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Two mode version of Hirota-Ramani equation and its abundant soliton structures

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Abstract

Wave propagation phenomena are connected to Hirota-Ramani equation (HRE) and as a member of the integrable PDEs, the HRE is used in many different domains. Considering this, we employ Kursonky's approach and developed the two-mode or Dual-mode version of the HRE that describes the propagation of two-wave solitons moving simultaneously in the same direction with mutual interaction that depends on an embedded phase-velocity parameter. The existing studies explore single mode version of HRE whereas two mode version remain unexplored. Motivated by this gap, we utilized the generalized Ricatti equation mapping method and the modified extended tanh-function method to find the different soliton structures of the developed model. By using these techniques we developed bright, dark, kink, anti-kink, singular and peakon shaped soliton solutions. Additionally, we performed a stability analysis of the LGH equation using the linear stability approach. A back substitution for each of the obtained result into the developed model was performed to ensure reliability and accuracy of the obtained solutions. The 3D and 2D graphical representation of some of the obtained results were portrayed. The efficacy and competence of the proposed methods in analyzing and obtaining soliton solutions for nonlinear partial differential equations (NLPDEs) are demonstrated through their implementation in this work. Additionally, the derived solutions are original and reflect contributions not previously documented in the literature

Keywords. Hirota-Ramani equation, Two-mode models, Stability analysis, Generalized Ricatti equation mapping method, Modified extended tanh-function method.

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1. INTRODUCTION

Nonlinear evolution equations (NLEEs) are a specific type of partial differential equations (PDEs). These equations are commonly used in numerous fields, including applied and pure mathematics, physics, chemistry, biology, and biochemistry, to model phenomena with physical significance [1]. Consequently, obtaining analytic solutions for NLEEs is essential for a proper understanding of the qualitative features of these occurrences. Analytical solutions to nonlinear wave equations can help decipher the mechanisms of various complex events. The importance of these analytical solutions lies in the fact that NLEEs represent a wide range of physical and mathematical phenomena. Thus, many innovative methods have been developed by mathematicians, engineers, and physicists, such as the Sine-Gordon expansion method [25], the Lie symmetry analysis [8, 31], the generalized tanh-coth method [14], tanh $\left(\phi(\frac{\xi}{2})\right)$-expansion

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method [26], generalized $\left(\frac{G'}{G}\right)$-expansion approach [12], tan $\left(\phi(\frac{\xi}{2})\right)$-expansion method [30], and many more [7, 16, 23]. Two-mode type is a new family of NLPDEs of the form [24]:

$$gv_{tt} - s^2 v_{xx} + \left(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x}\right) N(v, v_x, vv_x...) + \left(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x}\right) L(v_{nx}) = 0, \tag{1.1}$$

where $N(v, v_x, vv_x...)$, and $L(v_{nx})$; $n \geq 2$ are respectively the nonlinear terms and the linear terms that are involved in any given equation. $v(x,t)$ is the unknown field-function, $s > 0$ is the phase velocity, $|\beta| \leq 1$, $|\alpha| \leq 1$. α is called the parameter of nonlinearity while β is called the dispersion parameter. When $s = 0$ and we integrate with respect to t , the dual-mode problem is reduced to a PDE of first order in time t . That is

$$v_t + N(v, v_x, vv_x...) + L(v_{nx}). \tag{1.2}$$

In recent years, a few dual-mode equations have been constructed and studied. In [13, 27–29, 39], some authors have extracted abundant soliton solutions for the second-order Korteweg-de Vries equation. [34, 35] established a dual-mode Burger's and fourth-order Burger's and obtained their multiple soliton solutions by means of simplified Hirota approach or technique. In [19, 32], the tanh expansion approach and Hirota approach were carried out to seek possible solutions of two-mode coupled Burgers equation, coupled modified Korteweg-de Vries, and coupled Korteweg-de Vries equations. Also, dual-mode perturbed Burger's, the Ostrovsky, and the Schrodinger equations were constructed in [17, 18].

In addition, inspired by Korsunsky's technique, Wazwaz [35–38] has formulated two-mode versions of the Sharma-Tasso-Olver equation, the fourth-order Burgers' equation, the fifth-order KdV equation, higher-order modified KdV equations, and the KP equations, successfully deriving multiple-kink solutions through the simplified Hirota's method. Moreover, two additional two-mode models have been created using Korsunsky's framework, with their solutions obtained via the simplified bilinear method, the tanh-coth method, and the (G'/G)-expansion method. Such two-mode equations have been derived for the coupled Burgers equation, the coupled KdV equation, the coupled modified KdV equation, the KdV-Burgers equation, the third-order Fisher equation, the Kuramoto-Sivashinsky equation, and the higher-order Boussinesq-Burgers system [2, 5, 19–22, 32]. Furthermore, in [18, 40], the two-mode KdV equation and the two-mode Sharma-Tasso-Olver equation have been revisited, yielding additional solitary wave solutions. The two-mode concept has also been applied to Schrödinger equations [3**?**]. The dynamics of two-mode phenomena have also been explored in [4, 17]. It is important to highlight that the works mentioned above focus on presenting new two-mode equations derived from Korsunsky's method. We anticipate that further techniques will be developed to explore two-mode models.

Wave propagation phenomena are connected to Hirota-Ramani equation (HRE) and as a member of the integrable PDEs, the HRE is used in many different domains. Nonlinear Schrödinger-type equations, which are connected to the HRE, explain how light pulses can generate non-dispersive, persistent solitons in optical fibers. For long-distance communication networks, these solitons are essential. The HRE can represent internal waves in stratified fluids or shallow water waves, much like the Korteweg-de Vries (KdV) equation.

The Hirota-Ramani equation (HRE) is given as follows [6]

$$v_t - v_{xxt} + rv_x(1 - v_t) = 0, \tag{1.3}$$

where $r \neq 0$ and $v(x,t)$ is the amplitude of related to the wave mode; $v(x,t)$ represents wave amplitude, particle density, or another field quantity. The term of v_t is essential in describing the time-dependent behavior of the wave. It indicates that the system evolves dynamically rather than being static. The term of $-v_{xxt}$ reflects the influence of dispersion on the temporal evolution of the wave. In optics, it could describe group velocity dispersion, while in fluid dynamics, it represents the dispersive nature of shallow water waves or ion-acoustic waves in plasma. The nonlinear term $rv_x(1 - v_t)$ combines the spatial derivative of the wave v_x with a term that is dependent on v_t. The parameter r controls the strength of this nonlinearity. This term represents the interaction between the wave's spatial gradient (v_x) and its temporal evolution $(1 - v_t)$. The term rv_x reflects the influence of the spatial gradient on the system, often linked to nonlinear steepening or amplification of waves. The term of $1 - v_t$ acts as a modulation factor, dynamically adjusting the strength of the nonlinearity based on the wave's temporal behavior.

In this paper, the Kursonky's approach is applied to HRE (1.3) to develop two-mode version of the HRE, that is (TmHRE). The generalized Ricatti equation mappinhg method (GREMM) and the modified extended tanh-function method (METFM) are used to retrieve various soliton structures of the new model. These techniques have some limitations. The Riccati formulation of the GREMM allows one to convert a nonlinear PDE to the ODE form. This approach is particularly beneficial because it can lead to numerous types of solutions such as rational, hyperbolic, trigonometric, etc. while still allowing for an efficient way to create these solutions. One major drawback of the GREMM is that it generally requires very specific parameter constraints in order to extract appropriate solution. In contrast to METFM, the METFM is characterized by an extended tanh-function expansion that produces a variety of soliton, kink, anti-kink, and certain periodic solutions. It has the advantages of being simple and flexible for nonlinearities. However, the METFM may not generate as many complicated solution families as other methods, and it typically relies on a higher-order balance assumption which places limitations on the potential structures of solutions.

Applying Eq. (1.1) on Eq. (1.3), we have the two-mode version of HRE as follows:

$$v_{tt} - s^2 v_{xx} + \left(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x}\right)(r v_x(1 - v_t)) + \left(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x}\right)(-v_{xxt}) = 0. \tag{1.4}$$

The rest of the paper is organized as follows: Sections 2 and 3 provide the descriptions of the GREMM and METFM methods, respectively. Section 4 presents the applications of the methods. In section 5, a graphical representation of some of the obtained results is provided. Section 6 offers the stability analysis. Finally, section 7 concludes the study with some closing remarks.

2. Description of Methodologies

As with regards to GREMM, we assume that Eq. (1.4) is described by

$$v(\varsigma) = a_0 + \sum_{i=1}^{\varpi} a_i \psi^i(\varsigma), \; a_{\varpi} \neq 0, \tag{2.1}$$

where constants a_i, $i = 0, 1, 2, ...\varpi$ and ψ^i are parameters to be evaluated later. The function $\psi(\varsigma)$ satisfies the generalized Ricatti's equation defined by

$$\psi(\varsigma)' = \varrho_0 + \varrho_1 \psi(\varsigma) + \varrho_2 \psi^2, \; \varrho_2 \neq 0, \tag{2.2}$$

where $\varrho_0, \varrho_1,$ *and* ϱ_2 are constants. The resulting solutions of Equation (2.2) are categorized as follows: **Set 1**. For $\Omega = \varrho_1^2 - 4\varrho_0\varrho_1 > 0, \varrho_1\varrho_2 \neq 0$ and non-zero $\mathcal{M}$ and $\mathcal{N}$ are constants:

$$\psi_1(\varsigma) = -\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)\right),$$

$$\psi_2(\varsigma) = -\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)\right),$$

$$\psi_3(\varsigma) = -\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh(\sqrt{\Omega}\varsigma) \pm i\,\mathrm{sech}(\sqrt{\Omega}\varsigma)\right),$$

$$\psi_4(\varsigma) = -\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\coth(\sqrt{\Omega}\varsigma) \pm i\,\mathrm{csch}(\sqrt{\Omega}\varsigma)\right),$$

$$\psi_5(\varsigma) = -\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) + \coth\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)\right),$$

$$\psi_6(\varsigma) = \frac{1}{\varrho_2}\left(-\varrho_1 + \frac{\sqrt{\Omega(\mathcal{M}^2 + \mathcal{N}^2)} - \mathcal{M}\sqrt{\Omega}\cosh(\sqrt{\Omega}\varsigma)}{\mathcal{M}\sinh(\sqrt{\Omega}\varsigma) + \mathcal{N}}\right),$$

$$\psi_7(\varsigma) = \frac{1}{\varrho_2}\left(-\varrho_1 - \frac{\sqrt{\Omega(\mathcal{M}^2+\mathcal{N}^2)} - \mathcal{M}\sqrt{\Omega}\cosh(\sqrt{\Omega}\varsigma)}{\mathcal{M}\sinh(\sqrt{\Omega}\varsigma)+\mathcal{N}}\right),$$

$$\psi_8(\varsigma) = \frac{2\varrho_0\cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)}{\sqrt{\Omega}\sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right) - \varrho_1\cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)},$$

$$\psi_9(\varsigma) = \frac{-2\varrho_0\sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)}{\varrho_1\sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right) - \sqrt{\Omega}\cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)},$$

$$\psi_{10}(\varsigma) = \frac{2\varrho_0\cosh\left(\sqrt{\Omega}\varsigma\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\varsigma\right) - \varrho_1\cosh\left(\sqrt{\Omega}\varsigma\right) \pm i\sqrt{\Omega}},$$

$$\psi_{11}(\varsigma) = \frac{2\varrho_0\sinh\left(\sqrt{\Omega}\varsigma\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\varsigma\right) - \varrho_1\sinh\left(\sqrt{\Omega}\varsigma\right) \pm \sqrt{\Omega}},$$

$$\psi_{12}(\varsigma) = \frac{4\varrho_0\sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)\cosh\sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)}{2\sqrt{\Omega}\cosh^2\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) - 2\varrho_1\sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)\cosh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) - \sqrt{\Omega}}.$$

Set 2. For $\Omega = \varrho_1^2 - 4\varrho_0\varrho_1 < 0, \varrho_1\varrho_2 \neq 0$ and $\mathcal{M}$ and $\mathcal{N}$ are non-zero constants that satisfy $\mathcal{M}^2 - \mathcal{N}^2 > 0$:

$$\psi_{13}(\varsigma) = \frac{1}{2\varrho_2}\left(-\varrho_1 + \sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)\right),$$

$$\psi_{14}(\varsigma) = -\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)\right),$$

$$\psi_{15}(\varsigma) = \frac{1}{2\varrho_2}\left(-\varrho_1 + \sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}\varsigma) \pm \sec(\sqrt{-\Omega}\varsigma)\right)\right),$$

$$\psi_{16}(\varsigma) = -\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{-\Omega}\left(\cot(\sqrt{-\Omega}\varsigma) \pm \csc(\sqrt{-\Omega}\varsigma)\right)\right),$$

$$\psi_{17}(\varsigma) = \frac{1}{4\varrho_2}\left(-2\varrho_1 + \sqrt{-\Omega}\left(\tan(\frac{\sqrt{-\Omega}\varsigma)}{4} - \cot(\frac{\sqrt{-\Omega}\varsigma)}{4}\right)\right),$$

$$\psi_{18}(\varsigma) = \frac{1}{2\varrho_2}\left(-\varrho_1 + \frac{\pm\sqrt{\Omega(\mathcal{N}^2-\mathcal{M}^2)} \pm \mathcal{M}\sqrt{-\Omega}\cos(\sqrt{-\Omega}\varsigma)}{\mathcal{M}\sin(\sqrt{-\Omega}\varsigma)+\mathcal{N}}\right),$$

$$\psi_{19}(\varsigma) = \frac{2\varrho_0\cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)}{\sqrt{-\Omega}\sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right) + \varrho_1\cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)},$$

$$\psi_{20}(\varsigma) = \frac{2\varrho_0\sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)}{\sqrt{-\Omega}\cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right) - \varrho_1\sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)},$$

$$\psi_{21}(\varsigma) = -\frac{2\varrho_0\cos\left(\sqrt{-\Omega}\varsigma\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\varsigma\right) + \varrho_1\cos\left(\sqrt{-\Omega}\varsigma\right) \pm \sqrt{-\Omega}},$$

$$\psi_{22}(\varsigma) = -\frac{2\varrho_0\sin\left(\sqrt{-\Omega}\varsigma\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\varsigma\right) + \varrho_1\sin\left(\sqrt{-\Omega}\varsigma\right) \pm \sqrt{-\Omega}},$$

$$\psi_{23}(\varsigma) = \frac{4\varrho_0 \sin\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right)\cos\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right)}{2\sqrt{-\Omega}\cos^2\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right) - 2\varrho_1 \sin\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right)\cos\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right) - \sqrt{-\Omega}}.$$

Set 3. For $\varrho_0 = 0$ and $\varrho_1\varrho_2 \neq 0$:

$$\psi_{24}(\varsigma) = -\frac{\varrho_0\varsigma_0}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))},$$

$$\psi_{25}(\varsigma) = -\frac{\varrho_1(\cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))},$$

$$\psi_{26}(\varsigma) = -\frac{\varrho_1(\cosh(\varrho_1\varsigma) + \sinh(\varrho_1\varsigma))}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) + \sinh(\varrho_1\varsigma))}.$$

Set 4. For $\varrho_2 \neq 0$ and $\varrho_1 = \varrho_0 = 0$:

$$\psi_{27}(\varsigma) = -\frac{1}{\varrho_2\varsigma + \varsigma_0},$$

where ς_0 is an arbitrary constant.

3. THE DESCRIPTION OF THE MODIFIED EXTENDED TANH-FUNCTION METHOD

The fundamental principle of the modified extended tanh-function method (METFM) is expounded through the scrutiny of a specified partial differential equation (PDE) [10, 11]

$$P(v, v_t, v_x, v_y, \dots, v_t, \dots) = 0. \tag{3.1}$$

In the given context, P denotes a polynomial function that includes nonlinear components in its partial derivatives in the variables $v(x, y, \dots, t)$

$$v(x, y, \dots, t) = V(\varsigma),\ \zeta = kx + my + \cdots + gt. \tag{3.2}$$

Upon applying the transformation delineated in Eqs. (3.1) and (3.1) transforms into an ordinary differential equation (ODE) of the subsequent form

$$M\left(\psi(\varsigma), \psi'(\varsigma), \psi''(\varsigma), \dots\right) = 0. \tag{3.3}$$

Within the framework of the METFM, it is assumed that Eq. (3.3) can be represented by the following expression:

$$V(\varsigma) = \sum_{i=0}^{\varpi} a_i\psi(\varsigma)^i + \sum_{i=1}^{\varpi} b_i\psi(\varsigma)^{-i},\ a_\varpi \neq 0,\ b_\varpi \neq 0, \tag{3.4}$$

where constants a_i, b_i, $i = 0, 1, 2, \dots\varpi$ and ψ^i are parameters to be evaluated later. The function $\psi(\varsigma)$ satisfies the generalized Ricatti's equation defined by

$$\psi'(\varsigma) = \varrho_0 + \psi(\varsigma)^2. \tag{3.5}$$

Here, $\varrho_0 > 0$ is a parameter that will be found out afterward. Numerous solutions to Eq. (3.5) are possible, as may be shown below.

- If $\varrho_0 < 0$,

  $$\psi(\varsigma) = -\sqrt{-\varrho_0}tanh(\sqrt{-\varrho_0}\varsigma) \quad or \quad \psi(\varsigma) = -\sqrt{-\varrho_0}coth(\sqrt{-\varrho_0}\varsigma).$$

- If $\varrho_0 > 0$,

  $$\psi(\varsigma) = \sqrt{\varrho_0}tan(\sqrt{\varrho_0}\varsigma) \quad or \quad \psi(\varsigma) = -\sqrt{\varrho_0}cot(\sqrt{\varrho_0}\varsigma).$$

- If $\varrho_0 = 0$,

  $$\psi(\varsigma) = -\frac{1}{\varsigma}.$$

The highest-order derivative and the largest nonlinear variable are balanced to provide the positive integer ϖ in Eq. (3.4). By replacing Eq. (3.4), its derivative, and Eq. (3.5) into Eq. (3.3), as well as collecting all the terms of the same power ψ^i, $(i = 1, 2, \cdots, \varpi)$ and equating them to zero, one can use a symbolic computation tool to determine the values of a_i and b_i. By entering these values and the solutions into Eq. (3.4), one can obtain the exact solutions to Eq. (3.1).

4. APPLICATION OF THE METHODS

In this section, we employ both the GREMM and METFM to analyze the dual-mode Hirota-Ramani equation (TmHRE).

Using the transformation:

$$v(x,t) = V(\varsigma);\ \varsigma = x + ct, \tag{4.1}$$

the TmHRE becomes:

$$V''(\varsigma)\left(c^2 - 2cr(c - \alpha s)V'(\varsigma) + cr - s(\alpha r + s)\right) + cV^{(4)}(\varsigma)(\beta s - c). \tag{4.2}$$

Integrating (4.2) once and taking the constant of integration as zero, we have

$$V'(\varsigma)\left(c^2 + cr - s(\alpha r + s)\right) + cr(\alpha s - c)V'(\varsigma)^2 + cV^{(3)}(\varsigma)(\beta s - c). \tag{4.3}$$

Balancing between $V'(\varsigma)^2$ and $V^{(3)}(\varsigma)$ we have:

$$2(\varpi + 1) = \varpi + 3, \implies \varpi = 1. \tag{4.4}$$

4.1. Application of the GREMM.

Since $\varpi = 1$, from Eq. (4.4), then (2.1) becomes

$$V(\varsigma) = a_0 + a_1\psi(\varsigma). \tag{4.5}$$

Substituting equation (4.5) into Eq. (4.3) yields a polynomial in $\psi(\varsigma)$. By collecting like terms and solving the resulting system, we obtain the following cases:

Case 1: When $a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)}$; $\varrho_0 = \frac{c^2(-\varrho_1^2) + c^2 + cr + \beta c\varrho_1^2 s - \alpha rs - s^2}{4c\varrho_2(\beta s - c)}$.

Set 1. For $\Omega = \varrho_1^2 - 4\varrho_0\varrho_1 > 0, \varrho_1\varrho_2 \neq 0$ and non-zero $\mathcal{M}$ and $\mathcal{N}$ are constants:

$$V_1(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)\right)\right), \tag{4.6}$$

$$V_2(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)\right)\right), \tag{4.7}$$

$$V_3(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh(\sqrt{\Omega}\varsigma) \pm i\operatorname{sech}(\sqrt{\Omega}\varsigma)\right)\right), \tag{4.8}$$

$$V_4(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\coth(\sqrt{\Omega}\varsigma) \pm \operatorname{csch}(\sqrt{\Omega}\varsigma)\right)\right), \tag{4.9}$$

$$V_5(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) + \coth\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)\right)\right), \tag{4.10}$$

$$V_6(\varsigma) = a_0 + a_1\left(\frac{1}{\varrho_2}\left(-\varrho_1 + \frac{\sqrt{\Omega(\mathcal{M}^2 + \mathcal{N}^2)} - \mathcal{M}\sqrt{\Omega}\cosh(\sqrt{\Omega}\varsigma)}{\mathcal{M}\sinh(\sqrt{\Omega}\varsigma) + \mathcal{N}}\right)\right), \tag{4.11}$$

$$V_7(\varsigma) = a_0 + a_1\left(\frac{1}{\varrho_2}\left(-\varrho_1 - \frac{\sqrt{\Omega(\mathcal{M}^2 + \mathcal{N}^2)} - \mathcal{M}\sqrt{\Omega}\cosh(\sqrt{\Omega}\varsigma)}{\mathcal{M}\sinh(\sqrt{\Omega}\varsigma) + \mathcal{N}}\right)\right), \tag{4.12}$$

$$V_8(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)}{\sqrt{\Omega}\sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right) - \varrho_1 \cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)} \right), \tag{4.13}$$

$$V_9(\varsigma) = a_0 + a_1 \left(\frac{-2\varrho_0 \sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)}{\varrho_1\sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right) - \sqrt{\Omega} \cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)} \right), \tag{4.14}$$

$$V_{10}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \cosh\left(\sqrt{\Omega}\varsigma\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\varsigma\right) - \varrho_1 \cosh\left(\sqrt{\Omega}\varsigma\right) \pm i\sqrt{\Omega}} \right), \tag{4.15}$$

$$V_{11}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \sinh\left(\sqrt{\Omega}\varsigma\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\varsigma\right) - \varrho_1 \sinh\left(\sqrt{\Omega}\varsigma\right) \pm \sqrt{\Omega}} \right), \tag{4.16}$$

$$V_{12}(\varsigma) = a_0 + a_1 \left(\frac{4\varrho_0 \sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) \cosh\sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)}{2\sqrt{\Omega}\cosh^2\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) - 2\varrho_1 \sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) \cosh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) - \sqrt{\Omega}} \right), \tag{4.17}$$

where $v(x,t) = V(\varsigma)$, $a_0 = a_0$, $a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)}$, $\Omega = \varrho_1^2 - \frac{c^2\left(-\varrho_1^2\right) + c^2 + cr + \beta c \varrho_1^2 s - \alpha r s - s^2}{c(\beta s - c)}$, $\varrho_0 = \frac{c^2\left(-\varrho_1^2\right) + c^2 + cr + \beta c \varrho_1^2 s - \alpha r s - s^2}{4c\varrho_2(\beta s - c)}$, $\varrho_1 = \varrho_1$, $\varrho_2 = \varrho_2$.

Set 2. For $\Omega = \varrho_1^2 - 4\varrho_0\varrho_1 < 0, \varrho_1\varrho_2 \neq 0$ and $\mathcal{M}$ and $\mathcal{N}$ are non-zero constants that satisfy $\mathcal{M}^2 - \mathcal{N}^2 > 0$:

$$V_{13}(\varsigma) = a_0 + a_1 \left(\frac{1}{2\varrho_2} \left(-\varrho_1 + \sqrt{-\Omega} \tan\left(\frac{\sqrt{-\Omega}}{2}\varsigma \right) \right) \right), \tag{4.18}$$

$$V_{14}(\varsigma) = a_0 + a_1 \left(-\frac{1}{2\varrho_2} \left(\varrho_1 + \sqrt{-\Omega} \cot\left(\frac{\sqrt{-\Omega}}{2}\varsigma \right) \right) \right), \tag{4.19}$$

$$V_{15}(\varsigma) = a_0 + a_1 \left(\frac{1}{2\varrho_2} \left(-\varrho_1 + \sqrt{-\Omega} \left(\tan(\sqrt{-\Omega}\varsigma) \pm \sec(\sqrt{-\Omega}\varsigma) \right) \right) \right), \tag{4.20}$$

$$V_{16}(\varsigma) = a_0 + a_1 \left(-\frac{1}{2\varrho_2} \left(\varrho_1 + \sqrt{-\Omega} \left(\cot(\sqrt{-\Omega}\varsigma) \pm \csc(\sqrt{-\Omega}\varsigma) \right) \right) \right), \tag{4.21}$$

$$V_{17}(\varsigma) = a_0 + a_1 \left(\frac{1}{4\varrho_2} \left(-2\varrho_1 + \sqrt{-\Omega} \left(\tan(\frac{\sqrt{-\Omega}\varsigma)}{4} - \cot(\frac{\sqrt{-\Omega}\varsigma)}{4} \right) \right) \right), \tag{4.22}$$

$$V_{18}(\varsigma) = a_0 + a_1 \left(\frac{1}{2\varrho_2} \left(-\varrho_1 + \frac{\pm\sqrt{\Omega(\mathcal{N}^2 - \mathcal{M}^2)} \pm \mathcal{M}\sqrt{-\Omega}\cos(\sqrt{-\Omega}\varsigma)}{\mathcal{M}\sin(\sqrt{-\Omega}\varsigma) + \mathcal{N}} \right) \right), \tag{4.23}$$

$$V_{19}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)}{\sqrt{-\Omega}\sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right) + \varrho_1 \cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)} \right), \tag{4.24}$$

$$V_{20}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)}{\sqrt{-\Omega}\cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right) - \varrho_1 \sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)} \right), \tag{4.25}$$

$$V_{21}(\varsigma) = a_0 + a_1 \left(-\frac{2\varrho_0 \cos\left(\sqrt{-\Omega}\varsigma\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\varsigma\right) + \varrho_1 \cos\left(\sqrt{-\Omega}\varsigma\right) \pm \sqrt{-\Omega}} \right), \tag{4.26}$$

$$V_{22}(\varsigma) = a_0 + a_1\left(-\frac{2\varrho_0 \sin\left(\sqrt{-\Omega}\varsigma\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\varsigma\right) + \varrho_1 \sin\left(\sqrt{-\Omega}\varsigma\right) \pm \sqrt{-\Omega}}\right), \tag{4.27}$$

$$V_{23}(\varsigma) = a_0 + a_1\left(\frac{4\varrho_0 \sin\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right)\cos\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right)}{2\sqrt{-\Omega}\cos^2\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right) - 2\varrho_1 \sin\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right)\cos\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right) - \sqrt{-\Omega}}\right), \tag{4.28}$$

where $v(x,t) = V(\varsigma)$, $a_0 = a_0$, $a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)}$, $\Omega = \varrho_1^2 - \frac{c^2\left(-\varrho_1^2\right)+c^2+cr+\beta c\varrho_1^2 s-\alpha rs-s^2}{c(\beta s-c)}$, $\varrho_0 = \frac{c^2\left(-\varrho_1^2\right)+c^2+cr+\beta c\varrho_1^2 s-\alpha rs-s^2}{4c\varrho_2(\beta s-c)}$, $\varrho_1 = \varrho_1$, $\varrho_2 = \varrho_2$.

Set 3. For $\varrho_0 = 0$ and $\varrho_1\varrho_2 \neq 0$:

$$V_{24}(\varsigma) = a_0 + a_1\left(-\frac{\varrho_0\varsigma_0}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))}\right), \tag{4.29}$$

$$V_{25}(\varsigma) = a_0 + a_1\left(-\frac{\varrho_1(\cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))}\right), \tag{4.30}$$

$$V_{26}(\varsigma) = a_0 + a_1\left(-\frac{\varrho_1(\cosh(\varrho_1\varsigma) + \sinh(\varrho_1\varsigma))}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) + \sinh(\varrho_1\varsigma))}\right), \tag{4.31}$$

where $v(x,t) = V(\varsigma)$, $a_0 = a_0$, $a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)}$, $\varrho_0 = \frac{c^2\left(-\varrho_1^2\right)+c^2+cr+\beta c\varrho_1^2 s-\alpha rs-s^2}{4c\varrho_2(\beta s-c)}$, $\varrho_1 = \varrho_1$, $\varrho_2 = \varrho_2$, $\varsigma_0 =$ Any arbitrary constant.

Set 4. For $\varrho_2 \neq 0$ and $\varrho_1 = \varrho_0 = 0$:

$$V_{27}(\varsigma) = a_0 + a_1\left(-\frac{1}{\varrho_2\varsigma + \varsigma_0}\right), \tag{4.32}$$

where $v(x,t) = V(\varsigma)$, $a_0 = a_0$, $a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)}$, $\varrho_2 = \varrho_2$, $\varsigma_0 =$ Any arbitrary constant.

Case 2: When $a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)}$; $\varrho_0 = 0$; $\varrho_1 = \frac{\sqrt{c^2+cr-\alpha rs-s^2}}{\sqrt{c^2-\beta cs}}$;

Set 1. For $\Omega = \varrho_1^2 - 4\varrho_0\varrho_1 > 0, \varrho_1\varrho_2 \neq 0$ and non-zero $\mathcal{M}$ and $\mathcal{N}$ are constants:

$$V_{28}(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)\right)\right), \tag{4.33}$$

$$V_{29}(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)\right)\right), \tag{4.34}$$

$$V_{30}(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh(\sqrt{\Omega}\varsigma) \pm i\operatorname{sech}(\sqrt{\Omega}\varsigma)\right)\right), \tag{4.35}$$

$$V_{31}(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\coth(\sqrt{\Omega}\varsigma) \pm \operatorname{csch}(\sqrt{\Omega}\varsigma)\right)\right), \tag{4.36}$$

$$V_{32}(\varsigma) = a_0 + a_1\left(-\frac{1}{2\varrho_2}\left(\varrho_1 + \sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) + \coth\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)\right)\right), \tag{4.37}$$

$$V_{33}(\varsigma) = a_0 + a_1\left(\frac{1}{\varrho_2}\left(-\varrho_1 + \frac{\sqrt{\Omega(\mathcal{M}^2+\mathcal{N}^2)} - \mathcal{M}\sqrt{\Omega}\cosh(\sqrt{\Omega}\varsigma)}{\mathcal{M}\sinh(\sqrt{\Omega}\varsigma) + \mathcal{N}}\right)\right), \tag{4.38}$$

$$V_{34}(\varsigma) = a_0 + a_1\left(\frac{1}{\varrho_2}\left(-\varrho_1 - \frac{\sqrt{\Omega(\mathcal{M}^2+\mathcal{N}^2)} - \mathcal{M}\sqrt{\Omega}\cosh(\sqrt{\Omega}\varsigma)}{\mathcal{M}\sinh(\sqrt{\Omega}\varsigma) + \mathcal{N}}\right)\right), \tag{4.39}$$

$$V_{35}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)}{\sqrt{\Omega}\sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right) - \varrho_1 \cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)} \right), \tag{4.40}$$

$$V_{36}(\varsigma) = a_0 + a_1 \left(\frac{-2\varrho_0 \sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)}{\varrho_1\sinh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right) - \sqrt{\Omega} \cosh\left(\frac{\sqrt{\Omega}}{2}\varsigma\right)} \right), \tag{4.41}$$

$$V_{37}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \cosh\left(\sqrt{\Omega}\varsigma\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\varsigma\right) - \varrho_1 \cosh\left(\sqrt{\Omega}\varsigma\right) \pm i\sqrt{\Omega}} \right), \tag{4.42}$$

$$V_{38}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \sinh\left(\sqrt{\Omega}\varsigma\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\varsigma\right) - \varrho_1 \sinh\left(\sqrt{\Omega}\varsigma\right) \pm \sqrt{\Omega}} \right), \tag{4.43}$$

$$V_{39}(\varsigma) = a_0 + a_1 \left(\frac{4\varrho_0 \sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) \cosh\sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)}{2\sqrt{\Omega}\cosh^2\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) - 2\varrho_1 \sinh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right)\cosh\left(\frac{\sqrt{\Omega}}{4}\varsigma\right) - \sqrt{\Omega}} \right), \tag{4.44}$$

where $v(x,t) = V(\varsigma)$, $a_0 = a_0$, $a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)}$, $\Omega = \frac{c^2+cr-\alpha r s - s^2}{c^2-\beta c s}$, $\varrho_0 = 0$, $\varrho_1 = \frac{\sqrt{c^2+cr-\alpha r s-s^2}}{\sqrt{c^2-\beta c s}}$, $\varrho_2 = \varrho_2$.

Set 2. For $\Omega = \varrho_1^2 - 4\varrho_0\varrho_1 < 0, \varrho_1\varrho_2 \neq 0$ and $\mathcal{M}$ and $\mathcal{N}$ are non-zero constants that satisfy $\mathcal{M}^2 - \mathcal{N}^2 > 0$:

$$V_{40}(\varsigma) = a_0 + a_1 \left(\frac{1}{2\varrho_2} \left(-\varrho_1 + \sqrt{-\Omega} \tan\left(\frac{\sqrt{-\Omega}}{2}\varsigma \right) \right) \right), \tag{4.45}$$

$$V_{41}(\varsigma) = a_0 + a_1 \left(-\frac{1}{2\varrho_2} \left(\varrho_1 + \sqrt{-\Omega} \cot\left(\frac{\sqrt{-\Omega}}{2}\varsigma \right) \right) \right), \tag{4.46}$$

$$V_{42}(\varsigma) = a_0 + a_1 \left(\frac{1}{2\varrho_2} \left(-\varrho_1 + \sqrt{-\Omega} \left(\tan(\sqrt{-\Omega}\varsigma) \pm \sec(\sqrt{-\Omega}\varsigma) \right) \right) \right), \tag{4.47}$$

$$V_{43}(\varsigma) = a_0 + a_1 \left(-\frac{1}{2\varrho_2} \left(\varrho_1 + \sqrt{-\Omega} \left(\cot(\sqrt{-\Omega}\varsigma) \pm \csc(\sqrt{-\Omega}\varsigma) \right) \right) \right), \tag{4.48}$$

$$V_{44}(\varsigma) = a_0 + a_1 \Bigg(\frac{1}{4\varrho_2} \Bigg(-2\varrho_1 + \sqrt{-\Omega} \Bigg(\tan(\frac{\sqrt{-\Omega}\varsigma)}{4} - \cot(\frac{\sqrt{-\Omega}\varsigma)}{4} \Bigg) \Bigg) \Bigg), \tag{4.49}$$

$$V_{45}(\varsigma) = a_0 + a_1 \left(\frac{1}{2\varrho_2} \left(-\varrho_1 + \frac{\pm\sqrt{\Omega(\mathcal{N}^2 - \mathcal{M}^2)} \pm \mathcal{M}\sqrt{-\Omega}\cos(\sqrt{-\Omega}\varsigma)}{\mathcal{M}\sin(\sqrt{-\Omega}\varsigma) + \mathcal{N}} \right) \right), \tag{4.50}$$

$$V_{46}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)}{\sqrt{-\Omega}\sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right) + \varrho_1 \cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)} \right), \tag{4.51}$$

$$V_{47}(\varsigma) = a_0 + a_1 \left(\frac{2\varrho_0 \sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)}{\sqrt{-\Omega}\cos\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right) - \varrho_1 \sin\left(\frac{\sqrt{-\Omega}}{2}\varsigma\right)} \right), V_{48}(\varsigma) = a_0 + a_1 \Bigg(-\frac{2\varrho_0 \cos\left(\sqrt{-\Omega}\varsigma\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\varsigma\right) + \varrho_1 \cos\left(\sqrt{-\Omega}\varsigma\right) \pm \sqrt{-\Omega}} \tag{4.52}$$

$$V_{49}(\varsigma) = a_0 + a_1 \left(-\frac{2\varrho_0 \sin\left(\sqrt{-\Omega}\varsigma\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\varsigma\right) + \varrho_1 \sin\left(\sqrt{-\Omega}\varsigma\right) \pm \sqrt{-\Omega}} \right), \tag{4.53}$$

$$V_{50}(\varsigma) = a_0 + a_1 \left(\frac{4\varrho_0 \sin\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right) \cos\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right)}{2\sqrt{-\Omega}\cos^2\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right) - 2\varrho_1 \sin\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right) \cos\left(\frac{\sqrt{-\Omega}}{4}\varsigma\right) - \sqrt{-\Omega}} \right), \tag{4.54}$$

where $v(x,t) = V(\varsigma),\ \ a_0 = a_0,\ \ a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)},\ \ \Omega = \frac{c^2 + cr - \alpha r s - s^2}{c^2 - \beta c s},\ \ \varrho_0 = 0,\ \ \varrho_1 = \frac{\sqrt{c^2 + cr - \alpha r s - s^2}}{\sqrt{c^2 - \beta c s}},$ $\varrho_2 = \varrho_2$.

Set 3. For $\varrho_0 = 0$ and $\varrho_1\varrho_2 \neq 0$:

$$V_{51}(\varsigma) = a_0 + a_1 \left(-\frac{\varrho_0 \varsigma_0}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))} \right), \tag{4.55}$$

$$V_{52}(\varsigma) = a_0 + a_1 \left(-\frac{\varrho_1(\cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) - \sinh(\varrho_1\varsigma))} \right), \tag{4.56}$$

$$V_{53}(\varsigma) = a_0 + a_1 \left(-\frac{\varrho_1(\cosh(\varrho_1\varsigma) + \sinh(\varrho_1\varsigma))}{\varrho_2(\varsigma_0 + \cosh(\varrho_1\varsigma) + \sinh(\varrho_1\varsigma))} \right), \tag{4.57}$$

where $v(x,t) = V(\varsigma),\ a_0 = a_0,\ a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)},\ \varrho_0 = 0,\ \varrho_1 = \frac{\sqrt{c^2 + cr - \alpha r s - s^2}}{\sqrt{c^2 - \beta c s}},\ \varrho_2 = \varrho_2,\ \varsigma_0 =$ Any arbitrary constant.

Set 4. For $\varrho_2 \neq 0$ and $\varrho_1 = \varrho_0 = 0$:

$$V_{54}(\varsigma) = a_0 + a_1 \left(-\frac{1}{\varrho_2\varsigma + \varsigma_0} \right), \tag{4.58}$$

where $v(x,t) = V(\varsigma),\ \ a_0 = a_0,\ \ a_1 = -\frac{6(\beta\varrho_2 s - c\varrho_2)}{r(\alpha s - c)},\ \ \varrho_2 = \varrho_2,\ \ \varsigma_0 =$ Any arbitrary constant.

4.2. **Application of the METFM.** Since $\varpi = 1$, from Eq. (4.4), then (3.4) becomes

$$\psi(\varsigma) = a_0 + a_1\psi(\varsigma) + \frac{b_1}{\psi(\varsigma)}. \tag{4.59}$$

Substituting Eq. (4.59) into the Eq. (4.3) results in a polynomial in terms of $\psi(\varsigma)$. By collecting like powers and solving the corresponding system of equations, we obtain the following solution cases:

Case 1: When $a_1 = -\frac{6\alpha\left(\alpha r - \beta r + \beta^2(-s) + s\right)}{r(\alpha^2 r - \alpha\beta r - \alpha^2\beta s + 2\alpha s - \beta s)}; b_1 = \frac{3\left(\alpha^2 - 1\right)\left(\alpha r - \beta r + \beta^2(-s) + s\right)}{8r(\alpha - \beta)(\alpha^2(-r) + \alpha\beta r + \alpha^2\beta s - 2\alpha s + \beta s)}; c = \frac{(\alpha - \beta)(\alpha r + s)}{\alpha\beta - 1};$ $\varrho_0 = \frac{1 - \alpha^2}{16\alpha(\alpha - \beta)}$.

Set 1. For $\varrho_0 < 0$,

$$\begin{aligned} V_{55}(\varsigma) = a_0 &+ \frac{3\alpha\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\tanh\left(\frac{1}{4}\varsigma\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r - \beta r + \beta^2(-s) + s\right)}{2r\left(\alpha^2 r - \alpha\beta r - \alpha^2\beta s + 2\alpha s - \beta s\right)} \\ &- \frac{3\left(\alpha^2 - 1\right)\coth\left(\frac{1}{4}\varsigma\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r - \beta r + \beta^2(-s) + s\right)}{2r\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}(\alpha - \beta)\left(\alpha^2(-r) + \alpha\beta r + \alpha^2\beta s - 2\alpha s + \beta s\right)}, \end{aligned} \tag{4.60}$$

$$\begin{aligned} V_{56}(\varsigma) = a_0 &- \frac{3\left(\alpha^2 - 1\right)\tanh\left(\frac{1}{4}\varsigma\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r - \beta r + \beta^2(-s) + s\right)}{2r\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}(\alpha - \beta)\left(\alpha^2(-r) + \alpha\beta r + \alpha^2\beta s - 2\alpha s + \beta s\right)} \\ &+ \frac{3\alpha\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\coth\left(\frac{1}{4}\varsigma\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r - \beta r + \beta^2(-s) + s\right)}{2r\left(\alpha^2 r - \alpha\beta r - \alpha^2\beta s + 2\alpha s - \beta s\right)}, \end{aligned} \tag{4.61}$$

where $v(x,t) = V(\varsigma)$.

Set 2. For $\varrho_0 > 0$,

$$\begin{aligned} V_{57}(\varsigma) = a_0 &- \frac{3\alpha\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\tan\left(\frac{1}{4}\varsigma\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{2r\left(\alpha^2 r-\alpha\beta r-\alpha^2\beta s+2\alpha s-\beta s\right)} \\ &+ \frac{3\left(\alpha^2-1\right)\cot\left(\frac{1}{4}\varsigma\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{2r\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}(\alpha-\beta)\left(\alpha^2(-r)+\alpha\beta r+\alpha^2\beta s-2\alpha s+\beta s\right)}, \end{aligned} \tag{4.62}$$

$$\begin{aligned} V_{58}(\varsigma) = a_0 &+ \frac{3\alpha\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\cot\left(\frac{1}{4}\varsigma\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{2r\left(\alpha^2 r-\alpha\beta r-\alpha^2\beta s+2\alpha s-\beta s\right)} \\ &- \frac{3\left(\alpha^2-1\right)\tan\left(\frac{1}{4}\varsigma\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{2r\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}(\alpha-\beta)\left(\alpha^2(-r)+\alpha\beta r+\alpha^2\beta s-2\alpha s+\beta s\right)}, \end{aligned} \tag{4.63}$$

where $v(x,t) = V(\varsigma)$.

Case 2: When $a_1 = 0; b_1 = \frac{3\left(\alpha^2-1\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{2r(\alpha-\beta)(\alpha^2(-r)+\alpha\beta r+\alpha^2\beta s-2\alpha s+\beta s)}; c = \frac{(\alpha-\beta)(\alpha r+s)}{\alpha\beta-1}; \varrho_0 = \frac{1-\alpha^2}{4\alpha(\alpha-\beta)}$.

Set 1. For $\varrho_0 < 0$,

$$V_{59}(\varsigma) = a_0 - \frac{3\left(\alpha^2-1\right)\coth\left(\frac{1}{2}\varsigma\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{r\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}(\alpha-\beta)\left(\alpha^2(-r)+\alpha\beta r+\alpha^2\beta s-2\alpha s+\beta s\right)}, \tag{4.64}$$

$$V_{60}(\varsigma) = a_0 - \frac{3\left(\alpha^2-1\right)\tanh\left(\frac{1}{2}\varsigma\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{r\sqrt{-\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}(\alpha-\beta)\left(\alpha^2(-r)+\alpha\beta r+\alpha^2\beta s-2\alpha s+\beta s\right)}, \tag{4.65}$$

where $v(x,t) = V(\varsigma)$.

Set 2. For $\varrho_0 > 0$,

$$V_{61}(\varsigma) = a_0 + \frac{3\left(\alpha^2-1\right)\cot\left(\frac{1}{2}\varsigma\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{r\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}(\alpha-\beta)\left(\alpha^2(-r)+\alpha\beta r+\alpha^2\beta s-2\alpha s+\beta s\right)}, \tag{4.66}$$

$$V_{62}(\varsigma) = a_0 - \frac{3\left(\alpha^2-1\right)\tan\left(\frac{1}{2}\varsigma\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}\right)\left(\alpha r-\beta r+\beta^2(-s)+s\right)}{r\sqrt{\frac{1-\alpha^2}{\alpha(\alpha-\beta)}}(\alpha-\beta)\left(\alpha^2(-r)+\alpha\beta r+\alpha^2\beta s-2\alpha s+\beta s\right)}, \tag{4.67}$$

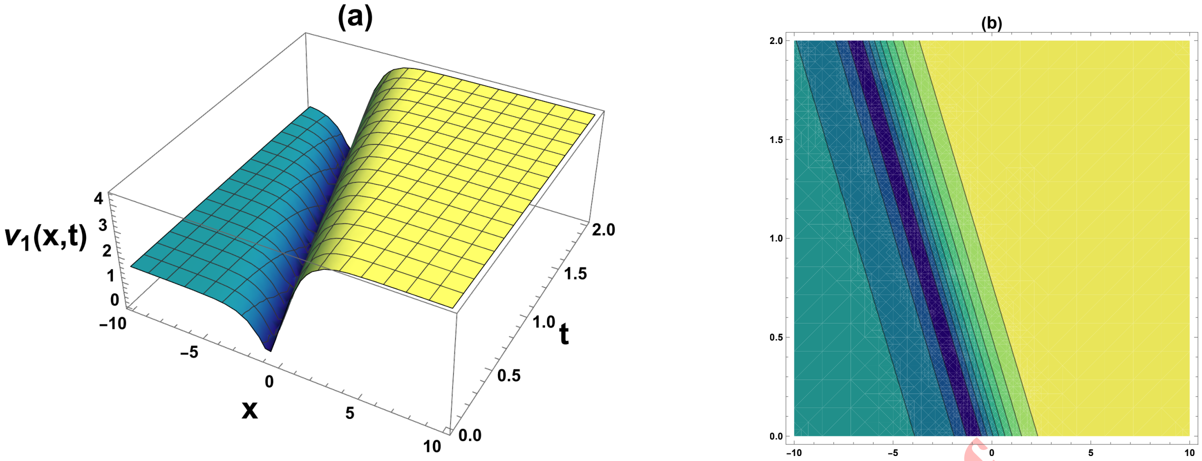
where $v(x,t) = V(\varsigma)$.

Case 3: When

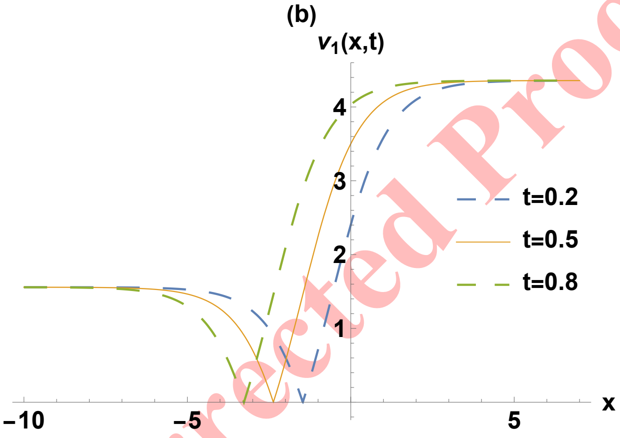
$$\begin{aligned} a_1 = &\frac{\frac{6\alpha\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c}+6\alpha c-6\beta c+6\alpha r-6\beta r-6\alpha\beta s+6s}{\frac{\alpha^2 rs(c^2+cr-\alpha rs-s^2)}{c(\beta s-c)}-\frac{\alpha\beta rs(c^2+cr-\alpha rs-s^2)}{c(\beta s-c)}+\alpha^2 rs-rs} \\ &- \frac{\frac{6\beta^2 s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}-\frac{6\beta\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c}-\frac{6\alpha\beta s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}}{\frac{\alpha^2 rs(c^2+cr-\alpha rs-s^2)}{c(\beta s-c)}-\frac{\alpha\beta rs(c^2+cr-\alpha rs-s^2)}{c(\beta s-c)}+\alpha^2 rs-rs}, \end{aligned}$$

$$b_1 = 0; \qquad \varrho_0 = \frac{c^2+cr-\alpha rs-s^2}{4c(\beta s-c)},$$

where $v(x,t) = V(\varsigma)$.



(A) 3D plot. (B) Contour plot.

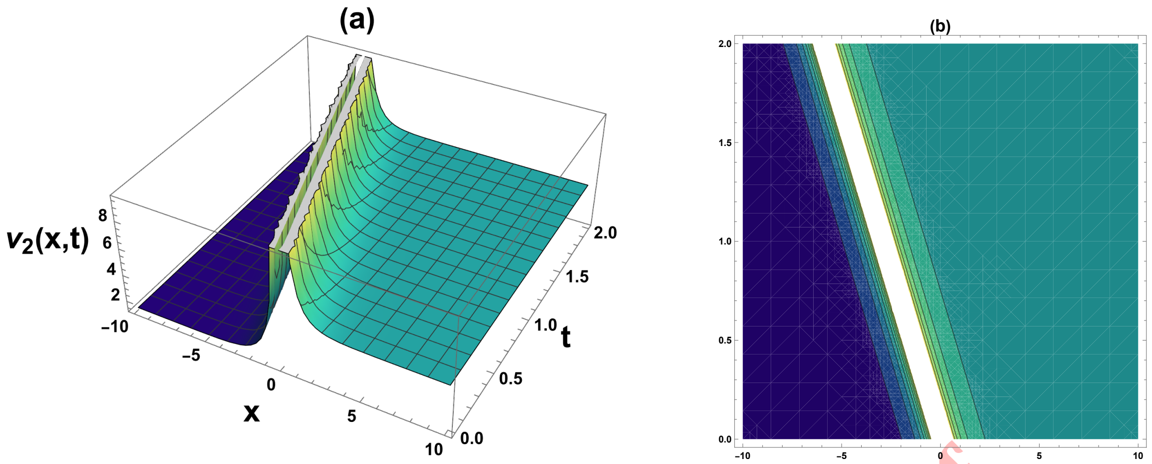


(C) 2D plot.

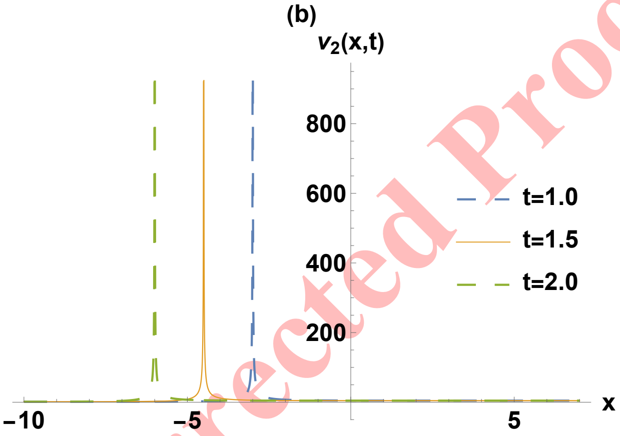
FIGURE 1. 3D, contour, and 2D absolute plot of $v_1(x,t)$ for parameters values $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0.$

Set 1. For $\varrho_0 < 0$,

$$\begin{aligned} V_{63}(\varsigma) = a_0 - &\frac{\sqrt{-\frac{c^2+cr-\alpha rs-s^2}{c(\beta s-c)}}}{2\left(\frac{\alpha^2 rs(c^2+cr-\alpha rs-s^2)}{c(\beta s-c)} - \frac{\alpha\beta rs(c^2+cr-\alpha rs-s^2)}{c(\beta s-c)} + \alpha^2 rs - rs\right)} \\ &\times \left(\frac{6\beta^2 s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)} - \frac{6\beta\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c} - \frac{6\alpha\beta s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}\right. \\ &\left. + \frac{6\alpha\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c} + 6\alpha c - 6\beta c + 6\ \alpha r - 6\beta r - 6\alpha\beta s + 6s\right) \\ &\times \tanh\left(\frac{1}{2}\varsigma\sqrt{-\frac{c^2+cr-\alpha rs-s^2}{c(\beta s-c)}}\right), \end{aligned} \tag{4.68}$$



(A) 3D plot. (B) Contour plot.

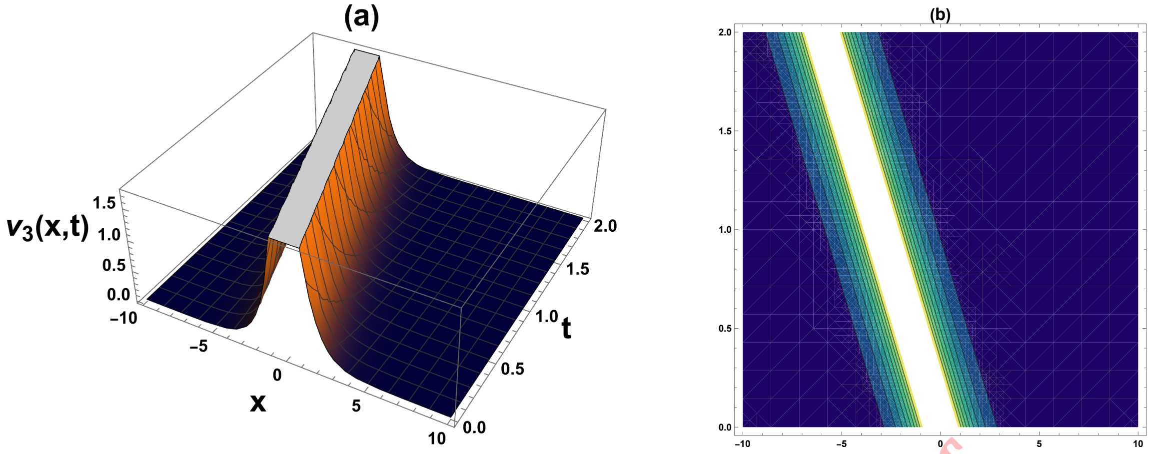


(C) 2D plot.

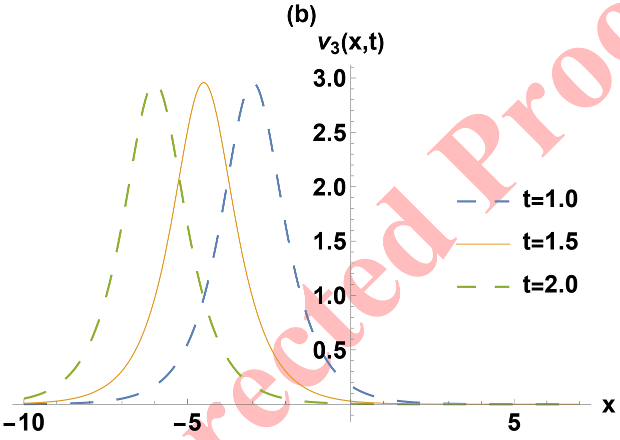
FIGURE 2. 3D, contour, and 2D absolute plot of $v_2(x,t)$ for parameters values $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0$.

$$\begin{aligned} V_{64}(\varsigma) = a_0 - \Bigg(\Bigg(\Bigg(\sqrt{-\frac{c^2+cr-\alpha rs-s^2}{c(\beta s-c)}}\Bigg(\frac{6\alpha\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c} + 6\alpha c - 6\beta c + 6\alpha r - 6\beta r - 6\alpha\beta s + 6s \\ - \frac{6\beta^2 s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)} - \frac{6\beta\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c} - \frac{6\alpha\beta s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}\Bigg) \\ \times \coth\left(\frac{1}{2}\varsigma\sqrt{-\frac{c^2+cr-\alpha rs-s^2}{c(\beta s-c)}}\right)\Bigg)\Bigg) \Big/ \Bigg(2\Bigg(\frac{\alpha^2 rs\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)} + \alpha^2 rs - rs \\ - \frac{\alpha\beta rs\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}\Bigg)\Bigg)\Bigg), \end{aligned} \tag{4.69}$$

where $v(x,t) = V(\varsigma)$.



(A) 3D plot. (B) Contour plot.

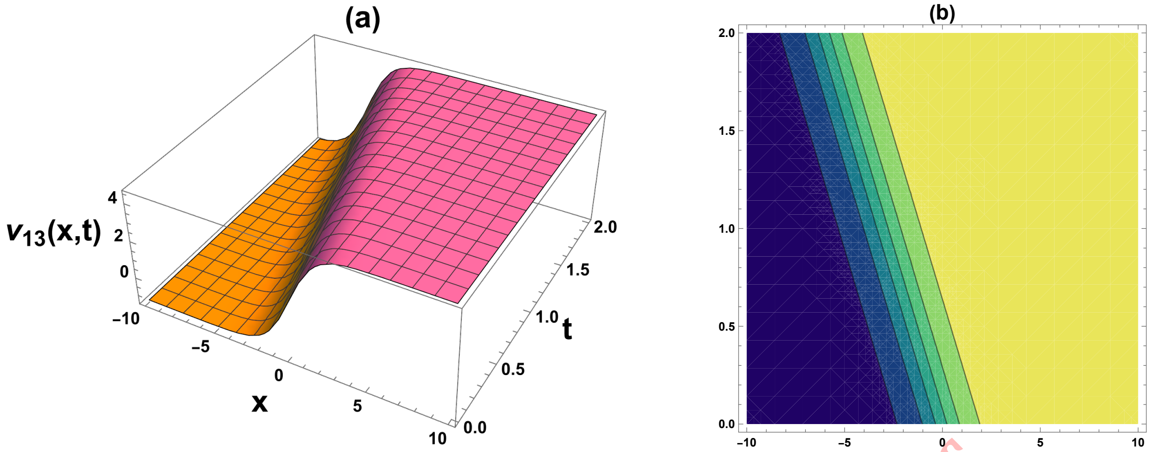


(C) 2D plot.

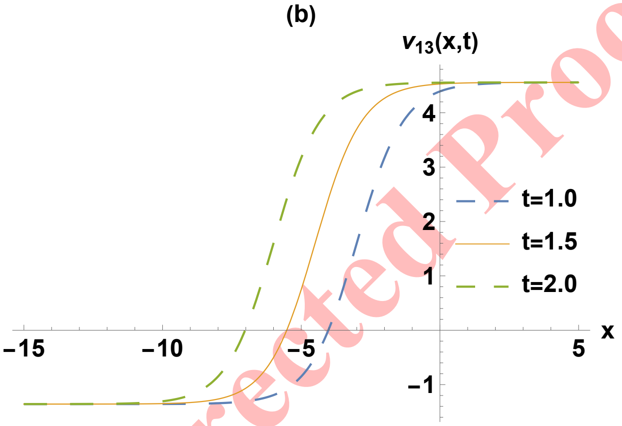
FIGURE 3. 3D, contour and 2D imaginary plot of $v_3(x,t)$ for parameters values $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0$.

Set 2. For $\varrho_0 > 0$,

$$V_{65}(\varsigma) = a_0 + \left(\sqrt{\frac{c^2+cr-\alpha rs-s^2}{c(\beta s-c)}} \left(\frac{6\alpha\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c} + 6\alpha c - 6\beta c + 6\alpha r - 6\beta r - 6\alpha\beta s + 6s \right.\right.$$
$$\left. - \frac{6\beta^2 s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)} - \frac{6\beta\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c} - \frac{6\alpha\beta s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)} \right)$$
$$\left.\times \tan\left(\frac{1}{2}\varsigma\sqrt{\frac{c^2+cr-\alpha rs-s^2}{c(\beta s-c)}}\right)\right) \Big/ \left(2\left(\frac{\alpha^2 rs\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)} + \alpha^2 rs - rs \right.\right.$$
$$\left.\left. - \frac{\alpha\beta rs\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}\right)\right), \tag{4.70}$$



(A) 3D plot. (B) Contour plot.

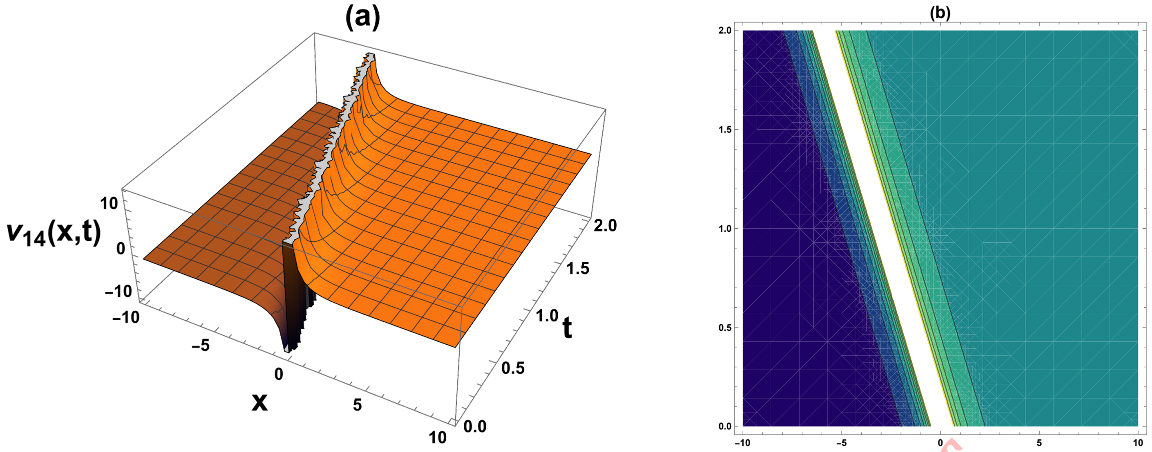


(C) 2D plot.

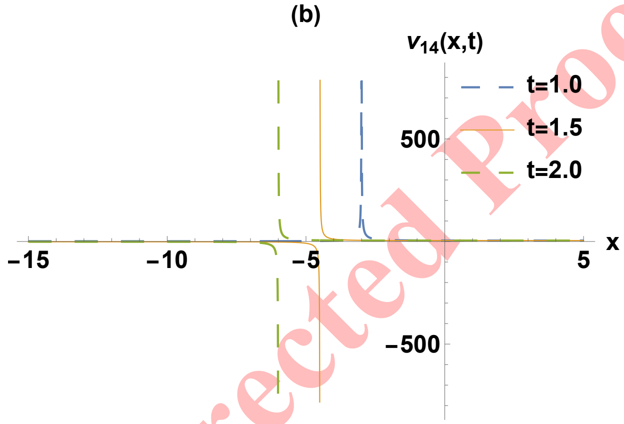
FIGURE 4. 3D, contour and 2D real plot of $v_4(x,t)$ for parameters values $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0$.

$$\begin{aligned}
V_{66}(\varsigma) = a_0 - {}&\Bigg(\Bigg(\Bigg(\sqrt{\frac{c^2+cr-\alpha rs-s^2}{c(\beta s-c)}}\Bigg(\frac{6\alpha\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c}+6\alpha c-6\beta c+6\alpha r-6\beta r-6\alpha\beta s+6s \\
&-\frac{6\beta^2 s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}-\frac{6\beta\left(c^2+cr-\alpha rs-s^2\right)}{\beta s-c}-\frac{6\alpha\beta s\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}\Bigg) \\
&\times\cot\left(\frac{1}{2}\varsigma\sqrt{\frac{c^2+cr-\alpha rs-s^2}{c(\beta s-c)}}\right)\Bigg)\Bigg)\Bigg/\Bigg(2\Bigg(\frac{\alpha^2 rs\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}+\alpha^2 rs-rs \\
&-\frac{\alpha\beta rs\left(c^2+cr-\alpha rs-s^2\right)}{c(\beta s-c)}\Bigg)\Bigg)\Bigg),
\end{aligned} \tag{4.71}$$

where $v(x,t) = V(\varsigma)$.



(A) 3D plot. (B) Contour plot.

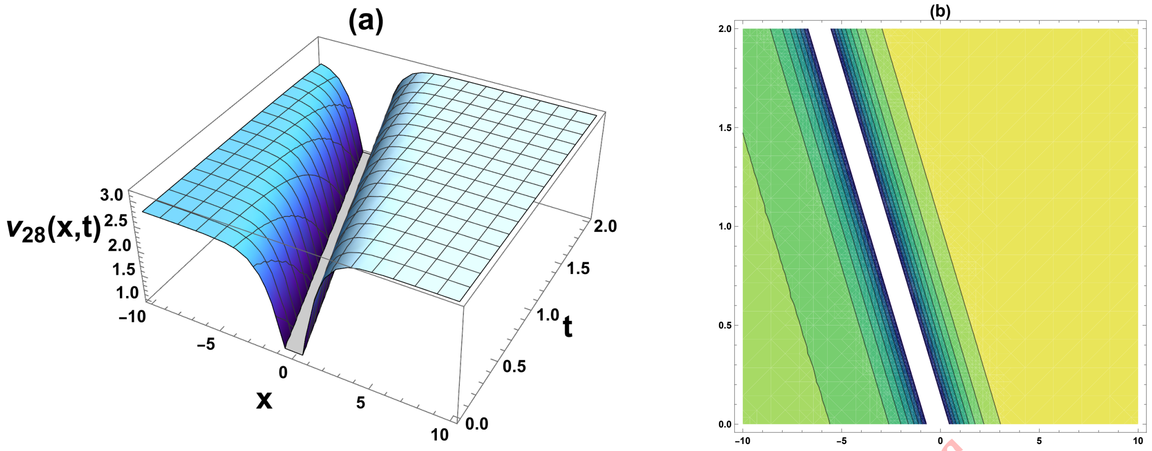


(C) 2D plot.

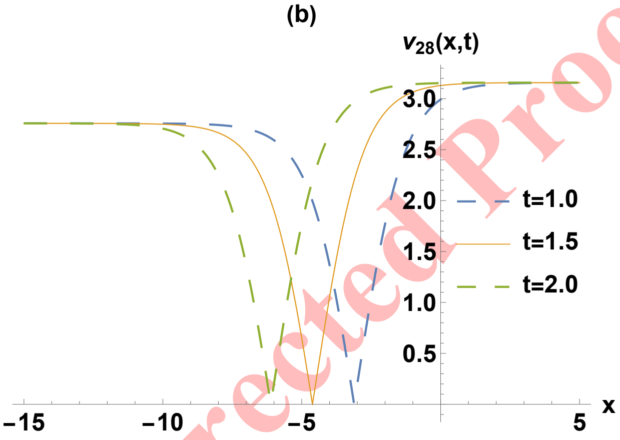
FIGURE 5. 3D, contour and 2D real plot of $v_5(x,t)$ for parameters values $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0$.

5. RESULTS AND DISCUSSION

The HRE is an integrable NLPDE that is mostly explored in the realm of soliton theory. In the context of integrable equations that yield exact solutions, it was determined by Hirota and Ramani that solitons remain steady, locally dispersed wave packets that keep their structure during propagation. The HRE is related to wave propagation occurrences. Wave propagation phenomena are intricately linked to the HRE. In the realm of physics, soliton equations serve to describe various types of waves that travel through space without undergoing dispersion. This includes waves in water, light traveling through optical fibers, and sound in certain media. In mathematics, soliton solutions emerge from a precise balance between dispersion where different wave frequencies move at different speeds—and nonlinearity, which signifies that the amplitude of a wave affects its velocity or form. Similar to other integrable models, the HRE encapsulates a particular nonlinear interaction that sustains the wave over time. In this section, the 3D and 2D graphical representation for some of the obtained results for the Tm-HRE, which is given by Equation (1.4) shall be presented by choosing suitable values of the parameters that are involved. In this study, we obtained several soliton solutions including kink, anti-kink, dark, bright, bright singular, and peakon type solutions for proposed model. The new Tm-HRE is a nonlinear model that describes the two coupled wave modes that interact physically



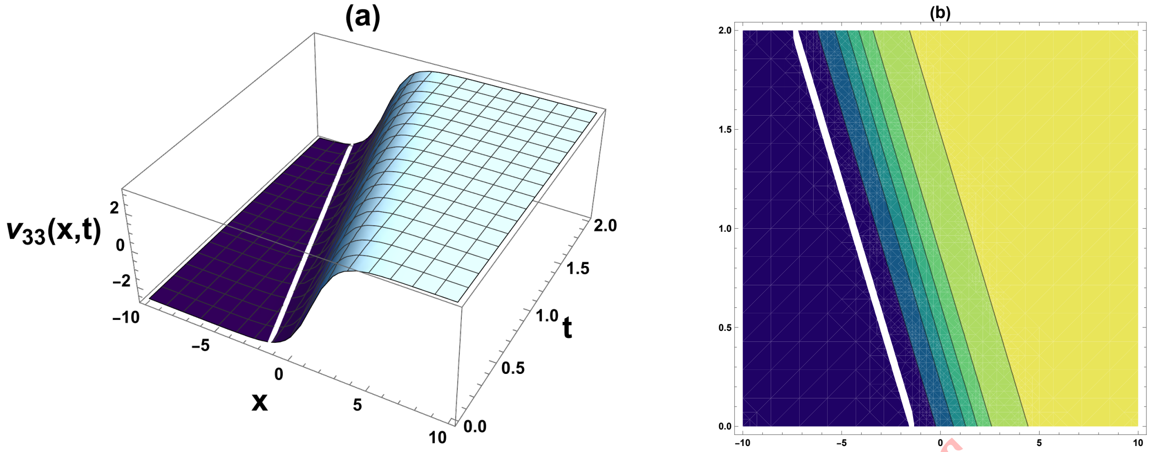
(A) 3D plot. (B) Contour plot.



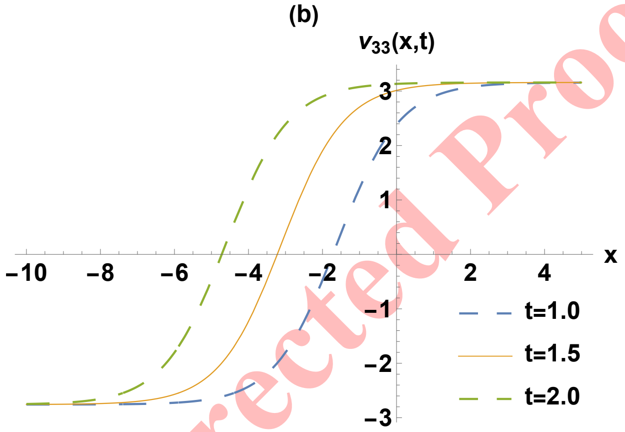
(C) 2D plot.

FIGURE 6. 3D, contour and 2D absolute plot of $v_6(x,t)$ for parameters values $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0$.

through their medium where they are propagated in. The proposed model has found applications in optical fibres, plasma physics, shallow water theory solid state physics and Bose Einstein condensates. Thus, the Tm-HRE provides for a deeper understanding and prediction of very complex forms of non-linear wave motion in multiple arenas of physical and engineering science. In Figure 1, we have the 3D, contour and 2D absolute plot of $v_1(x,t)$ by taking the following parameter values: $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0,$. The range of values for x-axis for the 3D, contour and 2D are [-10,10], [-10,10] and [-10,5] respectively. The 2D is plotted at $t = 0.2,\ 0.5,\ 0.8$. In Figure 2, we have the 3D, contour and 2D absolute plot of $v_2(x,t)$ by taking the following parameter values: $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0,$. The range of values for x-axis for the 3D, contour and 2D are [-10,10], [-10,10] and [-10,5] respectively. The 2D is plotted at $t = 1.0,\ 1.5,\ 2.0$. In Figure 3, we have the 3D, contour and 2D imaginary plot of $v_3(x,t)$ by taking the following parameter values: $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0,$. The range of values for the x-axis for the 3D, contour and 2D are [-10,10], [-10,10] and [-10,5], respectively. The 2D is plotted at $t = 1.0,\ 1.5,\ 2.0$. In Figure 4, we have the 3D, contour and 2D real plot of $v_{13}(x,t)$ by taking the following parameter values: $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0,$. The range of values for x-axis for the 3D, contour and 2D are [-10,10], [-10,10] and [-10,5] respectively. The 2D is plotted at



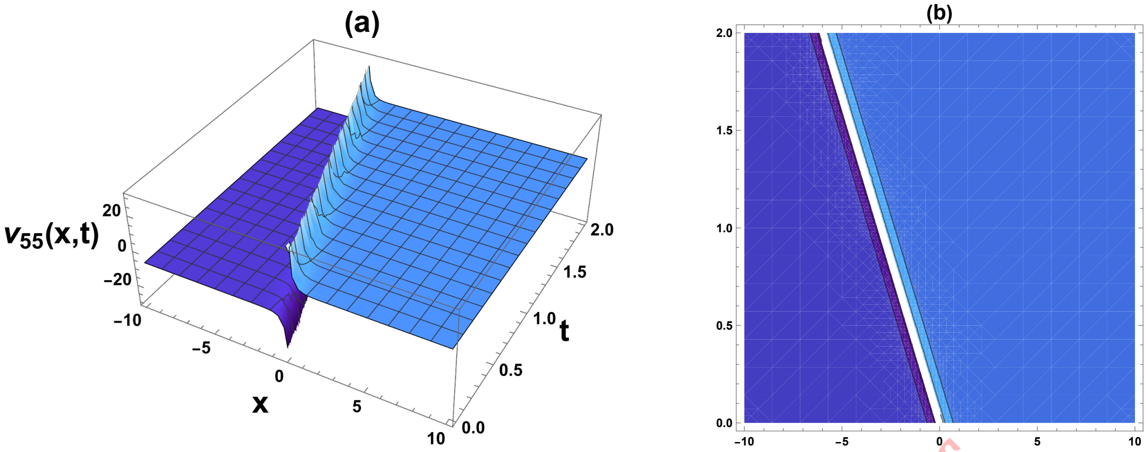
(A) 3D plot. (B) Contour plot.



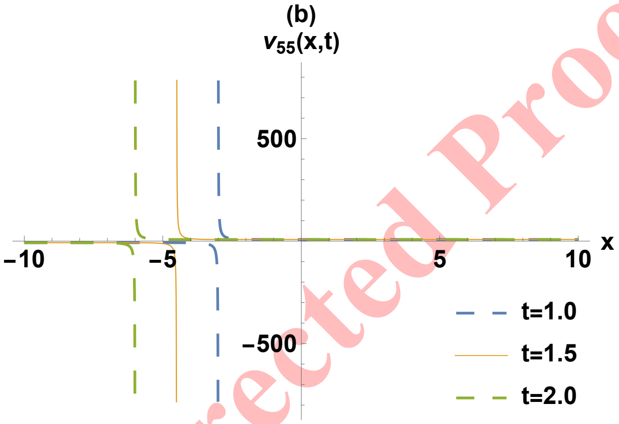
(C) 2D plot.

FIGURE 7. 3D, contour and 2D real plot of $v_7(x,t)$ for parameters values $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0,\ \mathcal{M} = 2,\ \mathcal{N} = 5.$

$t = 1.0,\ 1.5,\ 2.0$. In Figure 5, we have the 3D, contour and 2D real plot of $v_{14}(x,t)$ by taking the following parameter values: $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0,$. The range of values for x-axis for the 3D, contour and 2D are [-10,10], [-10,10] and [-10,5] respectively. The 2D is plotted at $t = 1.0,\ 1.5,\ 2.0$. In Figure 6, we have the 3D, contour and 2D absolute plot of $v_{28}(x,t)$ by taking the following parameter values: $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0,$. The range of values for x-axis for the 3D, contour and 2D are [-10,10], [-10,10] and [-10,5] respectively. The 2D is plotted at $t = 1.0,\ 1.5,\ 2.0$. In Figure 7 above, we have the 3D, contour and 2D real plot of $v_{33}(x,t)$ by taking the following parameter values: $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0,\ \mathcal{M} = 2,\ \mathcal{N} = 5$. The range of values for x-axis for the 3D, contour and 2D are [-10,10], [-10,10] and [-10,5] respectively. The 2D is plotted at $t = 1.0,\ 1.5,\ 2.0$. In Figure 8 above, we have the 3D, contour and 2D plot of $v_{55}(x,t)$ by taking the following parameter values: $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0$. The range of x-axis values for the 3D, contour, and 2D plots is [-10, 10]. The 2D is plotted at $t = 1.0,\ 1.5,\ 2.0$. In Figure 9 above, we have the 3D, contour and 2D plot of $v_{57}(x,t)$ by taking the following parameter values: $\alpha = -0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0$. The range of x-axis values for the 3D, contour, and 2D plots is [-10, 10]. The 2D is plotted at $t = 1.0,\ 1.5,\ 2.0$.



(A) 3D plot. (B) Contour plot.



(C) 2D plot.

FIGURE 8. 3D, contour and 2D plot of $v_{55}(x,t)$ for parameters values $\alpha = 0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0.$

6. STABILITY ANALYSIS

This section will describe the stability analysis [9, 15, 41] for the proposed model. Consider the perturbed solution of the subsequent form

$$v(x,t) = \mu w(x,t) + A_0. \tag{6.1}$$

It is obvious that the constant A_0 is a steady state solution to Eq. (1.4). By inserting Eq. (6.1) in Eq. (1.4), we obtain the following result

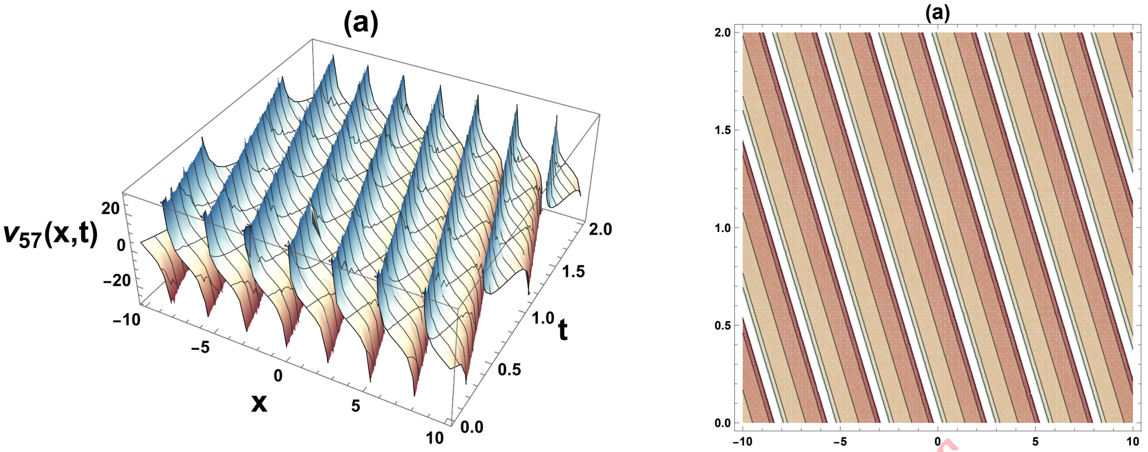
$$\mu w_{tt} - s^2\mu w_{xx} + \mu r w_{xt} - r\mu^2 w_x w_{tt} - r\mu^2 w_t w_{xx} - \alpha r s\mu w_{xx} + \alpha r s\mu 1^2 w_x w_{xt} + \alpha r s\mu^2 w_{xx} w_t - \mu w_{xxtt} + \mu\beta s w_{xxxt} = 0. \tag{6.2}$$

By linearizing the Eq. (6.2) in μ,

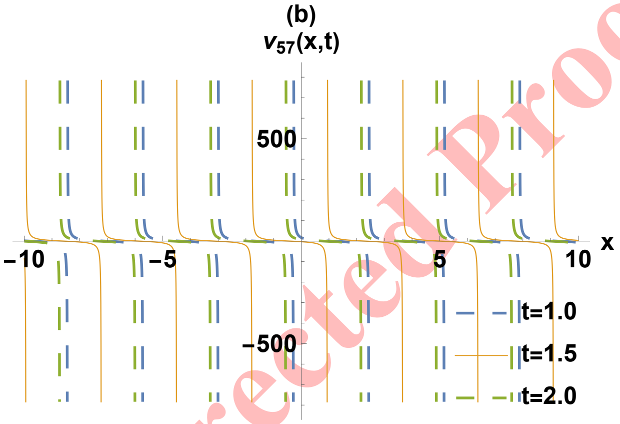
$$\mu w_{tt} - \mu s^2 w_{xx} + \mu r w_{xt} - \mu\alpha r s w_{xx} - \mu w_{xxtt} - \mu\beta s w_{xxxt} = 0. \tag{6.3}$$

Consider the subsequent solution for Eq. (6.3),

$$w(x,t) = e^{i(hx+\varpi)}, \tag{6.4}$$



(A) 3D plot. (B) Contour plot.



(C) 2D plot.

FIGURE 9. 3D, contour and 2D plot of $v_{57}(x,t)$ for parameters values $\alpha = -0.2,\ a_0 = 0.2,\ \beta = 0.7,\ c = 3,\ \varrho_1 = 0.56,\ \varrho_2 = 0.8,\ r = 1.2, s = 0$.

where h the normalized wave number and ϖ represents frequency. By inserting Eq. (6.4) into Eq. (6.3), we acquired the subsequent result

$$-\varpi^2 + s^2h^2 - rh\varpi + \alpha rsh^2 - h^2\varpi^2 - \beta sh^3\varpi = 0. \tag{6.5}$$

By solving Eq. (6.5) for ϖ,

$$\varpi = \frac{s^2h^2 + \alpha rsh^2}{\varpi + rh + h^2\varpi + \beta sh^3}. \tag{6.6}$$

It can be seen in Eq. (6.6) that the previously mentioned relation's sign is always positive for all values of $\frac{s^2h^2+\alpha rsh^2}{\varpi+rh+h^2\varpi+\beta sh^3} > 0$. As a result, the dispersion is unstable.

COMPARISON ANALYSIS

Here, we will compare our results with the solutions found in [6].

TABLE 1. Comparison analysis of our solutions with [6].

Solutions in [6]	Our solutions
(i) Utilized the modified Kudryashov method .	(i) Utilized two efficient techniques, the generalized Riccatti equation mapping method and the modified extended tanh-function method.
(ii) These solutions include trigonometric and hyperbolic trigonometric solutions.	(ii) These solutions included rational, hyperbolic trigonometric, and trigonometric forms.
(iii) By taking into account the proposed techniques, the kink, anti-kink, and singular solutions are generated.	(iii) It yields the bright, dark, kink, anti-kink, singular and peakon shaped soliton solutions. We also discussed the stability analysis of suggested model.

7. CONCLUSIONS

Wave propagation phenomena are fundamentally linked to the Hirota–Ramani equation (HRE), an integrable nonlinear partial differential equation that arises in various scientific and engineering applications. In this study, Kursonky's framework is utilized to construct a two-mode (dual-mode) extension of the HRE, which models the simultaneous propagation of two interacting solitons traveling in the same direction. The interaction is modulated by an embedded phase-velocity parameter. To derive exact soliton solutions of the proposed model, two analytical techniques, namely GREMM and METFM, are employed. These methods enable the construction of diverse soliton structures corresponding to the nonlinear dynamics of the system. By using these techniques we developed bright, dark, kink, anti-kink, singular and peakon shaped soliton solutions. The obtained results are remarkable and distinct from those found in the previously published literature. The obtained solutions are innovative and very interesting may be practically significant in diverse fields of interest. To have the clear and good understanding, absolute physical behavior of some reported solutions have been exhibited through 3D, and 2D graphs under the suitable values of parameters. The linear stability analysis is carried out to assess the dynamical stability of the obtained solutions. Furthermore, back-substitution of each derived solution into the original model is performed to confirm their correctness and analytical consistency. The reliability and robustness of both GREMM and METFM are further validated through symbolic computation using Mathematica. The findings presented herein are expected to be applicable to a wide range of physical contexts, including geophysics, civil engineering, pharmaceutical sciences, and other nonlinear wave-based systems. The methodologies adopted in this work can also be extended to other nonlinear models encountered in applied mathematics and engineering disciplines.

Data Availability Statement. All the data used is presented in the paper.

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