x_0, y_0), and depending on the location of the l^{th} grid point to the $-$ or $+$ side of the interface, shown by superscript

"*", will be denoted by superscript "+" or "-", respectively.

$$
\begin{aligned}
U_l = U^* &+ (\xi_l U^*_\xi + \eta_l U^*_\eta) + \frac{1}{2}(\xi_l^2 U^*_{\xi\xi} + 2\xi_l\eta_l U^*_{\xi\eta} + \eta_l^2 U^*_{\eta\eta}) \\
&+ \frac{1}{6}(\xi_l^3 U^*_{\xi\xi\xi} + 3\xi_l^2\eta_l U^*_{\xi\xi\eta} + 3\xi_l\eta_l^2 U^*_{\xi\eta\eta} + \eta_l^3 U^*_{\eta\eta\eta}) \\
&+ \frac{1}{24}(\xi_l^4 U^*_{\xi\xi\xi\xi} + 4\xi_l^3\eta_l U^*_{\xi\xi\xi\eta} + 6\xi_l^2\eta_l^2 U^*_{\xi\xi\eta\eta} \\
&+ 4\xi_l\eta_l^3 U^*_{\xi\eta\eta\eta} + \eta_l^4 U^*_{\eta\eta\eta\eta}) \\
&+ \frac{1}{120}(\xi_l^5 U^*_{\xi\xi\xi\xi\xi} + 5\xi_l^4\eta_l U^*_{\xi\xi\xi\xi\eta} + 10\xi_l^3\eta_l^2 U^*_{\xi\xi\xi\eta\eta} \\
&+ 10\xi_l^2\eta_l^3 U^*_{\xi\xi\eta\eta\eta} + 5\xi_l\eta_l^4 U^*_{\xi\eta\eta\eta\eta} + \eta_l^5 U^*_{\eta\eta\eta\eta\eta}) + O(k^6).
\end{aligned} \tag{3.8}
$$

Similarly we expand the time derivatives about (x_0, y_0)

$$
\begin{aligned}
\bar{U}_t =& - AU^*_\xi - BU^*_\eta - (\xi_c(AU^*_\xi + BU^*_\eta)_\xi + \eta_c(AU^*_\xi + BU^*_\eta)_\eta) \\
&- (\xi_c^2(AU^*_\xi + BU^*_\eta)_{\xi\xi} + 2\xi_c\eta_c(AU^*_\xi + BU^*_\eta)_{\xi\eta} \\
&+ \eta_c^2(AU^*_\xi + BU^*_\eta)_{\eta\eta}) \\
&- (\xi_c^3(AU^*_\xi + BU^*_\eta)_{\xi\xi\xi} + 3\xi_c^2\eta_c(AU^*_\xi + BU^*_\eta)_{\xi\xi\eta} \\
&+ 3\xi_c\eta_c^2(AU^*_\xi + BU^*_\eta)_{\xi\eta\eta} + \eta_c^3(AU^*_\xi + BU^*_\eta)_{\eta\eta\eta}) \\
&- (\xi_c^4(AU^*_\xi + BU^*_\eta)_{\xi\xi\xi\xi} + 4\xi_c^3\eta_c(AU^*_\xi + BU^*_\eta)_{\xi\xi\xi\eta} \\
&+ 6\xi_c^2\eta_c^2(AU^*_\xi + BU^*_\eta)_{\xi\xi\eta\eta} + 4\xi_c\eta_c^3(AU^*_\xi + BU^*_\eta)_{\xi\eta\eta\eta} \\
&+ \eta_c^4(AU^*_\xi + BU^*_\eta)_{\eta\eta\eta\eta}) + O(k^5) \\
\bar{U}_{tt} =& c^2((U^*_{\xi\xi} + U^*_{\eta\eta}) + \xi_c(U^*_{\xi\xi} + U^*_{\eta\eta})_\xi + \eta_c(U^*_{\xi\xi} + U^*_{\eta\eta})_\eta \\
&+ \xi_c^2(U^*_{\xi\xi} + U^*_{\eta\eta})_{\xi\xi} + 2\xi_c\eta_c(U^*_{\xi\xi} + U^*_{\eta\eta})_{\xi\eta} \\
&+ \eta_c^2(U^*_{\xi\xi} + U^*_{\eta\eta})_{\eta\eta} + \xi_c^3(U^*_{\xi\xi} + U^*_{\eta\eta})_{\xi\xi\xi} \\
&+ 3\xi_c^2\eta_c(U^*_{\xi\xi} + U^*_{\eta\eta})_{\xi\xi\eta} \\
&+ 3\xi_c\eta_c^2(U^*_{\xi\xi} + U^*_{\eta\eta})_{\xi\eta\eta} \\
&+ \eta_c^3(U^*_{\xi\xi} + U^*_{\eta\eta})_{\eta\eta\eta} + O(k^4) \\
\bar{U}_{ttt} =& - c^2((AU^*_\xi + BU^*_\eta)_{\xi\xi} + (AU^*_\xi + BU^*_\eta)_{\eta\eta} \\
&+ \xi_c((AU^*_\xi + BU^*_\eta)_{\xi\xi} + (AU^*_\xi + BU^*_\eta)_{\eta\eta})_\xi
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
&+ \eta_c((AU^*_\xi + BU^*_\eta)_{\xi\xi} + (AU^*_\xi + BU^*_\eta)_{\eta\eta})_\eta \\
&+ \xi_c^2((AU^*_\xi + BU^*_\eta)_{\xi\xi} + (AU^*_\xi + BU^*_\eta)_{\eta\eta})_{\xi\xi} \\
&+ 2\xi_c\eta_c((AU^*_\xi + BU^*_\eta)_{\xi\xi} + (AU^*_\xi + BU^*_\eta)_{\eta\eta})_{\xi\eta} \\
&+ \eta_c^2((AU^*_\xi + BU^*_\eta)_{\xi\xi} + (AU^*_\xi + BU^*_\eta)_{\eta\eta})_{\eta\eta}) + O(k^3)
\end{aligned}
$$

$$
\begin{aligned}
\bar{U}_{tttt} =& c^4\Big(U^*_{\xi\xi\xi\xi} + 2U^*_{\xi\xi\eta\eta} + U^*_{\eta\eta\eta\eta} \\
&+ \xi_c(U^*_{\xi\xi\xi\xi} + 2U^*_{\xi\xi\eta\eta} + U^*_{\eta\eta\eta\eta})_\xi \\
&+ \eta_c(U^*_{\xi\xi\xi\xi} + 2U^*_{\xi\xi\eta\eta} + U^*_{\eta\eta\eta\eta})_\eta\Big) + O(k^2) \\
\bar{U}_{ttttt} =& -c^4(AU^*_{\xi\xi\xi\xi\xi} + BU^*_{\xi\xi\xi\xi\eta} + 2AU^*_{\xi\xi\xi\eta\eta} + 2BU^*_{\xi\xi\eta\eta\eta} \\
&+ AU^*_{\xi\eta\eta\eta\eta} + BU^*_{\eta\eta\eta\eta\eta}) + O(k).
\end{aligned}
$$

Inserting all of these relations into (3.7) we obtain the local truncation error as a function of 42 values $U^\pm, U^\pm_\eta, \ldots, U^\pm_{\xi\xi\xi\xi\xi}$ in the both sides of the interface. We can now use the jump conditions (3.10), see the next section, to represent the local truncation error with respect to 21 values $U^-, U^+_\eta, \ldots, U^-_{\xi\xi\xi\xi\xi}$.

$$
\begin{aligned}
U^+ =& Q_1U^- \\
U^+_\eta =& Q_1U^-_\eta \\
U^+_\xi =& Q_2U^-_\xi + Q_3U^-_\eta \\
U^+_{\eta\eta} =& Q_1U^-_{\eta\eta} \\
U^+_{\xi\xi} =& (\frac{c^-}{c^+})^2Q_1U^-_{\xi\xi} + ((\frac{c^-}{c^+})^2 - 1)Q_1U^-_{\eta\eta} \\
U^+_{\xi\eta} =& Q_2U^-_{\xi\eta} + Q_3U^-_{\eta\eta} \\
U^+_{\eta\eta\eta} =& Q_1U^-_{\eta\eta\eta} \\
U^+_{\xi\eta\eta} =& Q_2U^-_{\xi\eta\eta} + Q_3U^-_{\eta\eta\eta} \\
U^+_{\xi\xi\eta} =& (\frac{c^-}{c^+})^2Q_1U^-_{\xi\xi\eta} + ((\frac{c^-}{c^+})^2 - 1)Q_1U^-_{\eta\eta\eta} \\
U^+_{\xi\xi\xi} =& Q_4U^-_{\xi\xi\xi} + Q_5U^-_{\xi\xi\eta} + Q_6U^-_{\xi\eta\eta} + Q_7U^-_{\eta\eta\eta} \\
U^+_{\eta\eta\eta\eta} =& Q_1U^-_{\eta\eta\eta\eta} \\
U^+_{\xi\eta\eta\eta} =& Q_2U^-_{\xi\eta\eta\eta} + Q_3U^-_{\eta\eta\eta\eta} \\
U^+_{\xi\xi\eta\eta} =& (\frac{c^-}{c^+})^2Q_1U^-_{\xi\xi\eta\eta} + ((\frac{c^-}{c^+})^2 - 1)Q_1U^-_{\eta\eta\eta\eta} \\
U^+_{\xi\xi\xi\eta} =& Q_4U^-_{\xi\xi\xi\eta} + Q_5U^-_{\xi\xi\eta\eta} + Q_6U^-_{\xi\eta\eta\eta} + Q_7U^-_{\eta\eta\eta\eta}
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
U^{+}_{\xi\xi\xi\xi} =& U^{-}_{\xi\xi\xi\xi} + 2(\frac{c^-}{c^+})^2((\frac{c^-}{c^+})^2 - Q_1)U^{-}_{\xi\xi\eta\eta} \\
& + ((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2 Q_1 + Q_1)U^{-}_{\eta\eta\eta\eta} \\
U^{+}_{\eta\eta\eta\eta\eta} =& Q_1 U^{-}_{\eta\eta\eta\eta\eta}
\end{aligned}
$$

$$
\begin{aligned}
U^{+}_{\xi\eta\eta\eta\eta} =& Q_2 U^{-}_{\xi\eta\eta\eta\eta} + Q_3 U^{-}_{\eta\eta\eta\eta\eta} \\
U^{+}_{\xi\xi\eta\eta\eta} =& (\frac{c^-}{c^+})^2 Q_1 U^{-}_{\xi\xi\eta\eta\eta} + ((\frac{c^-}{c^+})^2 - 1)Q_1 U^{-}_{\eta\eta\eta\eta\eta} \\
U^{+}_{\xi\xi\xi\eta\eta} =& Q_4 U^{-}_{\xi\xi\xi\eta\eta} + Q_5 U^{-}_{\xi\xi\eta\eta\eta} + Q_6 U^{-}_{\xi\eta\eta\eta\eta} + Q_7 U^{-}_{\eta\eta\eta\eta\eta} \\
U^{+}_{\xi\xi\xi\xi\eta} =& U^{-}_{\xi\xi\xi\xi\eta} + 2(\frac{c^-}{c^+})^2((\frac{c^-}{c^+})^2 - Q_1)U^{-}_{\xi\xi\eta\eta\eta} \\
& + ((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2 Q_1 + Q_1)U^{-}_{\eta\eta\eta\eta\eta} \\
U^{+}_{\xi\xi\xi\xi\xi} =& Q_2 U^{-}_{\xi\xi\xi\xi\xi} + (Q_9 + Q_{10})U^{-}_{\xi\xi\xi\eta\eta} \\
& + Q_8 U^{-}_{\xi\eta\eta\eta\eta} + Q_{11} U^{-}_{\eta\eta\eta\eta\eta},
\end{aligned}
$$

where the Q matrices are defined as follows

$$
\begin{aligned}
Q_1 =& \operatorname{diag}(1, \rho^-/\rho^+, 1) \\
Q_2 =& \operatorname{diag}(\kappa^-/\kappa^+, 1, \rho^+/\rho^-) \\
Q_3 =& (\frac{\rho^-}{\rho^+})((\frac{c^-}{c^+})^2 - 1)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
Q_4 =& \operatorname{diag}(\frac{\kappa^-}{\kappa^+}, 0, \frac{\rho^+}{\rho^-}(\frac{c^-}{c^+})^2) \\
Q_5 =& (\frac{c^-}{c^+})^2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
Q_6 =& \frac{\rho^+}{\rho^-}((\frac{c^-}{c^+})^2 - 1)\operatorname{diag}(0, 0, 1) \\
Q_7 =& ((\frac{c^-}{c^+})^2 - 1)\begin{pmatrix} 0 & \frac{\rho^-}{\rho^+} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{3.11}
$$

$$Q_8 =((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2 + 1)\begin{pmatrix} 3\frac{\kappa^-}{\kappa^+} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\rho^+}{\rho^-} \end{pmatrix}$$

$$Q_9 =2(\frac{c^-}{c^+})^2((\frac{c^-}{c^+})^2 - 1)\begin{pmatrix} 3\frac{\kappa^-}{\kappa^+} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\rho^+}{\rho^-} \end{pmatrix}$$

$$Q_{10} = \begin{pmatrix} \frac{\kappa^-}{\kappa^+} - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Q_{11} =\Big(\frac{\kappa^-}{\kappa^+}\Big((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2 + 1\Big) - \Big((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2\frac{\rho^-}{\rho^+} + \frac{\rho^-}{\rho^+}\Big)\Big)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now to obtain a fifth order method we force the coefficients of $U^-, U_\eta^+, \dots, U^-_{\xi\xi\xi\xi\xi}$ in the local truncation error to become zero. So we find the following linear systems

$$\sum_{l=1}^{21} \Gamma_l \hat{Q}_{il} = F_i, \quad (i = 1, 2, \dots, 21), \tag{3.12}$$

where the coefficient matrices $\hat{Q}$, when both the l point and the center point ij are located at the same side of the interface, are

$$\begin{aligned}
&\hat{Q}_{1l} = I, \quad \hat{Q}_{2l} = \xi_l I, \quad \hat{Q}_{3l} = \eta_l I, \quad \hat{Q}_{4l} = \xi_l^2 I, \\
&\hat{Q}_{5l} = 2\xi_l\eta_l I, \quad \hat{Q}_{6l} = \eta_l^2 I, \\
&\hat{Q}_{7l} = \xi_l^3 I, \quad \hat{Q}_{8l} = 3\xi_l^2\eta_l I, \quad \hat{Q}_{9l} = 3\xi_l\eta_l^2 I, \quad \hat{Q}_{10,l} = \eta_l^3 I, \\
&\hat{Q}_{11,l} = \xi_l^4 I, \quad \hat{Q}_{12,l} = 4\xi_l^3\eta_l I, \quad \hat{Q}_{13,l} = 6\xi_l^2\eta_l^2 I, \quad \hat{Q}_{14,l} = 4\xi_l\eta_l^3 I, \\
&\hat{Q}_{15,l} = \eta_l^4 I, \quad \hat{Q}_{16,l} = \xi_l^5 I, \quad \hat{Q}_{1,l} = 5\xi_l^4\eta_l I, \quad \hat{Q}_{18,l} = 10\xi_l^3\eta_l^2 I, \\
&\hat{Q}_{19,l} = 10\xi_l^2\eta_l^3 I, \quad \hat{Q}_{20,l} = 5\xi_l\eta_l^4 I, \quad \hat{Q}_{21,l} = \eta_l^5 I,
\end{aligned} \tag{3.13}$$

and are

$$
\begin{aligned}
\hat{Q}_{1l} =& Q_1,\\
\hat{Q}_{2l} =& \xi_l Q_2,\\
\hat{Q}_{3l} =& \eta_l Q_1 + \xi_l Q_3,\\
\hat{Q}_{4l} =& \xi_l^2 (\frac{c^-}{c^+})^2 Q_1,\\
\hat{Q}_{5l} =& 2\xi_l \eta_l Q_2,\\
\hat{Q}_{6l} =& \eta_l^2 Q_1 + 2\xi_l \eta_l Q_3 + \xi_l^2 ((\frac{c^-}{c^+})^2 - 1) Q_1,\\
\hat{Q}_{7l} =& \xi_l^3 Q_4,\\
\hat{Q}_{8l} =& \xi_l^3 Q_5 + 3\xi_l^2 \eta_l (\frac{c^-}{c^+})^2 Q_1,\\
\hat{Q}_{9l} =& \xi_l^3 Q_6 + 3\xi_l \eta_l^2 Q_2,\\
\hat{Q}_{10,l} =& \xi_l^3 Q_7 + 3\xi_l^2 \eta_l ((\frac{c^-}{c^+})^2 - 1) Q_1 + 3\xi_l \eta_l^2 Q_3 + \eta_l^3 Q_l,\\
\hat{Q}_{11,l} =& \xi_l^4 I,\\
\hat{Q}_{12,l} =& 4\xi_l^3 \eta_l Q_4,\\
\hat{Q}_{13,l} =& 2\xi_l^4 (\frac{c^-}{c^+})^2 ((\frac{c^-}{c^+})^2 - Q_1) + 4\xi_l^3 \eta_l Q_5 + 6\xi_l^2 \eta_l^2 (\frac{c^-}{c^+})^2 Q_1,\\
\hat{Q}_{14,l} =& 4\xi_l^3 \eta_l Q_6 + 4\xi_l \eta_l^3 Q_2,\\
\hat{Q}_{15,l} =& \xi_l^4 ((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2 Q_1 + Q_1) + 4\xi_l^3 \eta_l Q_7\\
&+ 6\xi_l^2 \eta_l^2 ((\frac{c^-}{c^+})^2 - Q_1) Q_1 + 4\xi \eta_l^3 Q_3 + \eta_l^4 Q_1,\\
\hat{Q}_{16,l} =& \xi_l^5 Q_2,\\
\hat{Q}_{17,l} =& 5\xi_l^4 \eta_l I,\\
\hat{Q}_{18,l} =& \xi_l^5 (Q_9 + Q_{10}) + 10\xi_l^3 \eta_l^2 Q_4,\\
\hat{Q}_{19,l} =& 20\xi_l^2 \eta_l^3 (\frac{c^-}{c^+})^2 ((\frac{c^-}{c^+})^2 - Q_1) + 10\xi_l^3 \eta_l^2 (\frac{c^-}{c^+})^2 Q_1,\\
\hat{Q}_{20,l} =& \xi_l^5 Q_8 + 10\xi_l^3 \eta_l^2 Q_6 + 5\xi_l \eta_l^4 Q_2,\\
\hat{Q}_{21,l} =& \xi_l^5 Q_{11} + 5\xi_l^4 \eta_l ((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2 Q_1 + Q_1) + 10\xi_l^3 \eta_l^2 Q_7,\\
&+ 10\xi_l^2 \eta_l^3 ((\frac{c^-}{c^+})^2 - 1) Q_1 + 5\xi_l \eta_l^4 Q_3 + \eta_l^5 Q_1,
\end{aligned}
\tag{3.14}
$$

otherwise.

The $F_i, \quad i = 1, \cdots, 21$, are also as follows

$$
\begin{aligned}
F_1 =& 0,\\
F_2 =& -\nu A,\\
F_3 =& -\nu B,\\
F_4 =& \nu^2 c^2 - 2\nu \xi_c A,\\
F_5 =& -2\nu(\eta_c A + \xi_c B),\\
F_6 =& \nu^2 c^2 - 2\nu \eta_c B,\\
F_7 =& -\nu^3 c^2 A + 3\nu^2 c^2 \xi_c - 6\nu \xi_c^2 A,\\
F_8 =& -\nu^3 c^2 B + 3\nu^2 c^2 \eta_c - 6\nu \xi_c(\xi_c B + 2\eta_c A),\\
F_9 =& -\nu^3 c^2 A + 3\nu^2 c^2 \xi_c - 6\nu \eta_c(\eta_c A + 2\xi_c B),\\
F_{10} =& -\nu^3 c^2 B + 3\nu^2 c^2 \eta_c - 6\nu \eta_c^2 B,\\
F_{11} =& \nu^4 c^4 - 4\nu^3 c^2 \xi_c A + 12\nu^2 c^2 \xi_c^2 - 24\nu \xi_c^3 A,\\
F_{12} =& -\nu^3 c^2(\xi_c B + \eta_c A) + 6\nu^2 c^2 \xi_c \eta_c - 6\nu \xi_c^2(\xi_c B + 3\eta_c A),\\
\\
F_{13} =& 2\nu^4 c^4 - 4\nu^3 c^2(\xi_c A + \eta_c B) + 12\nu^2 c^2(\xi_c^2 + \eta_c^2) - 72\nu \xi_c \eta_c(\xi_c B + \eta_c A),\\
F_{14} =& -\nu^3 c^2(\xi_c B + \eta_c A) + 6\nu^2 c^2 \xi_c^2 \eta_c - 6\nu \eta_c^2(3\xi_c B + \eta_c A),\\
F_{15} =& \nu^4 c^4 - 4\nu^3 c^2 \eta_c B + 12\nu^2 c^2 \eta_c^2 - 24\nu \eta_c^3 B,\\
F_{16} =& -\nu^5 c^4 A + 5\nu^4 c^4 \xi_c - 20\nu^3 c^2 \xi_c^2 A + 60\nu^2 c^2 \xi_c^3 - \nu \xi_c A,\\
F_{17} =& -\nu^5 c^4 B + 5\nu^4 c^4 \eta_c - 20\nu^3 c^2 \xi_c(\xi_c B + 2\eta_c A),\\
& + 180\nu^2 c^2 \xi_c^2 \eta_c - 120\nu \xi_c^3(\xi_c B + 4\eta_c A),\\
F_{18} =& -\nu^5 c^4 A + 20\nu^4 c^4 \xi_c - 10\nu^3 c^2 \eta_c^2 A - 10\nu^3 c^2 \xi_c(\xi_c A + 2\eta_c B)\\
& + 30\nu^2 c^2 \xi_c(\xi_c^2 + 3\eta_c^2) + 30\nu^2 c^2 - 120\xi_c^2 \eta_c(2\xi_c B + 3\eta_c A),\\
F_{19} =& -\nu^5 c^4 B + 5\nu^4 c^4 \eta_c - 10\nu^3 c^2 \eta_c^2 B - 10\nu^3 c^2 \xi_c(\xi_c B + 2\eta_c A),\\
& + 30\nu^2 c^2 \eta_c(3\xi_c^2 + \eta_c^2) - 120\nu \xi_c \eta_c^2(3\xi_c B + 2\eta_c A),\\
F_{20} =& -\nu^5 c^4 A + 5\nu^4 c^4 \xi_c - 20\nu^3 c^2 \eta_c(2\xi_c B + \eta_c A),\\
& + 180\nu^2 c^2 \xi_c \eta_c^2 - 120\nu \eta_c^3(4\xi_c B + \eta_c A),\\
F_{21} =& -\nu^5 c^4 B + 5\nu^4 c^4 \eta_c - 20\nu^3 c^2 \eta_c^2 + 60\nu^2 c^2 \eta_c^3 - 120\eta_c^4 B,
\end{aligned}
\tag{3.15}
$$

where (ξ_c, η_c) represents the coordinates of the center point (x_i, y_j) in transformed system. It is also possible to represent the linear systems (3.12) as a single linear system of equations with 189 unknowns. Taking transpose of both side of (3.12) gives

$$\sum_{l=1}^{21} \hat{Q}^t_{il}\Gamma^t_l = F^t_i, \quad (i = 1, 2, \ldots, 21),$$

which can be assembled to obtain the following single linear system of 189 unknowns

$$\hat{Q}\hat{\Gamma} = \hat{F}. \tag{3.16}$$

4. Jump Conditions

Derivation of special formulas in the interface requires high order jump relations to be imposed in the interface. We use the second order jump conditions in [5]

$$\begin{aligned}
U^+ &= Q_1 U^- \\
U^+_\eta &= Q_1 U^-_\eta \\
U^+_\xi &= Q_2 U^-_\xi + Q_3 U^-_\eta \\
U^+_{\eta\eta} &= Q_1 U^-_{\eta\eta} \\
U^+_{\xi\xi} &= (\frac{c^-}{c^+})^2 Q_1 U^-_{\xi\xi} + ((\frac{c^-}{c^+})^2 - 1) Q_1 U^-_{\eta\eta} \\
U^+_{\xi\eta} &= Q_2 U^-_{\xi\eta} + Q_3 U^-_{\eta\eta},
\end{aligned} \tag{4.1}$$

where Q matrices are defined in the new coordinates system and they were already introduced in (3.14). Differentiating (4.1) with respect to η we obtain,

$$\begin{aligned}
U^+_{\eta\eta\eta} &= Q_1 U^-_{\eta\eta\eta} \\
U^+_{\xi\eta\eta} &= Q_2 U^-_{\xi\eta\eta} + Q_3 U^-_{\eta\eta\eta} \\
U^+_{\xi\xi\eta} &= (\frac{c^-}{c^+})^2 Q_1 U^-_{\xi\xi\eta} + ((\frac{c^-}{c^+})^2 - 1) Q_1 U^-_{\eta\eta\eta} \\
U^+_{\eta\eta\eta\eta} &= Q_1 U^-_{\eta\eta\eta\eta} \\
U^+_{\xi\eta\eta\eta} &= Q_2 U^-_{\xi\eta\eta\eta} + Q_3 U^-_{\eta\eta\eta\eta} \\
U^+_{\xi\xi\eta\eta} &= (\frac{c^-}{c^+})^2 Q_1 U^-_{\xi\xi\eta\eta} + ((\frac{c^-}{c^+})^2 - 1) Q_1 U^-_{\eta\eta\eta\eta} \\
U^+_{\eta\eta\eta\eta\eta} &= Q_1 U^-_{\eta\eta\eta\eta\eta} \\
U^+_{\xi\eta\eta\eta\eta} &= Q_2 U^-_{\xi\eta\eta\eta\eta} + Q_3 U^-_{\eta\eta\eta\eta\eta} \\
U^+_{\xi\xi\eta\eta\eta} &= (\frac{c^-}{c^+})^2 Q_1 U^-_{\xi\xi\eta\eta\eta} + ((\frac{c^-}{c^+})^2 - 1) Q_1 U^-_{\eta\eta\eta\eta\eta}.
\end{aligned} \tag{4.2}$$

So, for a complete set of fifth order jump conditions we just need to provide some similar jump relations to the terms $U_{\xi\xi\xi}$, $U_{\xi\xi\xi\xi}$, $U_{\xi\xi\xi\xi\xi}$:

$$[u_{ttt}] = [c^2(u_{t\xi\xi} + u_{t\eta\eta})] = 0,$$
$$[p_{ttt}] = [c^2(p_{t\xi\xi} + p_{t\eta\eta})] = 0.$$

From (2.1) we obtain

$$[(\frac{c^2}{\rho})(p_{\xi\xi\xi} + p_{\xi\eta\eta})] = 0,$$
$$[\kappa c^2((u_\xi + v_\eta)_{\xi\xi} + (u_\xi + v_\eta)_{\eta\eta})] = 0,$$

which results in

$$\begin{aligned} p^+_{\xi\xi\xi} &= \frac{\rho^+}{\rho^-}(\frac{c^-}{c^+})^2(p^-_{\xi\xi\xi} + p^-_{\xi\eta\eta}) - p^+_{\xi\eta\eta} \\ u^+_{\xi\xi\xi} &= \frac{\kappa^-}{\kappa^+}(\frac{c^-}{c^+})^2(u^-_{\xi\xi\xi} + u^-_{\xi\eta\eta} + v^-_{\xi\xi\eta} + v^-_{\eta\eta\eta}) - u^+_{\xi\eta\eta} + v^+_{\xi\xi\eta} + v^+_{\eta\eta\eta}. \end{aligned} \tag{4.3}$$

Now dropping $p^+_{\xi\eta\eta}, u^+_{\xi\eta\eta}, v^+_{\xi\xi\eta}, v^+_{\eta\eta\eta}$ and using (4.2) gives

$$p^+_{\xi\xi\xi} = \frac{\rho^+}{\rho^-}(\frac{c^-}{c^+})^2 p^-_{\xi\xi\xi} + \frac{\rho^+}{\rho^-}((\frac{c^-}{c^+})^2 - 1)p^-_{\xi\eta\eta} \tag{4.4}$$

$$u^+_{\xi\xi\xi} = \frac{\kappa^-}{\kappa^+}u^-_{\xi\xi\xi} + \frac{\rho^-}{\rho^+}((\frac{c^-}{c^+})^2 - 1)v^-_{\eta\eta\eta}. \tag{4.5}$$

From $v_\xi = u_\eta$ it follows that $v^+_{\xi\xi\xi} = u^+_{\xi\xi\eta}$ from which by inserting $u^+_{\xi\xi\eta}$ into (4.2) we obtain

$$v^+_{\xi\xi\xi} = (\frac{c^-}{c^+})^2 u^-_{\xi\xi\eta} + ((\frac{c^-}{c^+})^2 - 1)u^-_{\eta\eta\eta}. \tag{4.6}$$

From (4.4),(4.5), and (4.6), we now obtain the following jump condition for $U_{\xi\xi\xi}$

$$U^+_{\xi\xi\xi} = Q_4 U^-_{\xi\xi\xi} + Q_5 U^-_{\xi\xi\eta} + Q_6 U^-_{\xi\eta\eta} + Q_7 U^-_{\eta\eta\eta}. \tag{4.7}$$

Now by differentiating (4.7) with respect to η we obtain

$$U^+_{\xi\xi\xi\eta} = Q_4 U^-_{\xi\xi\xi\eta} + Q_5 U^-_{\xi\xi\eta\eta} + Q_6 U^-_{\xi\eta\eta\eta} + Q_7 U^-_{\eta\eta\eta\eta}, \tag{4.8}$$

$$U^+_{\xi\xi\xi\eta\eta} = Q_4 U^-_{\xi\xi\xi\eta\eta} + Q_5 U^-_{\xi\xi\eta\eta\eta} + Q_6 U^-_{\xi\eta\eta\eta\eta} + Q_7 U^-_{\eta\eta\eta\eta\eta}. \tag{4.9}$$

We next obtain the jump conditions for $U_{\xi\xi\xi\xi}$:

$$[U_{tttt}] = 0 \qquad \text{gives} \qquad [c^4(U_{\xi\xi\xi\xi} + 2U_{\xi\xi\eta\eta} + U_{\eta\eta\eta\eta})] = 0,$$

and hence

$$U^+_{\xi\xi\xi\xi} = (\frac{c^-}{c^+})^4(U^-_{\xi\xi\xi\xi} + 2U^-_{\xi\xi\eta\eta} + U^-_{\eta\eta\eta\eta}) - 2U^+_{\xi\xi\eta\eta} - U^+_{\eta\eta\eta\eta}. \tag{4.10}$$

From (4.2), replacing $U^+_{\xi\xi\eta\eta}$ and $U^+_{\eta\eta\eta\eta}$ in the above formula gives

$$\begin{aligned} U^+_{\xi\xi\xi\xi} =& U^-_{\xi\xi\xi\xi} + 2(\frac{c^-}{c^+})^2((\frac{c^-}{c^+})^2 - Q_1)U^-_{\xi\xi\eta\eta} \\ &+ ((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2 Q_1 + Q_1)U^-_{\eta\eta\eta\eta}. \end{aligned} \tag{4.11}$$

Now differentiating this formula with respect to η we also obtain the following jump condition for $U^+_{\xi\xi\xi\xi\eta}$:

$$\begin{aligned} U^+_{\xi\xi\xi\xi\eta} =& U^-_{\xi\xi\xi\xi\eta} + 2(\frac{c^-}{c^+})^2((\frac{c^-}{c^+})^2 - Q_1)U^-_{\xi\xi\eta\eta\eta} \\ &+ ((\frac{c^-}{c^+})^4 - 2(\frac{c^-}{c^+})^2 Q_1 + Q_1)U^-_{\eta\eta\eta\eta\eta}. \end{aligned} \tag{4.12}$$

As the derivation of the jump condition for $U^+_{\xi\xi\xi\xi\xi}$ needs a long similar calculations we only here mention it without further calculations:

$$\begin{aligned} U^+_{\xi\xi\xi\xi\xi} =& Q_2 U^-_{\xi\xi\xi\xi\xi} + (Q_9 + Q_{10})U^-_{\xi\xi\xi\eta\eta} \\ &+ Q_8 U^-_{\xi\eta\eta\eta\eta} + Q_{11}U^-_{\eta\eta\eta\eta\eta}. \end{aligned} \tag{4.13}$$

5. Conclusions and discussions

High order interface methods are important tools to develop more efficient approximations for simulation of long time behavior of wave propagation in 2D with low evaluation costs. A fifth order method has been developed for 2D acoustic wave equation in heterogeneous media. We derived high order jump conditions to obtain a high order method at irregular points as well. Here the interface was assumed to be linear curve, However, in the case of an arbitrary smooth curve the derivatives of the interface come into the formulation and the situation is a bit more complicated from theoretical point of view. A similar method can be applied to Maxwell equations in a media with different conductivities.

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