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On the variable-order beta derivative with properties and applications

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Abstract

In this study, concepts of variable-order $\beta(\mu)$-derivative and $\beta(\mu)$-integral are presented. It is proven properties such as differentiability, continuity, linearity, commutative, associative etc. of variable-order $\beta(\mu)$-derivative. To prove that the generalized definition of the $\beta(\mu)$-derivative is effective, applicable and useful, it is considered a differential equation of variable-order. It is demonstrated solvability of the Rosenau-Hynam equation with variable-order $\beta(\mu)$-derivative as semi-analytical with the modified variational iteration method. It is examined the comparison of semi-analytical solutions with the exact solution and their oscillation. It is commented on the usefulness, effectiveness and reliability of modified variational iteration method for relevant equations. It is enriched semi-analytical solutions of the Rosenau-Hynam equation with variable $\beta(\mu)$-orders such as trigonometric, exponential and hyperbolic functions. The results are interpreted with the help of tables and figures.

Keywords. Variable-order $\beta(\mu)$-derivative, Variable-order $\beta(\mu)$-integral, Generalized variable-order $\beta(\mu)$-derivative, Modified variational iteration method, The Rosenau-Hynam equation.

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1. Introduction

Recently, interest in explaining physical and engineering problems with models containing fractional derivatives has been increasing day by day. Especially, it is used fractional derivatives as Riemann-Liouville, Caputo, Riesz, Caputo-Fabrizio, conformable, Atangana-Baleanu etc. to explain various problems in these fields. The search for fractional derivatives to better explain these models is still ongoing. Although most real world problems can be modelled with these derivative definitions, they are not sufficient to explain and interpret the problems. Several researches on new derivatives have emerged to eliminate these problems. Many fractional derivatives in the literature do not correspond to the classical definition of a derivative and its properties. These fractional derivatives have the disadvantage of being related to higher order derivatives. However, the concepts of conformable derivative and beta derivative are both highly compatible with the classical derivative and provide most of its properties. They also have advantages in areas where higher order derivatives are used. In 2014, Khalil et al. [24] introduced a new definition of the derivative called conformable derivative. It is put forward studies such as [6, 7, 30, 38, 39]. In recent years, the conformable derivative has been extended as variable-order and started to be used in many application areas such as boundary-value problem, initial-value problem [16, 27, 40]. In 2014, Atangana and Goufo [9] introduced a new derivative called the beta derivative to overcome the inadequacy of the conformable derivative. The definition of beta derivative belongs to the class of conformable derivatives. Beta derivative (β-derivative) is one of the derivative definitions used especially in better explain physical, biological, engineering problems, modeling of groundwater problems, mathematical modeling of infectious and fatal diseases in various regions, double chain DNA modeling etc. Considering current studies, the application areas where the β-derivative definition is used are increasing day by day. This derivative definition appears to be more effective in many areas. There are many

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studies regarding β-derivative definition in the literature. To illustrate and prove effectiveness of β-definition, various differential equations and models were introduced by researchers such as the Chen-Lee-Liu equation [45], nonlinear β-fractional PDEs [19], the coupled Schrdinger-Boussinesq system [22], nonlinear mathematical models [4], the nonlinear time-fractional model [3], time fractional ocean engineering models [43], higher order Sasa-Satsuma equation [17], the nonlinear Radhakrishnan-Kundu-Lakshmanan equation [33], nonlinear fractional partial differential equations [31], the time-fractional unstable nonlinear Schrdinger equation [15], nonlinear fractional model [11], time fractional Biswas-Arshed equation [32], Modeling the spread of computer virus [12], double-chain deoxyribonucleic acid model [26], GrossPitaevskii system with linear magnetic [44], Ebola hemorrhagic fever [10], Quantum Magnetoplasmas [25], optical fiber [5]. In the literature, variable-order differential equations have been employed to better represent real-world problems. Variable-order Caputo, Riemann-Liouville, Caputo-Fabrizio, Atangana-Baleanu etc. derivative definitions and properties are available in the literature. These derivative definitions have been applied to real world problems in many studies [14, 18, 20, 21, 29, 34, 37, 41, 42].

In this study, the definitions of variable-order $\beta(\mu)$-derivative, variable-order $\beta(\mu)$-integral and generalized variable-order $\beta(\mu)$-derivative are introduced as a generalization of beta derivative and integral. It will be examined variable-order $\beta(\mu)$-derivative and its properties such as differentiability, continuity, linearity, etc. in order to more realistically express the problems that cannot be fully explained by fractional models. Theorems regarding the commutative and associative etc. properties of variable-order $\beta(\mu)$-derivatives are included. Moreover, it is also shown that the concepts of variable-order $\beta(\mu)$-derivatives and integrals satisfy the Fundamental Theorems I and II of Calculus. Due to these properties, variable-order $\beta(\mu)$-derivative distinguishes itself from other defined fractional derivatives. It is not provided many of these properties fractional derivatives such as Caputo, Riemann-Liouville, Caputo-Fabrizio. Variable-order $\beta(\mu)$-derivative has been applied to the Rosenau-Hynam (R-H) equation to demonstrate its advantages and effectiveness due to these properties. Later, it is studied on semi-analytical solutions of the R-H equation with variable-order $\beta(\mu)$-derivative. Effect of the arbitrary constant c arising from the exact solution of the R-H equation on its semi-analytical solutions is investigated. Our results are compared with VIM solutions in sense of Caputo in the literature. [28] It is shown how semi-analytical solutions oscillate compared to the analytical solution in applications for variable $\beta(\mu)$-orders such as trigonometric, exponential and hyperbolic functions.

2. Theoretical Foundations of Variable-Order Beta Calculus

Definition 2.1. Consider that $u:[0,\infty)\to\mathbb{R}$ is a first order differentiable function. So, the β-derivative of u is written below

$$ {}_0^A\mathrm{D}_\mu^\beta u(\mu)=\begin{cases}\lim\limits_{\varepsilon\to 0}\dfrac{u(\mu+\varepsilon(\mu+\frac{1}{\Gamma(\beta)})^{1-\beta})-u(\mu)}{\varepsilon}, & \forall\mu\geq 0,\beta\in(0,1],\\ u(\mu), & \forall\mu\geq 0,\beta=0.\end{cases} \tag{2.1}$$

Here Γ is the Gamma function and $\Gamma(\mu)=\int_0^\infty x^{\mu-1}e^{-x}dx$ [8].

Definition 2.2. Suppose that $u:[0,\infty)\to\mathbb{R}$ is a function that is differentiable up to order $n+1$ for $\beta\in(n,n+1]$, $n\in\mathbb{N}$. The generalized definition of β-derivative of u is given below.

$$ {}_0^A\mathrm{D}_\mu^\beta u(\mu)=\begin{cases}\lim\limits_{\varepsilon\to 0}\dfrac{u^{(\lceil\beta\rceil-1)}(\mu+\varepsilon(\mu+\frac{1}{\Gamma(\beta)})^{\lceil\beta\rceil-\beta})-u^{(\lceil\beta\rceil-1)}(\mu)}{\varepsilon}, & \forall\mu\geq 0,\beta\in(n,n+1],\\ u(\mu), & \forall\mu\geq 0,\beta=0.\end{cases} \tag{2.2}$$

Here $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Definition 2.3. Consider that $u:[0,\infty)\to\mathbb{R}$ is a first order differentiable function and $\beta:[0,\infty)\to(0,1]$ is a continuous function. So, the variable-order $\beta(\mu)$-derivative of u is given as

$$ {}_0^A\mathrm{D}_\mu^{\beta(\mu)} u(\mu)=\begin{cases}\lim\limits_{\varepsilon\to 0}\dfrac{u(\mu+\varepsilon(\mu+\frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)})-u(\mu)}{\varepsilon}, & \forall\mu\geq 0,\beta(\mu)\in(0,1],\\ u(\mu), & \forall\mu\geq 0,\beta(\mu)=0.\end{cases} \tag{2.3}$$

Definition 2.4. Assume that $u : [0, \infty) \to \mathbb{R}$ is a function that is differentiable up to order $n + 1$ for $n \in \mathbb{N}$ and $\beta : [0, \infty) \to (n, n + 1]$ is a continuous function. The generalized definition of variable-order $\beta(\mu)$-derivative of u is given as below

$$ {}_0^A\mathrm{D}_\mu^{\beta(\mu)} u(\mu) = \begin{cases} \lim\limits_{\varepsilon \to 0} \dfrac{u^{(\lceil\beta(\mu)\rceil-1)}(\mu + \varepsilon(\mu + \frac{1}{\Gamma(\beta(\mu))})^{\lceil\beta(\mu)\rceil-\beta(\mu)}) - u^{(\lceil\beta(\mu)\rceil-1)}(\mu)}{\varepsilon}, & \forall\mu \geq 0, \\ u(\mu), & \forall\mu \geq 0, \beta(\mu) = 0. \end{cases} \tag{2.4}$$

Definition 2.5. The β-integral of a continuous function $u : [a, b] \to \mathbb{R}$ is written as below [8].

$$ {}_a^A\mathrm{I}_\mu^\beta u(\mu) = \int_a^\mu (x + \frac{1}{\Gamma(\beta)})^{\beta-1} u(x) dx. \tag{2.5}$$

Definition 2.6. The variable-order $\beta(\mu)$-integral of u is defined as below

$$ {}_a^A\mathrm{I}_\mu^{\beta(\mu)} u(\mu) = \int_a^\mu (x + \frac{1}{\Gamma(\beta(x))})^{\beta(x)-1} u(x) dx, \tag{2.6}$$

where $u : [a, b] \to \mathbb{R}, \beta : [a, b] \to (0, 1]$ are continuous functions.

Theorem 2.7. *Assume that $u : [0, \infty) \to \mathbb{R}$ is a function and $\beta : [0, \infty) \to (0, 1]$ is a continuous function. If u is $\beta(\mu_0)$-differentiable at a point $\mu_0 > 0$ then u is continuous at point $\mu = \mu_0$.*

Proof. Let u be a $\beta(\mu_0)$-differentiable function. Then, following definition can be written

$$ {}_0^A\mathrm{D}_{\mu_0}^{\beta(\mu_0)} u(\mu_0) = \lim_{\varepsilon \to 0} \frac{u(\mu_0 + \varepsilon(\mu_0 + \frac{1}{\Gamma(\beta(\mu_0))})^{1-\beta(\mu_0)}) - u(\mu_0)}{\varepsilon}.$$

By substituting with $h = \varepsilon(\mu_0 + \frac{1}{\Gamma(\beta(\mu_0))})^{1-\beta(\mu_0)}$ transformation, the following equality is obtained:

$$ {}_0^A\mathrm{D}_{\mu_0}^{\beta(\mu_0)} u(\mu_0) = \lim_{h \to 0} \frac{u(\mu_0 + h) - u(\mu_0)}{h(\mu_0 + \frac{1}{\Gamma(\beta(\mu_0))})^{\beta(\mu_0)-1}}.$$

Also, it can be written as

$$ u(\mu_0 + h) = u(\mu_0) + h(\mu_0 + \frac{1}{\Gamma(\beta(\mu_0))})^{\beta(\mu_0)-1} \frac{u(\mu_0 + h) - u(\mu_0)}{h(\mu_0 + \frac{1}{\Gamma(\beta(\mu_0))})^{\beta(\mu_0)-1}},$$

by taking as limit for $h \to 0$ both sides of this equality, we get

$$ \lim_{h \to 0} u(\mu_0 + h) = u(\mu_0) + 0. {}_0^A\mathrm{D}_{\mu_0}^{\beta(\mu_0)} u(\mu_0) = u(\mu_0).$$

This means that the function u is continuous at point μ_0. $\square$

Theorem 2.8. *Suppose that $u : [0, \infty) \to \mathbb{R}$ is a function and $\beta : [0, \infty) \to (0, 1]$ is a continuous function. It is said that u is $\beta(\mu)$-differentiable once u is locally differentiable function.*

Proof. Let u be a differentiable function on the interval $[0, \infty)$, it has following limit

$$ \lim_{h \to 0} \frac{u(\mu + h) - u(\mu)}{h}.$$

So, $\lim\limits_{h \to 0} \left\{ (\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)} \left(\frac{u(\mu + h) - u(\mu)}{h} \right) \right\}$ has a real value. It gets following equality by substituting with $h = \varepsilon(\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}$ transformation

$$ {}_0^A\mathrm{D}_\mu^{\beta(\mu)} u(\mu) = \lim_{\varepsilon \to 0} \frac{u(\mu + \varepsilon(\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}) - u(\mu)}{\varepsilon}.$$

This means that the function u is $\beta(\mu)$-differentiable. $\square$

Theorem 2.9. *Let $u, v : [0, \infty) \to \mathbb{R}$ be locally differentiable functions for $\forall \mu > 0$ and $\beta : [0, \infty) \to (0, 1]$ be a continuous function. Then,*

- ${}_0^A D_\mu^{\beta(\mu)} u(\mu) = \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)} u'(\mu)$,
- ${}_0^A D_\mu^{\beta(\mu)} (au + bv)(\mu) = a {}_0^A D_\mu^{\beta(\mu)} u(\mu) + b {}_0^A D_\mu^{\beta(\mu)} v(\mu), \quad \forall a, b \in \mathbb{R}$,
- ${}_0^A D_\mu^{\beta(\mu)} (uv)(\mu) = v(\mu) {}_0^A D_\mu^{\beta(\mu)} u(\mu) + u(\mu) {}_0^A D_\mu^{\beta(\mu)} v(\mu)$,
- ${}_0^A D_\mu^{\beta(\mu)} (\frac{u}{v})(\mu) = \frac{v(\mu) {}_0^A D_\mu^{\beta(\mu)} u(\mu) - u(\mu) {}_0^A D_\mu^{\beta(\mu)} v(\mu)}{v(\mu)^2}, v(\mu) \neq 0$.

Proof. These properties can be proven by using definition of the variable-order $\beta(\mu)$-derivative (Definition 2.3). □

Remark 2.10. Variable-order $\beta(\mu)$-derivatives of some functions given below can be easily obtained from Definition 2.3 and Theorem 2.9:

- For $\forall m \in \mathbb{R}$, ${}_0^A \mathrm{D}_\mu^{\beta(\mu)} (\mu^m) = m\mu^{m-1} (\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}$.
- For $u(\mu) = c$, c constant, ${}_0^A \mathrm{D}_\mu^{\beta(\mu)} u(\mu) = 0$.
- For $\forall m \in \mathbb{R}$, ${}_0^A \mathrm{D}_\mu^{\beta(\mu)} (e^{m\mu}) = me^{m\mu} (\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}$.
- For $\forall m \in \mathbb{R}$,${}_0^A \mathrm{D}_\mu^{\beta(\mu)} sin(m\mu) = mcos(m\mu)(\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}$.
- For $\forall m \in \mathbb{R}$, ${}_0^A \mathrm{D}_\mu^{\beta(\mu)} cos(m\mu) = -msin(m\mu)(\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}$.

Theorem 2.11. *Suppose that $u : [0, \infty) \to \mathbb{R}, v : [0, \infty) \to [0, \infty)$ are differentiable functions on the interval $[0, \infty)$ and $\beta : [0, \infty) \to (0, 1]$ is a continuous function. So, the variable-order $\beta(\mu)$-derivative obeys the chain rule, meaning*

$$ {}_0^A D_\mu^{\beta(\mu)} (uov)(\mu) = \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)} v'(\mu).u'(v(\mu)). \tag{2.7} $$

Proof. Let u and v be differentiable functions on the interval $[0, \infty)$. Then, it can be written following equality

$$ {}_0^A \mathrm{D}_\mu^{\beta(\mu)} (uov)(\mu) = \lim_{\varepsilon \to 0} \frac{(uov)(\mu + \varepsilon(\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}) - (uov)(\mu)}{\varepsilon}. $$

By substituting with $h = \varepsilon(\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}$ transformation and by performing the necessary operations, the following expression is easily obtained.

$$ {}_0^A \mathrm{D}_\mu^{\beta(\mu)} (uov)(\mu) = \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)} v'(\mu).u'(v(\mu)). $$

□

Theorem 2.12. *Assume that $u : [0, \infty) \to \mathbb{R}$ is a second order differentiable and non-constant function on the interval $[0, \infty)$, $\beta : [0, \infty) \to (0, 1)$ is a first order differentiable function on the interval $[0, \infty)$ and $\gamma : [0, \infty) \to (0, 1)$ is a continuous function. Then,*

$$ {}_0^A D_\mu^{\gamma(\mu)+\beta(\mu)} u(\mu) \neq {}_0^A D_\mu^{\gamma(\mu)} \left({}_0^A D_\mu^{\beta(\mu)} u(\mu)\right). \tag{2.8} $$

Proof. Considering the differentiability properties of u, it can be written following equality

$$ {}_0^A \mathrm{D}_\mu^{\gamma(\mu)} \left({}_0^A \mathrm{D}_\mu^{\beta(\mu)} u(\mu)\right) = {}_0^A \mathrm{D}_\mu^{\gamma(\mu)} \left(\lim_{\varepsilon \to 0} \frac{u(\mu + \varepsilon(\mu + \frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}) - u(\mu)}{\varepsilon}\right). $$

By substituting with $h=\varepsilon(\mu+\frac{1}{\Gamma(\beta(\mu))})^{1-\beta(\mu)}$ transformation and by making the necessary arrangements,

$$
{}_0^A\mathrm{D}_\mu^{\gamma(\mu)}\left({}_0^A\mathrm{D}_\mu^{\beta(\mu)}u(\mu)\right)=\left[\left((1-\beta(\mu))\frac{1-\frac{\beta'(\mu)\Gamma'(\beta(\mu))}{\Gamma^2(\beta(\mu))}}{\left(\mu+\frac{1}{\Gamma(\beta(\mu))}\right)}-\beta'(\mu)ln(\mu+\frac{1}{\Gamma(\beta(\mu))})\right)u'(\mu)+u''(\mu)\right]
\left(\mu+\frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)}\left(\mu+\frac{1}{\Gamma(\gamma(\mu))}\right)^{1-\gamma(\mu)}.
$$

To determine the ${}_0^A\mathrm{D}_\mu^{\gamma(\mu)+\beta(\mu)}u(\mu)$ derivative, two cases should be considered.

Case I: if $0<\gamma(\mu)+\beta(\mu)\leq 1$ then

$$
{}_0^A\mathrm{D}_\mu^{\gamma(\mu)+\beta(\mu)}u(\mu)=\left(\mu+\frac{1}{\Gamma(\gamma(\mu)+\beta(\mu))}\right)^{1-\gamma(\mu)-\beta(\mu)}u'(\mu).
$$

Case II: if $1<\gamma(\mu)+\beta(\mu)\leq 2$ then

$$
{}_0^A\mathrm{D}_\mu^{\gamma(\mu)+\beta(\mu)}u(\mu)=\left(\mu+\frac{1}{\Gamma(\gamma(\mu)+\beta(\mu))}\right)^{2-\gamma(\mu)-\beta(\mu)}u''(\mu).
$$

Consequently,

$$
{}_0^A\mathrm{D}_\mu^{\gamma(\mu)+\beta(\mu)}u(\mu)\neq{}_0^A\mathrm{D}_\mu^{\gamma(\mu)}\left({}_0^A\mathrm{D}_\mu^{\beta(\mu)}u(\mu)\right).
$$

□

Theorem 2.13. *Consider that $u:[0,\infty)\to\mathbb{R}$ is a second order differentiable function and $\beta,\gamma:[0,\infty)\to(0,1)$ are first order differentiable functions on the interval $[0,\infty)$. Then,*

$$
{}_0^A D_\mu^{\gamma(\mu)}\left({}_0^A D_\mu^{\beta(\mu)}u(\mu)\right)\neq{}_0^A D_\mu^{\beta(\mu)}\left({}_0^A D_\mu^{\gamma(\mu)}u(\mu)\right). \tag{2.9}
$$

Proof. By the definition of variable-order $\beta(\mu)$-derivative and Theorem 2.9,

$$
{}_0^A\mathrm{D}_\mu^{\gamma(\mu)}\left({}_0^A\mathrm{D}_\mu^{\beta(\mu)}u(\mu)\right)=\left[\left((1-\beta(\mu))\frac{1-\frac{\beta'(\mu)\Gamma'(\beta(\mu))}{\Gamma^2(\beta(\mu))}}{\left(\mu+\frac{1}{\Gamma(\beta(\mu))}\right)}-\beta'(\mu)ln(\mu+\frac{1}{\Gamma(\beta(\mu))})\right)u'(\mu)+u''(\mu)\right]
\left(\mu+\frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)}\left(\mu+\frac{1}{\Gamma(\gamma(\mu))}\right)^{1-\gamma(\mu)}.
$$

Similarly,

$$
{}_0^A\mathrm{D}_\mu^{\beta(\mu)}\left({}_0^A\mathrm{D}_\mu^{\gamma(\mu)}u(\mu)\right)=\left[\left((1-\gamma(\mu))\frac{1-\frac{\gamma'(\mu)\Gamma'(\gamma(\mu))}{\Gamma^2(\gamma(\mu))}}{\left(\mu+\frac{1}{\Gamma(\gamma(\mu))}\right)}-\gamma'(\mu)ln(\mu+\frac{1}{\Gamma(\gamma(\mu))})\right)u'(\mu)+u''(\mu)\right]
\left(\mu+\frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)}\left(\mu+\frac{1}{\Gamma(\gamma(\mu))}\right)^{1-\gamma(\mu)}.
$$

Consequently,

$$
{}_0^A\mathrm{D}_\mu^{\gamma(\mu)}\left({}_0^A\mathrm{D}_\mu^{\beta(\mu)}u(\mu)\right)\neq{}_0^A\mathrm{D}_\mu^{\beta(\mu)}\left({}_0^A\mathrm{D}_\mu^{\gamma(\mu)}u(\mu)\right).
$$

□

Theorem 2.14. *Assume that $u:[0,\infty)\to\mathbb{R}$ is a second order differentiable function on the interval $[0,\infty)$ and $\beta:[0,\infty)\to(0,1)$ is a continuous function. Then,*

$$
{}_0^A D_\mu^{1+\beta(\mu)}u(\mu)\neq{}_0^A D_\mu^{\beta(\mu)}u'(\mu). \tag{2.10}
$$

Proof. By employing definition of variable-order $\beta(\mu)$-derivative, it can be written following equality

$$ {}_0^A\mathrm{D}_\mu^{\beta(\mu)} u'(\mu) = \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)} u''(\mu). $$

For $1 < 1+\beta(\mu) < 2$,

$$ {}_0^A\mathrm{D}_\mu^{1+\beta(\mu)} u(\mu) = \left(\mu + \frac{1}{\Gamma(1+\beta(\mu))}\right)^{2-\beta(\mu)} u''(\mu). $$

Consequently,

$$ {}_0^A\mathrm{D}_\mu^{1+\beta(\mu)} u(\mu) \neq {}_0^A\mathrm{D}_\mu^{\beta(\mu)} u'(\mu). $$

□

Theorem 2.15. *Suppose that* $u : [0,\infty) \to \mathbb{R}$ *is a second order differentiable function and* $\beta : [0,\infty) \to (0,1)$ *is a first order differentiable function on the interval* $[0,\infty)$*. Then,*

$$ \frac{d}{d\mu}\left({}_0^A D_\mu^{\beta(\mu)} u(\mu)\right) \neq {}_0^A D_\mu^{\beta(\mu)} u'(\mu). \tag{2.11} $$

Proof. By applying definition of variable-order $\beta(\mu)$-derivative, it can be written following as

$$ {}_0^A\mathrm{D}_\mu^{\beta(\mu)} u'(\mu) = \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)} u''(\mu). $$

Also,

$$ \frac{d}{d\mu}\left({}_0^A\mathrm{D}_\mu^{\beta(\mu)} u(\mu)\right) = \left[\left((1-\beta(\mu))\frac{1-\frac{\beta'(\mu)\Gamma'(\beta(\mu))}{\Gamma^2(\beta(\mu))}}{\left(\mu+\frac{1}{\Gamma(\beta(\mu))}\right)} - \beta'(\mu) ln(\mu + \frac{1}{\Gamma(\beta(\mu))})\right) u'(\mu) + u''(\mu)\right] \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)}. $$

Consequently,

$$ \frac{d}{d\mu}\left({}_0^A\mathrm{D}_\mu^{\beta(\mu)} u(\mu)\right) \neq {}_0^A\mathrm{D}_\mu^{\beta(\mu)} u'(\mu). $$

□

Theorem 2.16. *Let* $u : [0,b) \to \mathbb{R}, \beta : [0,b) \to (0,1]$ *be continuous functions. Then, for* $\forall\mu \geq 0$*,*

$$ {}_0^A D_\mu^{\beta(\mu)}\left({}_0^A I_\mu^{\beta(\mu)} u(\mu)\right) = u(\mu). \tag{2.12} $$

Proof. If $u : [0,b) \to \mathbb{R}$ be a continuous function, ${}_0^A\mathrm{I}_\mu^{\beta(\mu)} u(\mu)$ is differentiable. Then, by using definition of variable-order $\beta(\mu)$-derivative, it can be written as

$$ \begin{aligned} {}_0^A\mathrm{D}_\mu^{\beta(\mu)}\left({}_0^A\mathrm{I}_\mu^{\beta(\mu)} u(\mu)\right) &= \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)} \frac{d}{d\mu}\left(\int_0^\mu \left(x + \frac{1}{\Gamma(\beta(x))}\right)^{\beta(x)-1} u(x)dx\right) \\ &= \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{1-\beta(\mu)} \left(\mu + \frac{1}{\Gamma(\beta(\mu))}\right)^{\beta(\mu)-1} u(\mu) = u(\mu). \end{aligned} $$

□

Theorem 2.17. *Let* $u : [0,b) \to \mathbb{R}, \beta : [0,b) \to (0,1]$ *be continuous functions. If* u *is variable-order* $\beta(\mu)$*-differentiable for* $\forall\mu \geq 0$ *then*

$$ {}_0^A I_\mu^{\beta(\mu)}\left({}_0^A D_\mu^{\beta(\mu)} u(\mu)\right) = u(\mu) - u(0). \tag{2.13} $$

Proof. With the definition of variable-order $\beta(\mu)$-derivative and its properties, the equality can be easily proven. □

3. Main Idea of Modified Variational Iteration Method (MVIM)

A partial differential equation $F(x, \mu, u(x,\mu), \frac{\partial u(x,\mu)}{\partial x}, \frac{\partial u(x,\mu)}{\partial \mu}, ...) = 0$ can be given as

$$L_\mu u + R_x u + Nu = 0. \tag{3.1}$$

Where L_μ is the linear operator of time derivative, R_x is the linear operator and Nu is the non-linear part of the equation. The correction functional for Eq. (3.1) is given as

$$u_{n+1}(x,\mu) = u_0(x,\mu) + \int_0^\mu \lambda \left[R_x u_n(x,s) + Nu_n(x,s)\right] ds. \tag{3.2}$$

Here λ is a Lagrange multiplier. The Lagrange multiplier for $L_\mu = \frac{\partial^m}{\partial \mu^m}, m \geq 1$ is given as follows. [1]

$$\lambda(x,\mu) = \frac{(-1)^m}{(m-1)!}(\mu - x)^{m-1}. \tag{3.3}$$

The initial approximation function u_0 is generally taken as $u(x, 0)$. In order to get rid of unnecessary terms and transaction volume in the Eq. (3.2), the equation can be rearranged as

$$u_{n+1} = u_0 + \int_0^\mu \lambda \left[R_x u_{n-1} + G_{n-1}\right] ds + \int_0^\mu \lambda \left[R_x \left(u_n - u_{n-1}\right) + \left(G_n - G_{n-1}\right)\right] ds,$$

$$\Rightarrow u_{n+1} = u_n + \int_0^\mu \lambda \left[R_x \left(u_n - u_{n-1}\right) + \left(G_n - G_{n-1}\right)\right] ds. \tag{3.4}$$

Here, Eq. (3.4) is called the modified correction functional. Convergence polynomials G_n can be obtained from the equation

$$Nu_n = G_n + o(\mu^{n+1}). \tag{3.5}$$

For $n \geq 0$, the iteration process can be continued until the desired precision in Eq. (3.4) is achieved. Thus, the components u_0, u_1,u_2,u_3,... are found. Semi-analytical solutions without any convergence condition are expected to approach the exact solution as the iteration process increases. That is, $\lim_{n\to\infty} u_n = u(x,\mu)$. Unnecessary terms and processing load resulting from the variational iteration method are minimized. [2] VIM is a widely utilized technique in the literature and there are many modified versions of it. Some of these are used operational simplicity, to extend the range and improve accuracy. [23, 35, 36]

4. Applications

In this section, a nonlinear PDE with variable-order $\beta(\mu)$-derivative will be examined to test the solvability of equation as semi-analytical and the applicability of proposed method.

Example 4.1. *We consider the Rosenau-Hynam (R-H) equation with variable-order $\beta(\mu)$-derivative given as*

$$\begin{cases} {}_0^A D_\mu^{\beta(\mu)} u = uu_{xxx} + uu_x + 3u_x u_{xx}, \\ u(x,0) = -\frac{8c}{3} cos^2\left(\frac{x}{4}\right), \end{cases} \quad \beta : [0,\infty) \to (0,1]. \tag{4.1}$$

Here c is an arbitrary constant. The R-H equation emerges in the examination of the nonlinear dispersive of pattern stripe formation in liquid drops. By taking $\beta(\mu)$-function in the interval $(0, 1]$, the R-H equation with variable-order $\beta(\mu)$-derivative can be rewritten in the form

$$u_\mu - (\mu + \frac{1}{\Gamma(\beta(\mu))})^{\beta(\mu)-1} \left(uu_{xxx} + uu_x + 3u_x u_{xx}\right) = 0. \tag{4.2}$$

For $L_\mu = u, Nu = -(\mu + \frac{1}{\Gamma(\beta(\mu))})^{\beta(\mu)-1} \left(uu_{xxx} + uu_x + 3u_x u_{xx}\right)$ the R-H equation is written in the form $L_\mu u + Nu = 0$. The expression $G_n$is also obtained from the equation

$$-(\mu + \frac{1}{\Gamma(\beta(\mu))})^{\beta(\mu)-1} \left(uu_{xxx} + uu_x + 3u_x u_{xx}\right) = G_n + o(\mu^{n+1}). \tag{4.3}$$

TABLE 1. Comparison of exact solution with present solution for $\beta_0(\mu) = 1$.

μ	$c=0.5$ *Exact*	*MVIM*	*Absolute Error*	$c=1$ *Exact*	*MVIM*	*Absolute Error*
0	-0.6666666664	-0.6666666664	0	-1.333333333	-1.333333333	0
0.2	-0.6999861125	-0.6999861125	$1.8585E-10$	-1.466444556	-1.466444556	$2.6449E-11$
0.4	-0.7332222776	-0.7332222775	$1.9394E-10$	-1.598225774	-1.598225779	$3.3843E-09$
0.6	-0.7662920881	-0.7662920882	$4.3701E-12$	-1.727360275	-1.727360333	$5.7784E-08$
0.8	-0.7991128869	-0.7991128885	$1.4368E-09$	-1.852557790	-1.852558222	$4.3247E-07$
1.0	-0.8316026391	-0.8316026473	$7.7844E-09$	-1.972567385	-1.972569444	$2.0596E-06$

TABLE 2. Comparison of VIM solutions in sense of Caputo with present solutions for $c = 0.5$.

x	μ	$\alpha=0.7$ VIM [28]	$\beta(\mu)=0.7$ MVIM	$\alpha=0.8$ VIM [28]	$\beta(\mu)=0.8$ MVIM	$\alpha=0.9$ VIM [28]	$\beta(\mu)=0.9$ MVIM
$x=\frac{\pi}{4}$	0.2	-1.3027037	-1.2950527	-1.2995666	-1.2946667	-1.2968978	-1.2945300
	0.6	-1.3163161	-1.3136761	-1.3159022	-1.3132343	-1.3150924	-1.3133362
	1.0	-1.3194666	-1.3293006	-1.3228476	-1.3265973	-1.3251352	-1.3261255
$x=\frac{\pi}{2}$	0.2	-1.1781765	-1.1620129	-1.1715349	-1.1612407	-1.1658775	-1.1609678
	0.6	-1.2168471	-1.2026496	-1.2124978	-1.2017553	-1.2080227	-1.2020699
	1.0	-1.2404292	-1.2404689	-1.2415934	-1.2371960	-1.2413894	-1.2376901
$x=\pi$	0.2	-0.7263173	-0.7014323	-0.7160797	-0.7002813	-0.7073528	-0.6998750
	0.6	-0.7945239	-0.7651264	-0.7847888	-0.7637838	-0.7754812	-0.7643551
	1.0	-0.8456349	-0.8275599	-0.8421984	-0.8246536	-0.8374790	-0.8267183

The modified correction functional corresponding for Eq. (4.1) is obtained as

$$\begin{cases} u_{n+1} = u_n - \int_0^{\mu} (G_n - G_{n-1})\, ds, \\ u(x,0) = -\frac{8c}{3} cos^2\left(\frac{x}{4}\right), \end{cases}., G_{-1} = 0, n \geq 0. \tag{4.4}$$

For $\beta(\mu) = 1$, the exact solution of Eq. (4.1) is $u(x,\mu) = \frac{8c}{3} cos^2\left(\frac{x-c\mu}{4}\right)$. [13]

It is chosen exponential, hyperbolic and trigonometric functions in the range $(0,1]$ to demonstrate the effectiveness of $\beta(\mu)$-functions. Variable-order $\beta(\mu)$-functions used for R-H equation are as follows:

$$\beta_0(\mu) = 1, \quad \beta_1(\mu) = 1 - \frac{e^{-\mu}}{2}, \quad \beta_2(\mu) = 1 - \frac{cosh(\mu)}{2}, \quad \beta_3(\mu) = 1 - \frac{sin(\mu)^2}{4}, \quad \beta_4(\mu) = \frac{2}{3} + \frac{1}{3} sin^2(e^{\mu}).$$

In Table 1, it is shown comparison of semi-analytical solution and exact solution for c=0.5 and c=1 taking $\beta_0(\mu) = 1$. Also, it is seen that the exact solution and the semi-analytical solutions match one to one. In Tables 2-3, it is compared VIM solutions in sense of Caputo in [28] with present solutions for $c = 0.5, c = 1$, respectively. When compared to VIM solutions in the sense of Caputo, it can be said that there are equivalent solutions alternatively.

In Tables 4-5, it is shown comparisons of oscillations of the semi-analytical solutions obtained for various $\beta(\mu)$-functions and their changes with respect to μ as using 5 iterations and taking $x = \pi$ for $c = 0.5, c = 1$, respectively. It can be observed that the semi-analytical solutions with variable-order $\beta(\mu)$-derivative exhibit a stable behavior compared to the exact solution and are in accordance with each other.

In Figures 1-2, it is shown the changes of semi-analytical solutions according to μ for (a) $x = \pi, 0 \leq \mu \leq 1$, the changes of semi-analytical solutions according to x for (b) $\mu = 1, -2\pi \leq x \leq 2\pi$ when $c = 0.5, c = 1$, respectively.

TABLE 3. Comparison of VIM solutions in sense of Caputo with present solutions for $c = 1$.

x	μ	$\alpha = 0.7$ VIM [28]	$\beta(\mu) = 0.7$ MVIM	$\alpha = 0.8$ VIM [28]	$\beta(\mu) = 0.8$ MVIM	$\alpha = 0.9$ VIM [28]	$\beta(\mu) = 0.9$ MVIM
	0.2	-2.6345127	-2.6116190	-2.6251671	-2.6102992	-2.6172579	-2.6098297
$x = \frac{\pi}{4}$	0.6	-2.6345745	-2.6616555	-2.6474590	-2.6603316	-2.6563851	-2.6602928
	1.0	-2.5722042	-2.6848619	-2.6080979	-2.6695617	-2.6365766	-2.6631510
	0.2	-2.4277172	-2.3692138	-2.4036916	-2.3663043	-2.3832715	-2.3652745
$x = \frac{\pi}{2}$	0.6	-2.5350335	-2.5112297	-2.5303337	-2.5081314	-2.5232013	-2.5090430
	1.0	-2.5563768	-2.6251593	-2.5831444	-2.6098826	-2.6015950	-2.6084332
	0.2	-1.5711438	-1.4722061	-1.5304096	-1.4676207	-1.4957251	-1.4660018
$x = \pi$	0.6	-1.8301141	-1.7226074	-1.7949603	-1.7175738	-1.7612479	-1.7198462
	1.0	-1.9968865	-1.9547780	-1.9951864	-1.9461679	-1.9871083	-1.9545141

TABLE 4. Oscillations of semi-analytical solutions for various $\beta(\mu)$-functions when $c = 0.5$.

μ	$\beta_1(\mu)$	$\beta_2(\mu)$	$\beta_3(\mu)$	$\beta_4(\mu)$
0	-0.6666666664	-0.6666666664	-0.6666666664	-0.6666666664
0.2	-0.7031284153	-0.7066898187	-0.6999383643	-0.6981061562
0.4	-0.7361823038	-0.7430072971	-0.7329729820	-0.7309313598
0.6	-0.7670198550	-0.7761924659	-0.7653061898	-0.7639221894
0.8	-0.7968325911	-0.8068186895	-0.7964736588	-0.7958585556
1.0	-0.8268120369	-0.8354593320	-0.8260110591	-0.8255203697

TABLE 5. Oscillations of semi-analytical solutions for various $\beta(\mu)$-functions when $c = 1$.

μ	$\beta_1(\mu)$	$\beta_2(\mu)$	$\beta_3(\mu)$	$\beta_4(\mu)$
0	-1.333333333	-1.333333333	-1.333333333	-1.333333333
0.2	-1.478834848	-1.493115879	-1.466361173	-1.459156966
0.4	-1.609773241	-1.636563000	-1.597277202	-1.589303838
0.6	-1.730280776	-1.765489230	-1.723486283	-1.718059239
0.8	-1.844489726	-1.881709108	-1.842393280	-1.839708453
1.0	-1.956532369	-1.987037170	-1.951403058	-1.948536760

5. Conclusion

In this study, to show how variable-order behaves when comparing constant and variable-order, the concepts of variable-order $\beta(\mu)$-derivative and integral are introduced as a generalization of beta derivative and integral. Theoretical foundations of variable-order $\beta(\mu)$-derivative and integral definition are presented. These include features such as differentiability, continuity, linearity etc. in order to more realistically express the problems that cannot be fully explained by fractional models. Theorems regarding the commutative and associative etc. properties of variable-order $\beta(\mu)$-derivatives are included. Moreover, it is also shown that the concepts of variable-order $\beta(\mu)$-derivatives and integrals satisfy the Fundamental Theorems I and II of Calculus. It is discussed an application in order to verify the applicability of the variable-order $\beta(\mu)$-derivative. Semi-analytical solutions have been successfully obtained via MVIM. It is shown how semi-analytical solutions oscillate compared to the exact solution in application for variable $\beta(\mu)$-orders such as trigonometric, exponential and hyperbolic functions.

Effect of the arbitrary constant c arising from the solution of the R-H equation on its semi-analytical solutions is investigated. In Table 1, semi-analytical solutions of the R-H equation are obtained via MVIM for $c = 0.5, c = 1$

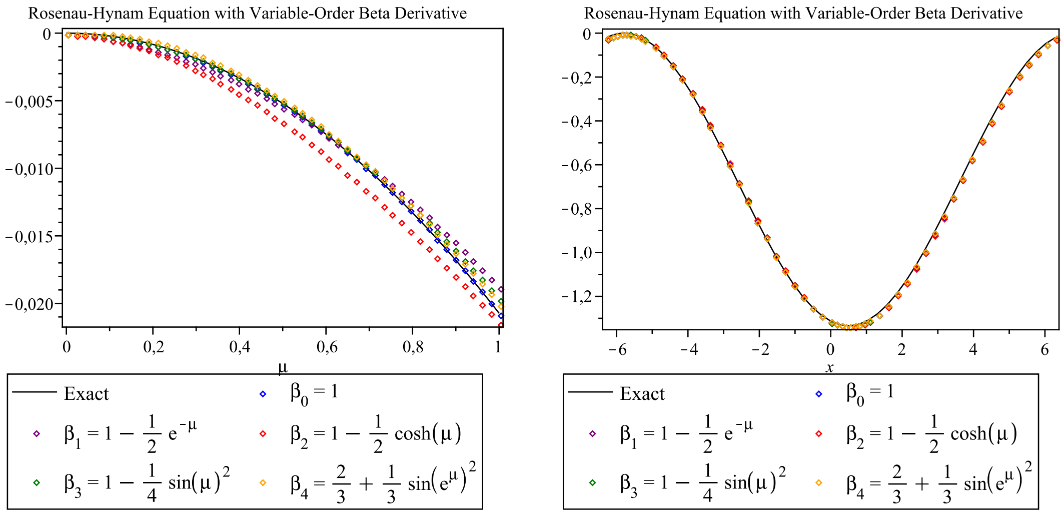

FIGURE 1. Comparison of semi-analytical and exact solutions for (a) $x = \pi, 0 \leq \mu \leq 1$ and (b) $\mu = 1, -2\pi \leq x \leq 2\pi$ via MVIM when $c = 0.5$.

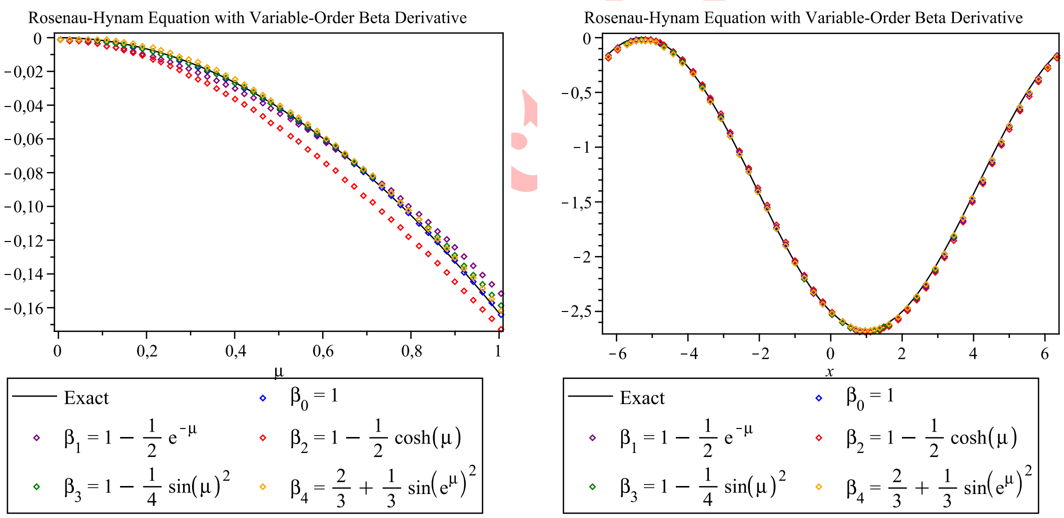

FIGURE 2. Comparison of semi-analytical and exact solutions for (a) $x = \pi, 0 \leq \mu \leq 1$ and (b) $\mu = 1, -2\pi \leq x \leq 2\pi$ via MVIM when $c = 1$.

when $0 < \beta(\mu) \leq 1$. In Tables 2-3, it is compared VIM solutions in sense of Caputo in [28] with present solutions for $c = 0.5, c = 1$, respectively. Results for $x = \pi$ and various $\beta(\mu)$ variable-orders are shown in Tables 4-5. For $c = 0.5$, maximum error for $\beta_0(\mu) = 1$ is around 10^{-9}. It has been observed that solutions oscillate around $10^{-3} - 10^{-5}$ when the solutions for variable-orders $\beta_1(\mu), \beta_2(\mu), \beta_3(\mu), \beta_4(\mu)$ are compared with each other. For $c = 1$, maximum error for $\beta_0(\mu) = 1$ is around 10^{-6}. It has been observed that solutions oscillate around $10^{-2} - 10^{-3}$ when the solutions for variable-orders $\beta_1(\mu), \beta_2(\mu), \beta_3(\mu), \beta_4(\mu)$ are compared with each other. In addition, it is shown comparison of exact and semi-analytical solutions of the R-H equation for $c = 0.5$ in Figure 1 and for $c = 1$ in Figure 2. Comparing our results with the VIM solution in [28] for various values of x and μ, it is seen that the MVIM solutions obtained are much better than the VIM solutions in sense of Caputo. Therefore, it is observed that the variable-order $\beta(\mu)$-derivative chosing as an alternative to the Caputo fractional derivative gives more efficient and convergent results to the exact solution when $\beta_0(\mu) = 1$ in the R-H equation. Also, semi-analytical solutions of the R-H equation with variable-order $\beta(\mu)$-derivative allow us to interpret its behaviors, oscillations and fluctuations compared to the exact solution.

For different values of arbitrary constant c, it has been determined that MVIM is fast and effective in finding semi-analytical solutions of the R-H equation with variable-order $\beta(\mu)$-derivative and that semi-analytical solutions are highly consistent with exact solutions. Solvability of the R-H equation with variable-order $\beta(\mu)$-derivatives has been demonstrated with MVIM. Finally, MVIM is extremely convenient to use, fast and finding convergent solutions for the R-H equation with variable-order $\beta(\mu)$-derivatives. MVIM is a method that can be adapted to variable-order differential equations and is quite free from processing load. It can be seen that MVIM can be used without overflow error even if different variable order functions other than trigonometric, exponential, hyperbolic variable order functions are taken.

Declarations

- **Conflict of interest**: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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