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A piecewise constant argument method for solving second order nonlinear differential equations

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Abstract

The objective of this study is to develop a numerical approach for solving initial value problems to second-order nonlinear differential equations, employing the Piecewise Constant Argument Method (PCAM). To this end, a differential equations with piecewise constant arguments (DEPCA) is constructed, parameterized by a positive integer n, representing the number of subintervals. The explicit form of the unique solution to the initial value problem for the second-order DEPCA is established. Furthermore, it is proven that this solution approximates the solution to the stated problem as n becomes large. Several problems derived from physical models with prescribed initial conditions are solved to validate the method. The results confirm the efficiency and high accuracy of the proposed PCAM approach.

Keywords. Approximate solution, Initial value problems, second order nonlinear differential equations, Piecewise constant argument.
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1. INTRODUCTION

The focus of this work is on applying the piecewise constant argument technique to numerically solve the initial value problem

$$\begin{cases} x''(t) = \phi(t, x(t), x'(t)), & t \in [0,1] \\ x(0) = x_0, & x'(0) = x'_0, \end{cases} \tag{1.1}$$

where $\phi(\cdot,\cdot,\cdot)$ is given real valued differentiable function in $\mathbb{R}^3$, x_0 and x'_0 are real constants.

The initial value problem (1.1) has been a subject of interest to many scientists for years. In particular, in [5], the homotopy perturbation method (HPM) and the variational iteration method (VIM) are employed to analyze and obtain solutions for nonlinear oscillator differential equations. Initial value problems for nonlinear differential equations such as the Duffing, Van der Pol, Blasius, and jerk equations were solved numerically using the operational integration matrix of Bernoulli polynomials and the inverse Laplace transform method in [16]. In [2], the Algebraic Method (AGM) is utilized to thoroughly investigate and solve three intricate nonlinear differential equations associated with vibrational systems, specifically the Van der Pol, Rayleigh, and Duffing equations. In [9] review paper highlights the latest developments in nonlinear vibration theory, focusing on coupled damping nonlinear oscillators, and encompasses the following topics: (a) Some fallacies in the study of non-conservative issues; (b) non-conservative Duffing oscillator with three expansions; (c)the non-conservative oscillators through the modified homotopy expansion; (d) the HPM for fractional nonconservative oscillators; (e) the HPM for delay non-conservative oscillators; and (f) quasi-exact solution based on He's frequency formula. The modifications HPM using either the parameter expansion technique or the enhanced perturbation method are succinctly discussed by Anjum [4]. These approaches incorporate either

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the parameter-expansion scheme or an improved perturbation framework. Both strategies demonstrate high efficiency when applied to nonlinear oscillatory systems.

This method is used to solve various nonlinear ordinary differential equations, including the Bernoulli, Lane-Emden, and Duffing equations [14, 17**?**].

This note provides an approximate solution to the initial value problem associated with Eq. (1.1). First, we construct an initial value problem for differential equations with piecewise constant arguments (DEPCA), given by Eq. (1.1), where equation depends on the number of subintervals n. It is demonstrated that this problem admits a unique piecewise-smooth solution, which can be explicitly expressed. Moreover, we establish that this solution closely approximates the original problem's solution as n becomes large. A simplified algorithm is introduced to compute the approximate solution efficiently. Numerical examples are provided to illustrate the accuracy and effectiveness of the proposed approach, along with corresponding error bounds.

2. DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

Let $x = x(t)$ be solution of the initial value problem (1.1). Then there exists $L > 0$ such that $|x(t)| \leq L$ for all $t \in [0, 1]$. In that follows, we suppose that the function $\phi(t, x, x')$ has the compact support $D = [0, 1] \times [-L, L]^2$, i.e. we assume that

$$\phi(t, x, x') = 0 \quad \text{for} \quad (t, x, x') \notin D. \tag{2.1}$$

Moreover, under this assumption on the function ϕ, the solution x of the initial value problem (1.1) is preserved.

We remark that the assumption (2.1) ensures the boundedness of the function $\phi(t, x, x')$ in D. Thus, under these conditions, we can construct a numerical approximation algorithm for the solution.

Let n be a positive integer number and $\frac{[nt]}{n} = t_m, \quad t \in [t_{m-1}, t_m), t_m = \frac{m}{n}, \ m = 0, 1, ..., n$, where $[t]$ denotes floor function. Consider the following DEPCA of the form

$$\begin{cases} x''(t) = \phi(\frac{[nt]}{n}, x(\frac{[nt]}{n}), x'(\frac{[nt]}{n})), \\ x(0) = x_0, \quad x'(0) = x'_0. \end{cases} \tag{2.2}$$

A solutions of the Eq. (2.2) is defined as following [**? ?**]:

Definition 2.1. A function $x(t) := x_n(t)$ is called a solution of the initial value problem (2.2) if the following conditions are satisfied:

(i) $x(t)$, $x'(t)$ are continuous on $[0, 1]$;

(ii) $x''(t)$ exist and continuous on $[0, 1]$ with possible exception at points $t_m, \quad m = 0, 1, ..., n$, where one-sided derivatives exist;

(iii) $x(t)$ satisfy the initial value problem (2.2) in $(0, 1)$, with the possible exception at the points $t_m, \quad m = 0, 1, ..., n$.

The Eq. (2.2) is called DEPCA corresponding to the non-linear differential equation (1.1).

Theorem 2.2. *For any positive integer number n the initial value problem (2.2) has a unique solution $x(t) := x_n(t)$ expressed in each interval $[t_{m-1}, t_m), \quad m = 1, ..., n$ as*

$$x(t) = \frac{1}{2}\phi(t_{m-1}, x(t_{m-1}), x'(t_{m-1}))(t - t_{m-1})^2 + x'(t_{m-1})(t - t_{m-1}) + x(t_{m-1}), \tag{2.3}$$

where

$$x(t_{m-1}) = \lim_{t \to t_{m-1} - 0} x(t), \quad x'(t_{m-1}) = \lim_{t \to t_{m-1} - 0} x'(t).$$

Proof. Let n be a positive integer. For $t \in [0, t_1)$ the Eq. (2.2) is

$$x''(t) = \phi(t_0, x(t_0), x'(t_0)).$$

The integrations of this equation yield

$$x'(t) = \phi(t_0, x(t_0), x'(t_0))t + x'(0), \quad t \in [0, t_1) \tag{2.4}$$

and

$$x(t) = \frac{1}{2}\phi(t_0, x(t_0), x'(t_0))t^2 + x'(0)t + x(0), \quad t \in [0, t_1). \tag{2.5}$$

There exist the following limits $x(t_1) = \lim_{t \to t_1 - 0} x(t)$, $x'(t_1) = \lim_{t \to t_1 - 0} x'(t)$. Then we solve Eq. (2.2) for $t \in [t_{m-1}, t_m)$, $m = 2, \dots, n$, as

$$x'(t) = \phi(t_{m-1}, x(t_{m-1}), x'(t_{m-1}))(t - t_{m-1}) + x'(t_{m-1}), \tag{2.6}$$

hence

$$x(t) = \frac{1}{2}\phi(t_{m-1}, x(t_{m-1}), x'(t_{m-1}))(t - t_{m-1})^2 + x'(t_{m-1})(t - t_{m-1}) + x(t_{m-1}), \tag{2.7}$$

where

$$x(t_{m-1}) = \lim_{t \to t_{m-1} - 0} x(t), \quad x'(t_{m-1}) = \lim_{t \to t_{m-1} - 0} x'(t).$$

By construction the functions $x(t)$ and $x'(t)$ are continuous on $[0, 1]$ and $x''(t)$ exists on $(0, t_1) \cup (t_1, t_2) \cup \cdots \cup (t_{m-1}, 1)$ and continuous on $[0, 1]$ with possible exception at points t_m, $m = 0, 1, ..., n$, where one-sided derivatives exist. It is clear that this function is the unique solution of the Eq. (2.2). □

Lemma 2.3. *There exists a positive number C such that for any n and $m \in \{1, 2, ..., n\}$ the inequalities*

$$|x'(t_m)| \leq C, \tag{2.8}$$

$$|x(t_m)| \leq C. \tag{2.9}$$

hold, where $x(t_m)$ and $x'(t_m)$ are defined by (2.6), (2.7).

Proof. Let

$$M = \max_{(t,x,x') \in D} |\phi(t, x, x')|. \tag{2.10}$$

Then for $t \in [0, t_1)$ from (2.4) and (2.5) we obtain

$$|x'(t_1)| \leq \frac{1}{n}M + |x'(0)| \tag{2.11}$$

and

$$|x(t_1)| \leq \frac{1}{2n^2}M + \frac{1}{n}|x'(0)| + |x(0)|, \tag{2.12}$$

respectively.

For $t \in [t_1, t_2)$ (2.6) and (2.7) have the forms

$$x'(t) = \phi(t_1, x(t_1), x'(t_1))(t - t_1) + x'(t_1), \tag{2.13}$$

and

$$x(t) = \frac{1}{2}\phi(t_1, x(t_1), x'(t_1))(t - t_1)^2 + x'(t_1)(t - t_1) + x(t_1). \tag{2.14}$$

Applying the equality (2.10), from (2.13), (2.14), we obtain

$$|x'(t_2)| \leq \frac{1}{n}M + |x'(t_1)| \tag{2.15}$$

and

$$|x(t_2)| \leq \frac{1}{2n^2}M + \frac{1}{n}|x'(t_1)| + |x(t_1)|. \tag{2.16}$$

Analysing similarly, from (2.6), (2.7) for $t \in [t_{m-1}, t_m)$, $m = 3, 4, ..., n$, we obtain the following inequalities

$$|x'(t_m)| \leq \frac{1}{n}M + |x'(t_{m-1})| \tag{2.17}$$

and

$$|x(t_m)| \le \frac{1}{2n^2}M + \frac{1}{n}|x'(t_{m-1})| + |x(t_{m-1})|. \tag{2.18}$$

Using the recurrence inequality (2.17) and taking account (2.11), we get

$$|x'(t_m)| \le \frac{M}{n} + |x'(t_{m-1})| \le \frac{(m-1)M}{n} + |x'(t_1)| \le \frac{mM}{n} + |x'(0)|. \tag{2.19}$$

Hence

$$|x'(t_m)| \le \frac{mM}{n} + |x'(0)| \le M + |x'(0)|. \tag{2.20}$$

Similarly, using (2.18), we obtain the following evaluations

$$\begin{aligned}
|x(t_m)| &\le \frac{1}{2n^2}M + \frac{1}{n}|x'(t_{m-1})| + |x(t_{m-1})| \\
&\le \frac{1}{2n^2}M + \frac{1}{n}|x'(t_{m-1})| + \frac{1}{2n^2}M + \frac{1}{n}|x'(t_{m-2})| + |x(t_{m-2})| \\
&\le \frac{1}{2n^2}M + \frac{1}{n}|x'(t_{m-1})| + \frac{1}{2n^2}M + \frac{1}{n}|x'(t_{m-2})| \\
&+ ... + \frac{1}{2n^2}M + \frac{1}{n}|x'(t_1)| + |x(t_1)| \\
&\le \frac{m}{2n^2}M + \frac{1}{n}|x'(t_{m-1})| + \frac{1}{n}|x'(t_{m-2})| + ... + \frac{1}{n}|x'(0)| + |x(0)|.
\end{aligned}$$

Then applying (2.19), we have

$$\begin{aligned}
|x(t_m)| \le \frac{m}{2n^2}M &+ \frac{1}{n}\Big(\frac{(m-1)M}{n} + |x'(0)| + \frac{(m-2)M}{n} + |x'(0)| \\
&+ ... + \frac{M}{n} + |x'(0)|\Big) + |x(0)| \\
&\le \frac{m}{2n^2}M + \frac{1}{n}\Big(\frac{m(m-1)M}{2n} + m|x'(0)|\Big) + |x(0)| \\
&\le \frac{m}{2n^2}M + \frac{m(m-1)}{2n^2}M + \frac{m}{n}|x'(0)| + |x(0)|.
\end{aligned}$$

Hence

$$|x(t_m)| \le \frac{m}{2n^2}M + \frac{m(m-1)}{2n^2}M + \frac{m}{n}|x'(0)| + |x(0)| \le \frac{M}{2} + |x'(0)| + |x(0)|. \tag{2.21}$$

Since $m \le n$, from (2.19) and (2.21) we obtain the inequalities (2.8), (2.9). ☐

The following theorem claims that for a large n the function x_n, defined by (2.3), presents an approximate solution for the initial value problem (1.1).

Theorem 2.4. *For any $\varepsilon > 0$ there exists a positive number $n_0 = n_0(\varepsilon)$ such that for any n with $n > n_0$ the inequality*

$$\sup_{t\in[0,1]} |x''_n(t) - \phi(t, x_n(t), x'_n(t))| < \varepsilon \tag{2.22}$$

holds, where $x_n(t)$ is solution of the initial value problem (2.2).

Proof. Let x_n be solution of (2.2). Then for $t \in [t_{m-1}, t_m)$

$$x''_n(t) = \phi(t_{m-1}, x_n(t_{m-1}), x'_n(t_{m-1})).$$

This gives

$$\begin{aligned} R_n(t) &:= x_n''(t) - \phi(t, x_n(t), x_n'(t)) \\ &= \phi(t, x_n(t_{m-1}), x_n'(t_{m-1})) - \phi(t, x_n(t), x_n'(t)) \quad \text{for } t \in [t_{m-1}, t_m). \end{aligned}$$

Since the function ϕ is differentiable in D, there exists $F > 0$ such that

$$|\phi(t, x_n(t), x_n'(t)) - \phi(t, x_n(t_{m-1}), x_n'(t_{m-1}))| \le F\Big(|x_n(t) - x_n(t_{m-1})| + |x_n'(t) - x_n'(t_{m-1})|\Big). \tag{2.23}$$

Equations (2.6), (2.7) represented as, respectively

$$x_n'(t) - x_n'(t_{m-1}) = \phi(t_{m-1}, x(t_{m-1}), x'(t_{m-1}))(t - t_{m-1})$$

and

$$x_n(t) - x_n(t_{m-1}) = \frac{1}{2}\phi(t_{m-1}, x(t_{m-1}), x'(t_{m-1}))(t - t_{m-1})^2 + x'(t_{m-1})(t - t_{m-1})$$

for $t \in [t_{m-1}, t_m)$. These yield

$$|x_n'(t) - x_n'(t_{m-1})| \le \frac{1}{n}M$$

and

$$|x_n(t) - x_n(t_{m-1})| \le \frac{1}{n^2}M + \frac{1}{n}C$$

for any $t \in [t_{m-1}, t_m)$, $m = 1, 2, \dots, n$. Consequently, the residual term satisfies the inequality

$$|R_n(t)| \le F\Big(\frac{1}{n^2}M + \frac{1}{n}(M + C)\Big), \quad t \in [0, 1].$$

This proves the assertion of the theorem. □

3. STABILITY ANALYSIS

We introduce Zero-stability for the proposed piecewise constant argument method.

Definition 3.1. (Zero-stability) The numerical method applied in (2.3) for approximating problem (1.1) is said to be *zero-stable* if there exists a constant C such that, for every $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$ for which the inequality

$$|x_\varepsilon(t) - x(t)| \le C\varepsilon,$$

holds for all $t \in [0, 1]$ and positive integer n. Here, $x = x_n$ and $x_\varepsilon = x_{\varepsilon n}$ denote, respectively, the solutions of the exact and the perturbed problems

$$\begin{cases} x''(t) = \phi\Big(\frac{[nt]}{n}, x(\frac{[nt]}{n}), x'(\frac{[nt]}{n})\Big), \\ x(0) = x_0, \quad x'(0) = x_0', \end{cases} \tag{3.1}$$

$$\begin{cases} x_\varepsilon''(t) = \phi\Big(\frac{[nt]}{n}, x_\varepsilon(\frac{[nt]}{n}), x_\varepsilon'(\frac{[nt]}{n})\Big), \\ x_\varepsilon(0) = x_0 + \delta_0, \qquad x_\varepsilon'(0) = x_0' + \delta_0, \end{cases} \tag{3.2}$$

where $|\delta_0| \le \varepsilon$.

Theorem 3.2. *(Zero-stability) Assume that ϕ is differentiable function. Then the numerical method applied in (2.3) for approximating problem (1.1) is zero-stable.*

Proof. Let n be a positive integer. Since for $t \in [0, t_1)$ (3.2) is

$$x_\varepsilon''(t) = \phi(t_0, x_\varepsilon(t_0), x_\varepsilon'(t_0)),$$

the integrations of this equation yield

$$x_\varepsilon'(t) = \int_{t_0}^{t} \phi(t_0, x_\varepsilon(t_0), x_\varepsilon'(t_0))ds + x_\varepsilon'(t_0), \tag{3.3}$$

and

$$x_\varepsilon(t) = \int_{t_0}^{t}\int_{t_0}^{s} \phi(t_0, x_\varepsilon(t_0), x'_\varepsilon(t_0))drds + \int_{t_0}^{t} x'_\varepsilon(t_0)ds + x_\varepsilon(t_0). \tag{3.4}$$

Similarly, from Eq. (3.1) we have

$$x'(t) = \int_{t_0}^{t} \phi(t_0, x(t_0), x'(t_0))ds + x'(t_0), \tag{3.5}$$

and

$$x(t) = \int_{t_0}^{t}\int_{t_0}^{s} \phi(t_0, x(t_0), x'(t_0))drds + \int_{t_0}^{t} x'(t_0)ds + x(t_0). \tag{3.6}$$

The subtraction (3.3) and (3.5) gives

$$x'_\varepsilon(t) - x'(t) = \int_{t_0}^{t} \Big(\phi(t_0, x_\varepsilon(t_0), x'_\varepsilon(t_0)) - \phi(t_0, x(t_0), x'(t_0))\Big)ds + x'_\varepsilon(t_0) - x'(t_0).$$

According (2.23) the inequality

$$|\phi(t_0, x_\varepsilon(t_0), x'_\varepsilon(t_0)) - \phi(t_0, x(t_0), x'(t_0))| \le F\Big(|x_\varepsilon(t_0) - x(t_0)| + |x'_\varepsilon(t_0) - x'(t_0))|\Big)$$

holds for some $F > 0$. Then we get

$$|x'_\varepsilon(t_1) - x'(t_1)| \le \frac{F}{n}|x_\varepsilon(t_0) - x(t_0)| + (\frac{F}{n} + 1)|x'_\varepsilon(t_0) - x'(t_0))| < (\frac{2F}{n} + 1)\varepsilon,$$

since by the initial values in (3.1) and (3.2)

$$|x_\varepsilon(t_0) - x(t_0)| \le \varepsilon, \quad |x'_\varepsilon(t_0) - x'(t_0)| \le \varepsilon.$$

Similarly for

$$\begin{aligned} &x_\varepsilon(t) - x(t) \\ &= \int_{t_0}^{t}\int_{t_0}^{s} \Big(\phi(t_0, x_\varepsilon(t_0), x'_\varepsilon(t_0)) - \phi(t_0, x(t_0), x'(t_0))\Big)drds + \int_{t_0}^{t}(x'_\varepsilon(t_0) - x'(t_0))ds + x_\varepsilon(t_0) - x(t_0), \end{aligned}$$

we have

$$|x_\varepsilon(t_1) - x(t_1)| \le (\frac{F}{n^2} + 1)|x_\varepsilon(t_0) - x(t_0)| + (\frac{F}{n^2} + \frac{1}{n})|x'_\varepsilon(t_0) - x'(t_0)| \le (\frac{2F}{n^2} + \frac{1}{n} + 1)\varepsilon.$$

Thus, for a large n the inequalities

$$|x'_\varepsilon(t_1) - x'(t_1)| \le (\frac{2F + 1}{n} + 1)\varepsilon, \tag{3.7}$$

and

$$|x_\varepsilon(t_1) - x(t_1)| \le (\frac{2F + 1}{n} + 1)\varepsilon \tag{3.8}$$

hold.

For $t \in [t_1, t_2)$ from equations (3.2) and (3.1) yield

$$x'_\varepsilon(t) = \int_{t_1}^{t} \phi(t_1, x_\varepsilon(t_1), x'_\varepsilon(t_1))ds + x'_\varepsilon(t_1), \tag{3.9}$$

$$x'(t) = \int_{t_1}^{t} \phi(t_1, x(t_1), x'(t_1))ds + x'(t_1), \tag{3.10}$$

and

$$x_\varepsilon(t) = \int_{t_1}^{t}\int_{t_1}^{s} \phi(t_1, x_\varepsilon(t_1), x'_\varepsilon(t_1))drds + \int_{t_1}^{t} x'_\varepsilon(t_1)ds + x_\varepsilon(t_1), \tag{3.11}$$

$$x(t)=\int_{t_1}^{t}\int_{t_1}^{s}\phi(t_1,x(t_1),x'(t_1))drds+\int_{t_1}^{t}x'(t_1)ds+x(t_1). \tag{3.12}$$

Then for $t\in[t_1,t_2)$ the Equations (3.9), (3.10) and (3.11), (3.12) give, respectively,

$$x'_\varepsilon(t)-x'(t)=\int_{t_1}^{t}\Big(\phi(t_1,x_\varepsilon(t_1),x'_\varepsilon(t_1))-\phi(t_1,x(t_1),x'(t_1))\Big)ds+x'_\varepsilon(t_1)-x'(t_1),$$

and

$$\begin{aligned}&x_\varepsilon(t)-x(t)\\&=\int_{t_1}^{t}\int_{t_1}^{s}\Big(\phi(t_1,x_\varepsilon(t_1),x'_\varepsilon(t_1))-\phi(t_1,x(t_1),x'(t_1))\Big)drds+\int_{t_1}^{t}(x'_\varepsilon(t_1)-x'(t_1))ds+x_\varepsilon(t_1)-x(t_1).\end{aligned}$$

Again using the inequality (2.23) to the right hand side of these equation and $|t_1-t|\le\frac{1}{n}$ for $t\in[t_1,t_2)$, we evaluate

$$|x'_\varepsilon(t_2)-x'(t_2)|\le\frac{F}{n}|x_\varepsilon(t_1)-x(t_1)|+(\frac{F}{n}+1)|x'_\varepsilon(t_1)-x'(t_1))|,$$

$$|x_\varepsilon(t_2)-x(t_2)|\le(\frac{F}{n^2}+1)|x_\varepsilon(t_1)-x(t_1)|+(\frac{F}{n^2}+\frac{1}{n})|x'_\varepsilon(t_1)-x'(t_1)|.$$

Using (3.7), (3.8), we get

$$|x'_\varepsilon(t_2)-x'(t_2)|\le(\frac{2F+1}{n}+1)^2\varepsilon,$$

$$|x_\varepsilon(t_2)-x(t_2)|\le(\frac{2F+1}{n}+1)^2\varepsilon.$$

Let the following estimates hold for $t\in[t_{m-2},t_{m-1})$:

$$|x'_\varepsilon(t_{m-1})-x'(t_{m-1})|\le\Big(\frac{2F+1}{n}+1\Big)^{m-1}\varepsilon, \tag{3.13}$$

$$|x_\varepsilon(t_{m-1})-x(t_{m-1})|\le\Big(\frac{2F+1}{n}+1\Big)^{m-1}\varepsilon. \tag{3.14}$$

Using Equations (3.2) and (3.1) and performing calculations analogous to those above, for $t\in[t_{m-1},t_m)$, $m=3,4,...,n$, we obtain

$$\begin{aligned}x_\varepsilon(t)-x(t)&=\int_{t_{m-1}}^{t}\int_{t_{m-1}}^{s}\Big(\phi(t_{m-1},x_\varepsilon(t_{m-1}),x'_\varepsilon(t_{m-1}))-\phi(t_{m-1},x(t_{m-1}),x'(t_{m-1}))\Big)drds\\&+\int_{t_{m-1}}^{t}(x'_\varepsilon(t_{m-1})-x'(t_{m-1}))ds+x_\varepsilon(t_{m-1})-x(t_{m-1}).\end{aligned}$$

Then applying (2.23) and using the inequality $|t_{m-1}-t|\le\frac{1}{n}$ for $t\in[t_{m-1},t_m)$, we get

$$|x_\varepsilon(t_m)-x(t_m)|\le(\frac{F}{n^2}+1)|x_\varepsilon(t_{m-1})-x(t_{m-1})|+(\frac{F}{n^2}+\frac{1}{n})|x'_\varepsilon(t_{m-1})-x'(t_{m-1}))|.$$

It follows from here and (3.13), (3.14) that

$$|x_\varepsilon(t_m)-x(t_m)|\le(\frac{2F+1}{n}+1)^m\varepsilon.$$

Since $(\frac{2F+1}{n}+1)^m<e^{2F+1}$, the piecewise constant argument method is zero stable. □

Table 1. Comparisons exact and proposed approximate solutions for Example 4.1 are presented in this table, as determined by HPM, VIM [5], and the present method, where the number of subintervals is chosen $n = 10^5$.

t	AE of HPM [5]	AE of VIM [5]	AE of Present method
0.0	0	0	0
0.1	5.50×10^{-10}	2.73×10^{-8}	2.41×10^{-8}
0.2	7.04×10^{-8}	1.71×10^{-6}	9.28×10^{-8}
0.3	1.15×10^{-6}	1.90×10^{-5}	1.99×10^{-7}
0.4	8.24×10^{-6}	1.04×10^{-4}	3.38×10^{-7}
0.5	3.71×10^{-5}	3.86×10^{-4}	4.99×10^{-7}
0.6	1.24×10^{-4}	1.11×10^{-3}	6.73×10^{-7}
0.7	3.43×10^{-4}	2.71×10^{-3}	8.53×10^{-7}
0.8	8.11×10^{-4}	5.80×10^{-3}	1.02×10^{-6}
0.9	1.70×10^{-3}	1.12×10^{-2}	1.19×10^{-6}
1.0	3.27×10^{-3}	2.02×10^{-2}	1.33×10^{-6}

4. Numerical experiments and Discussions

In order to derive an approximate solution to Eq. (1.1), it is necessary to compute $x(t) = x_n(t)$, which is explicitly defined by (2.3). Additionally, Theorem 2.4 establishes the convergence of the algorithm, with inequality (2.22) demonstrating the reduction of the residual error as n becomes large.

The procedures of the PCAM are outlined as follows:
Input: n is a positive integer number, initial conditions.
Step 1: Compute $x(t)$, $x'(t)$ for $t \in [0, t_1)$;
Step 2: Find limits $x(t_1) = \lim\limits_{t \to t_1 - 0} x(t)$, $x'(t_1) = \lim\limits_{t \to t_1 - 0} x'(t)$;
Step 3: Let $x(t)$ be solution of Eq. (2.2) in $[t_{m-2}, t_{m-1})$ and

$$x(t_{m-2}) = \lim_{t \to t_{m-2} - 0} x(t), \qquad x'(t_{m-2}) = \lim_{t \to t_{m-2} - 0} x'(t).$$

Compute $x(t)$, $x'(t)$ for $t \in [t_{m-1}, t_m)$, $m = 2, \ldots, n$ using Eq. (2.6) and Eq. (2.7), and find corresponding limits:

$$x(t_{m-1}) = \lim_{t \to t_{m-1} - 0} x(t), \qquad x'(t_{m-1}) = \lim_{t \to t_{m-1} - 0} x'(t),$$

Step 4: Construct an approximate solution from (2.3).
Output: Errors are found.
Let us define absolute and residual errors as follows, respectively

$$\text{Absolute Error (AE)} = |X_i - x_i|, \quad i = 1, 2, ..., n,$$

$$\text{Residual Error (RE)} = |x_i'' - \phi(t_i, x_i, x_i')|, \quad i = 1, 2, ..., n,$$

where X and x are exact and approximate solutions respectively.

We present numerical results obtained using the PCAM, based on the approximate solution formula (2.3), and compare them with the problems provided in references [5]-[10].

Example 4.1. Consider the Van Der Pol Oscillator problem is expressed in [5] as

$$\begin{aligned} &x''(t) + x'(t) + x(t) + x^2(t)x'(t) = 2cost - cos^3 t, \quad t \in [0, 1], \\ &x(0) = 0, \quad x'(0) = 1. \end{aligned} \tag{4.1}$$

The exact solution of the problem (4.1) is $x(t) = sint$.

Example 4.2. Consider the nonlinear oscillator differential equation is given by [5]

$$\begin{aligned} &x''(t) - x(t) + x^2(t) + x'^2(t) - 1 = 0, \quad t \in [0,1], \\ &x(0) = 2, \quad x'(0) = 0. \end{aligned} \tag{4.2}$$

An exact solution of Eq. (4.2), which is given in [5] has the form $x(t) = 1 + cost$. The comparisons presented in Table 2 for Example 4.2 show that PCAM demonstrates significantly higher accuracy than the results obtained by HPM and VIM in [5].

Table 2. The absolute errors associated with the approximate solutions for Example 4.2 are shown in this table, as obtained using HPM, VIM [5], and the proposed method, with the number of subintervals selected as $n = 10^5$.

t	AE of HPM [5]	AE of VIM [5]	AE of Present method
0.0	0	0	0
0.1	8.33×10^{-6}	8.33×10^{-6}	8.34×10^{-10}
0.2	1.33×10^{-4}	1.33×10^{-4}	6.69×10^{-9}
0.3	6.76×10^{-4}	6.75×10^{-4}	2.27×10^{-8}
0.4	2.13×10^{-3}	2.13×10^{-3}	5.42×10^{-8}
0.5	5.22×10^{-3}	5.20×10^{-3}	1.06×10^{-7}
0.6	1.08×10^{-2}	1.08×10^{-2}	1.86×10^{-7}
0.7	2.01×10^{-2}	2.00×10^{-2}	3.01×10^{-7}
0.8	3.44×10^{-2}	3.41×10^{-2}	4.57×10^{-7}
0.9	5.54×10^{-2}	5.46×10^{-2}	6.64×10^{-7}
1.0	8.46×10^{-2}	8.33×10^{-2}	9.32×10^{-7}

Example 4.3. The unforced Van der Pol oscillator is expressed in [2, 16] as

$$\begin{aligned} &x''(t) + 0.15(1 - x^2(t))x'(t) + 1.44x(t) = 0, \quad t \in [0,1], \\ &x(0) = 0.2, \quad x'(0) = 0. \end{aligned} \tag{4.3}$$

The exact solution to this problem is not known. The approximate solution of (4.3) obtained using the PCAM is presented in the form of Eq. (2.3), where the number of subintervals is selected $n = 10^5$. The approximate solution obtained using the AGM is given in [2] as $x(t) = 0.20036e^{-0.072t}cos(1.19784t - 0.060036)$, where the comparisons with approximate solutions are demonstrated in [2, 16]. The approximate solution values and the residual errors for Eq. (4.3) at selected points, obtained using the PCAM and as the AGM and ILTM methods are demonstrated in the Table 3, where the residual errors for the approximate solutions obtained by PCAM are indicating a higher degree of accuracy.

Table 3. Presents numerical results and residual error comparison between the PCAM and other methods obtained in [2, 16] for Example 4.3.

t	AGM [2]	ILTM [16]	Present method	RE of AGM [2]	RE of Present method
0.0	0.199999028	0.200000333	0.20000000000000	9.33×10^{-7}	0
0.2	0.194321218	0.194324429	0.19432221121652	1.97×10^{-5}	1.29×10^{-16}
0.4	0.177824159	0.17784224	0.17782620520601	1.36×10^{-4}	1.20×10^{-17}
0.6	0.151747193	0.151804379	0.15175592335904	3.86×10^{-4}	7.00×10^{-17}
0.8	0.117835809	0.117963527	0.11786599210517	7.28×10^{-4}	4.00×10^{-17}
1.0	0.078228637	0.078460834	0.07830643659475	1.05×10^{-3}	3.12×10^{-6}

Example 4.4. The Duffing oscillator equation is given by [9]

$$\begin{aligned} &x''(t) + x(t) + x^3(t) = 0, \quad t \in [0,1], \\ &x(0) = x_0, \quad x'(0) = 0. \end{aligned} \tag{4.4}$$

The exact solution to this problem is not available. The approximate solution of Eq. (4.4), derived using the PCAM, is expressed in the form of Eq. (2.3), where the number of subintervals is chosen $n = 10^5$. The approximate solution obtained via the HPM is presented in [9] as $x(t) = \left(1 - \frac{1}{32a^2}x_0^2\right)x_0 cos(at) + \frac{1}{32a^2}x_0^3 cos(3at)$, where $a = \sqrt{1 + \frac{3}{4}x_0^2}$. For $x_0 = 0.01$ and $x_0 = 0.1$ the Tables 4 and 5, respectively, display the values and residual errors of the approximate solution for Eq. (4.4) at selected points, calculated using both the PCAM and HPM. The results in Tables 4 and 5 show that the difference between the residual errors obtained using the HPM and PCAM is significantly notable.

Table 4. For $x_0 = 0.01$, the table presents the values of the approximate solution and the residual errors for Example 4.4.

t	HPM [9]	Present method	RE of HPM [9]	RE of Present method
0.0	0.01000000000000	0.01000000000000	5.93×10^{-12}	0
0.1	0.00995003666944	0.00995003666108	5.63×10^{-12}	3.12×10^{-18}
0.2	0.00980064604300	0.00980064597646	4.78×10^{-12}	7.59×10^{-19}
0.3	0.00955332121662	0.00955332099331	3.59×10^{-12}	1.08×10^{-18}
0.4	0.00921053406850	0.00921053354305	2.33×10^{-12}	3.81×10^{-18}
0.5	0.00877571051397	0.00877570949716	1.27×10^{-12}	3.43×10^{-18}
0.6	0.00825319621484	0.00825319447743	6.46×10^{-13}	7.17×10^{-19}
0.7	0.00764821309119	0.00764821036860	5.54×10^{-13}	3.24×10^{-18}
0.8	0.00696680707574	0.00696680307352	9.72×10^{-13}	2.98×10^{-18}
0.9	0.00621578763886	0.00621578203885	1.74×10^{-12}	6.35×10^{-19}

Example 4.5. The cubic-quintic-septic Duffing equation is expressed in [4] as

$$\begin{aligned} &x''(t) + x(t) + x^3(t) + x^5(t) + x^7(t) = 0, \quad t \in [0,1], \\ &x(0) = x_0, \quad x'(0) = 0. \end{aligned} \tag{4.5}$$

The exact solution is also not known. The approximate solution of Eq. (4.5), obtained using the PCAM, is expressed in the form of Eq. (2.3), where the number of subintervals is selected $n = 10^5$. The approximate solution obtained using the HPM is $x(t) = x_0 cos(\sqrt{1 + \frac{3}{4}x_0^2 + \frac{5}{8}x_0^4 + \frac{35}{64}x_0^6}t)$. Tables 6 and 7 present the approximate solution values for Eq. (4.5) at selected points, computed using both the PCAM and HPM, along with the residual errors for the

Table 5. For $x_0 = 0.1$, the table presents the values of the approximate solution and the residual errors for Example 4.4.

t	HPM [9]	Present method	RE of HPM [9]	RE of Present method
0.0	0.10000000000000	0.10000000000000	1.15×10^{-12}	0
0.1	0.09949543326085	0.09949543319750	2.75×10^{-8}	7.98×10^{-17}
0.2	0.09798692383504	0.09798692350628	1.02×10^{-7}	1.31×10^{-17}
0.3	0.09548998180563	0.09548998123793	2.06×10^{-7}	4.28×10^{-17}
0.4	0.09203025120225	0.09203025094106	3.12×10^{-7}	1.41×10^{-17}
0.5	0.08764320609771	0.08764320723383	3.91×10^{-7}	1.50×10^{-17}
0.6	0.08237373532312	0.08237373924576	4.22×10^{-7}	3.44×10^{-17}
0.7	0.07627562521798	0.07627563315544	3.93×10^{-7}	2.77×10^{-17}
0.8	0.06941095150550	0.06941096396954	3.10×10^{-7}	4.73×10^{-17}
0.9	0.06184939241547	0.06184940869191	1.88×10^{-7}	4.53×10^{-17}

approximate solutions derived from both methods at $x_0 = 0.05$ and $x_0 = 0.1$. As shown in Tables 6 and 7, the PCAM produces solutions with exceptionally high accuracy.

TABLE 6. For $x_0 = 0.05$, the table presents the values of the approximate solution and the residual errors for Example 4.5.

t	HPM [4]	Present method	RE of HPM [4]	RE of Present method
0.0	0.05000000000000	0.05000000000000	3.13×10^{-5}	0
0.1	0.04974973931842	0.04974958374617	2.99×10^{-5}	2.45×10^{-17}
0.2	0.04900146249005	0.04900085553331	2.58×10^{-5}	6.06×10^{-20}
0.3	0.04776266008565	0.04776135077633	1.94×10^{-5}	9.64×10^{-18}
0.4	0.04604573304658	0.04604354076521	1.13×10^{-5}	9.57×10^{-19}
0.5	0.04386786854607	0.04386470211673	2.15×10^{-6}	1.08×10^{-18}
0.6	0.04125086793831	0.04124673657558	7.19×10^{-6}	8.42×10^{-18}
0.7	0.03822092851752	0.03821594362293	1.58×10^{-5}	1.22×10^{-17}
0.8	0.03480838127188	0.03480274890183	2.31×10^{-5}	9.69×10^{-18}
0.9	0.03104738725738	0.03104139192325	2.83×10^{-5}	7.02×10^{-18}

TABLE 7. For $x_0 = 0.1$, the table presents the values of the approximate solution and the residual errors for Example 4.5.

t	HPM [4]	Present method	RE of HPM [4]	RE of Present method
0.0	0.10000000000000	0.10000000000000	2.53×10^{-4}	0
0.1	0.09949664132756	0.09949538295315	2.42×10^{-4}	1.43×10^{-17}
0.2	0.09799163270931	0.09798672554820	2.08×10^{-4}	2.42×10^{-17}
0.3	0.09550012532805	0.09548954679935	1.56×10^{-4}	3.38×10^{-17}
0.4	0.09204720162074	0.09202950463768	9.03×10^{-5}	9.70×10^{-19}
0.5	0.08766762276924	0.08764209026991	1.59×10^{-5}	1.79×10^{-17}
0.6	0.08240547875348	0.08237221092782	5.98×10^{-5}	2.29×10^{-17}
0.7	0.07631374448999	0.07627367076908	1.30×10^{-4}	1.64×10^{-17}
0.8	0.06945374652419	0.06940856131280	1.88×10^{-4}	1.94×10^{-17}
0.9	0.06189454564546	0.06184657371772	2.30×10^{-4}	9.54×10^{-18}

Example 4.6. Consider the following relativistic harmonic oscillator equation is given by [1]

$$\begin{aligned} &x''(t) + \tfrac{x(t)}{\sqrt{1+x^2(t)}} = 0, \quad t \in [0,1], \\ &x(0) = 0.5, \quad x'(0) = 0. \end{aligned} \tag{4.6}$$

In this case, an exact solution is not available. An estimated solution derived using the Parker-Sochacki with Laplace-Pade method (PSLPM) is presented in [1] as

$$x(t) = 0.502825cos(0.961462t) - 0.00282495cos(2.4962t).$$

Table 8 provides the values of the approximate solutions and residual errors obtained via the PCAM and PSLPM techniques at selected points.

Example 4.7. Duffing harmonic oscillator equation is expressed in [10] as

$$\begin{aligned} &x''(t) + x(t) + x^3(t) + \tfrac{x(t)}{1+x^2(t)} = 0, \quad t \in [0,1], \\ &x(0) = 0.1, \quad x'(0) = 0. \end{aligned} \tag{4.7}$$

Table 8. Approximate values and residual errors using PCAM and PSLPM are presented for Example 4.6.

t	PSLPM [1]	Present method	RE of PSLPM [1]	RE of Present method
0.0	0.50000005000000	0.50000000000000	1.09×10^{-7}	1.00×10^{-16}
0.1	0.49776531502788	0.49776526548618	1.12×10^{-7}	3.00×10^{-16}
0.2	0.49107713372091	0.47998366742458	2.71×10^{-7}	2.00×10^{-16}
0.3	0.47998371519052	0.47998366742458	1.90×10^{-6}	1.00×10^{-16}
0.4	0.46456584987633	0.46456582295873	9.85×10^{-6}	0
0.5	0.44493750711942	0.44493760770902	3.57×10^{-5}	1.00×10^{-16}
0.6	0.42124654566233	0.42124715394139	1.00×10^{-4}	2.00×10^{-16}
0.7	0.39367542121515	0.39367759007586	2.38×10^{-4}	0
0.8	0.36244175603157	0.36244793126437	4.90×10^{-4}	0
0.9	0.32779862545778	0.32781377997598	9.06×10^{-4}	1.00×10^{-16}

An exact solution is not also available. The equivalent linearization method employing weighted averaging (ELM-WA) for the approximate solution of the Duffing-harmonic oscillator is denoted in [10] as

$$x(t) = x_0 cos\left(\sqrt{1 + 0.72x_0^2 + \frac{1}{1 + 0.72x_0^2}}\, t\right),$$

where $x_0 = 0.1$. Table 9 displays the approximate solution values and residual errors obtained through the PCAM and ELM-WA approaches at chosen points.

Table 9. The table shows present method and ELM-WA results with residual errors at specific points for Example 4.7.

t	ELM-WA [10]	Present method	RE of ELM-WA [10]	RE of Present method
0.0	0.10000000000000	0.10000000000000	4.75×10^{-6}	1.00×10^{-17}
0.1	0.09900163990693	0.09900161620786	4.32×10^{-6}	6.00×10^{-17}
0.2	0.09602649408526	0.09602640264484	3.14×10^{-6}	4.00×10^{-17}
0.3	0.09113396787217	0.09113377551584	1.54×10^{-6}	4.00×10^{-17}
0.4	0.08442175132616	0.08442144069844	8.73×10^{-8}	8.00×10^{-17}
0.5	0.07602386862994	0.07602343896769	1.38×10^{-6}	4.00×10^{-17}
0.6	0.06610800200252	0.06610746612411	2.14×10^{-6}	8.00×10^{-17}
0.7	0.05487214355447	0.05487152251737	2.32×10^{-6}	1.00×10^{-17}
0.8	0.04254064193952	0.04253995921774	2.05×10^{-6}	9.00×10^{-17}
0.9	0.02935972273965	0.02935899944435	1.48×10^{-6}	6.00×10^{-17}

Table 10. The table shows absolute error of present method at specific points for Example 4.8.

t	$a = 1$ $n = 10^3$	$a = 10$ $n = 10^3$	$a = 100$ $n = 10^3$	$a = 1$ $n = 10^4$	$a = 10$ $n = 10^4$	$a = 100$ $n = 10^4$	$a = 1$ $n = 10^5$	$a = 10$ $n = 10^5$	$a = 100$ $n = 10^5$
0.0	0	0	0	0	0	0	0	0	0
0.1	2.66×10^{-6}	2.56×10^{-5}	1.70×10^{-4}	2.67×10^{-7}	2.66×10^{-6}	2.56×10^{-5}	2.67×10^{-8}	2.67×10^{-7}	2.66×10^{-6}
0.2	1.14×10^{-5}	1.12×10^{-4}	9.03×10^{-4}	1.14×10^{-6}	1.14×10^{-5}	1.12×10^{-4}	1.14×10^{-7}	1.14×10^{-6}	1.14×10^{-5}
0.3	2.76×10^{-5}	2.72×10^{-4}	2.31×10^{-3}	2.77×10^{-6}	2.76×10^{-5}	2.72×10^{-4}	2.77×10^{-7}	2.77×10^{-6}	2.76×10^{-5}
0.4	5.29×10^{-5}	5.21×10^{-4}	4.54×10^{-3}	5.30×10^{-6}	5.29×10^{-5}	5.21×10^{-4}	5.30×10^{-7}	5.30×10^{-6}	5.29×10^{-5}
0.5	8.91×10^{-5}	8.80×10^{-4}	7.76×10^{-3}	8.92×10^{-6}	8.91×10^{-5}	8.80×10^{-4}	8.92×10^{-7}	8.92×10^{-6}	8.91×10^{-5}
0.6	1.38×10^{-4}	1.36×10^{-3}	1.21×10^{-2}	1.38×10^{-5}	1.38×10^{-4}	1.36×10^{-3}	1.38×10^{-6}	1.38×10^{-5}	1.38×10^{-4}
0.7	2.04×10^{-4}	2.01×10^{-3}	1.79×10^{-2}	2.04×10^{-5}	2.04×10^{-4}	2.01×10^{-3}	2.04×10^{-6}	2.04×10^{-5}	2.04×10^{-4}
0.8	2.89×10^{-4}	2.86×10^{-3}	2.55×10^{-2}	2.89×10^{-5}	2.89×10^{-4}	2.86×10^{-3}	2.89×10^{-6}	2.89×10^{-5}	2.89×10^{-4}
0.9	3.98×10^{-4}	3.93×10^{-3}	3.51×10^{-2}	3.98×10^{-5}	3.98×10^{-4}	3.93×10^{-3}	3.99×10^{-6}	3.98×10^{-5}	3.98×10^{-4}
1.0	5.34×10^{-4}	5.28×10^{-3}	4.71×10^{-2}	5.35×10^{-5}	5.34×10^{-4}	5.28×10^{-3}	5.35×10^{-6}	5.35×10^{-5}	5.34×10^{-4}

TABLE 11. The computation time (seconds) for selected values of a and n for Example 4.8.

n	$a=1$	$a=10$	$a=100$
1000	8.57	9.67	10.01
10000	103.1	109.26	106.73
100000	2032.06	1989.75	1785.75

Example 4.8. The nonlinear second order initial value problem is given

$$\begin{aligned} &x''(t)+\frac{x^2(t)}{1+x(t)}+x'(t)=2-2t-\frac{(e^t+t^2)^2}{1+e^t+t^2}, \quad t\in[0,a], \\ &x(0)=1, \quad x'(0)=1. \end{aligned} \tag{4.8}$$

An exact solution is $x(t)=e^t+t^2$. In this example, numerical results are presented for selected values of n, namely $n=10^3,10^4,10^5$. The absolute errors are evaluated at 11 representative points. The example clearly demonstrates that the proposed method is not only effective on the interval $[0,1]$, but also remains efficient on the extended intervals $[0,10]$ and $[0,100]$. Table 10 reports the absolute errors obtained for different values of n over these intervals, whereas Table 11 provides the corresponding computational times.

5. CONCLUSION

In this study, a robust numerical algorithm utilizing the PCAM technique is presented for obtaining approximate solutions to nonlinear second-order differential equations with initial value conditions. Initially, the PCAM approach is formulated for second-order differential equations, and the uniqueness of the solution for the DEPCA (2.3) is established. Subsequently, it is demonstrated that the residual error of the approximation $x_n(t)$ tends to zero as the value of n becomes sufficiently large. Several examples, showing the absolute and residual errors for approximate solutions to nonlinear second-order differential equations, are provided to demonstrate the effectiveness of the proposed method. The results obtained using present method are compared with those from other methods, such as VIM [5], HPM [4, 5, 9], ILTM [16], AGM [2], PSLPM [1], and ELM-WA [10]. The findings indicate that the absolute error of the proposed approach significantly decreases with an increasing number of points n, clearly outperforming the aforementioned techniques. The numerical results obtained in the article is calculated using the `Maple 12` software package on a computer with an Intel® Celeron® CPU N2840 @ 2.16GHz.

Data Availability. The datasets generated during and analyzed during the current study are available from the corresponding author on reasonable request.

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