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Optimal control in professional Gambling: A fractional logistic dynamics approach with Caputo-Fabrizio derivatives

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Abstract

The growing prevalence of excessive gambling has become a major concern in the 21st century, requiring urgent attention to mitigate its potential negative consequences. To address this problem, various preventive strategies, such as public awareness campaigns and educational programs, aim to highlight the detrimental effects of gambling addiction. Advanced mathematical modeling plays a pivotal role in promoting healthier behavior and preventing addiction-related problems. This article investigates a fractional-order mathematical model to understand gambling addiction across different user categories, including non-gamblers, exposed individuals, addicted gamblers, professional gamblers, and recovered gamblers. In the context of qualitative analysis, the study establishes the existence, uniqueness, nonnegativity, and boundedness of model solutions. Key elements of the fractional-order model are identified, including equilibrium points and the basic reproduction number. To assess the stability of the gambling addiction model in these user groups, the study applies the fractional Routh-Hurwitz criterion. Moreover, the global asymptotic stability of all equilibria is demonstrated through the construction of innovative Lyapunov functions. Numerical simulations and optimal control analysis of the fractional-order model further provide a detailed understanding of gambling addiction dynamics within these distinct user classes.

Keywords. Caputo–Fabrizio derivative, Gambling dynamics, Optimal control, Fractional logistic model.
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1. Introduction

Gambling, both as a recreational activity and as a professional pursuit, has long been a subject of fascination and concern. Although recreational gambling is widespread, a distinct group of individuals, known as professional gamblers, engage in gambling as a career, using strategic decision making and risk management to profit from their activities. However, professional gamblers are often juxtaposed with those suffering from gambling addiction, a condition that can have severe psychological, financial, and social consequences. Understanding the dynamic between professional gambling and addiction is crucial for developing better intervention strategies and improving our understanding of gambling behaviors.

Gambling, the act of wagering money or valuables on uncertain outcomes, has been a pervasive element of human activity throughout history [18]. This practice includes various forms such as casino games, sports betting, lotteries, and online gambling [2, 28, 33]. Although gambling holds the promise of substantial rewards, it is fraught with significant risks. The attraction of gambling often lies in the thrill of risk taking and the potential for large financial gains, but these attractions come with complex psychological, social, and economic repercussions [15].

The effects of gambling can differ greatly depending on the age of the individual. For young people, gambling can be particularly tempting due to cognitive biases, peer pressure, and the desire for quick financial rewards. However, gambling among young people is associated with a variety of risks, including academic difficulties and mental health problems, which can lead to problematic behaviors later in life [21, 22, 32]. However, adults often engage in gambling

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as a form of entertainment or as a coping mechanism for financial stress. While it can provide enjoyment and social interaction, it also poses significant risks, such as financial instability, relationship problems, and broader societal impacts [23, 30, 35].

Several factors influence adult gambling behavior, including economic pressures, the psychological appeal of risk-taking, and prevailing social norms. Additionally, historical and cultural contexts play a crucial role in shaping gambling practices and their regulation. For example, during British rule in India, gambling became widespread due to its revenue potential despite its adverse social effects. This led to the enactment of The Public Gambling Act of 1867 [14], an early legislative effort to address the societal impact of gambling.

A fundamental element in understanding gambling behavior is the role of memory [38]. Memory influences how individuals recall their past gambling experiences, which significantly affects their future gambling decisions [34]. Positive memories of wins can strengthen gambling habits, while the tendency to minimize or forget losses can lead to continued gambling despite adverse outcomes. Cognitive distortions associated with memory, such as the gambler's fallacy and the near-miss effect, reveal how memory biases can sustain gambling behavior. Consequently, memory not only shapes perceptions of gambling results but is also integral to the development and maintenance of gambling disorders [20].

Several recent studies have focused on modeling COVID-19 dynamics using advanced differential and fractional operators. Araz et al. [3] introduced a COVID-19 transmission model formulated through differential and integral operators, incorporating stability analysis, optimal control strategies, and non-local operators to explore system dynamics under various scenarios. Arik et al. [4] proposed a six-compartment epidemiological model that accounts for environmental pathogens and social distancing, employing fractional and piecewise differential operators, and analyzed the system using optimal control theory alongside Newton polynomial-based numerical simulations. Basti et al. [13] investigated a fractional SECIR model to assess the impact of infectious diseases on individuals with chronic conditions, addressing the existence and stability of solutions, estimating model parameters for the Algerian context, and demonstrating that targeted interventions can reduce the basic reproduction number below one. Furthermore, Iugret et al. [24] developed a stochastic COVID-19 model using fractal-fractional derivatives (Caputo–Fabrizio, Caputo, and Atangana–Baleanu), examining equilibrium points, solution properties, the basic reproduction number, and infection extinction, with numerical simulations emphasizing the effects of memory, stochasticity, and fractal dynamics.

Recent developments in gambling behavior research have increasingly utilized advanced mathematical models to unravel its complexities and enhance predictive accuracy. Traditionally, integer-order models have been employed, which assume that each gambling decision is made independently of prior outcomes [19, 27]. In contrast, fractional-order models incorporate a memory component, recognizing that past experiences and outcomes significantly influence current behavior [11, 12]. This transition from integer-order to fractional-order models represents a notable advancement, offering a more refined understanding of how gambling decisions evolve over time. By integrating memory effects into these models, researchers can achieve deeper insights into gambling behavior, resulting in more precise predictions and more effective intervention strategies.

The logistic growth model is employed to analyze and forecast gambling behavior in contexts where growth is influenced by various constraints. This model refines the exponential growth model by integrating factors such as resource limitations, competition, and environmental influences that affect growth rates. In the realm of gambling behavior, the logistic growth model helps understand how gambling participation evolves over time under realistic conditions [36]. It posits that as the number of gamblers increases, factors like financial constraints, regulatory measures, and psychological barriers gradually reduce the growth rate. Ultimately, the system approaches a carrying capacity (K), which signifies the maximum sustainable level of gambling activity. The logistic growth model is expressed mathematically as: $P(t) = \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)e^{-rt}}$ where $P(t)$ represents the number of gamblers at time t, K is the carrying capacity, P_0 is the initial number of gamblers, and r is the growth rate. Initially, gambling participation may expand rapidly during an exponential phase. However, as it approaches the carrying capacity K, the growth rate decelerates and eventually stabilizes. By considering limiting factors, the logistic growth model offers a more realistic portrayal of gambling dynamics, enhancing our ability to predict and manage gambling behavior effectively in real-world scenarios.

Most conventional models of gambling addiction focus primarily on the behavior of addicted individuals and tend to overlook the complex interplay between professional gamblers and those at risk of addiction. These models also often fail to account for the role of memory, past behavior, and long-term experiences in shaping gambling behavior. Such factors play a critical role in both addiction and professional gambling, as decisions are not solely based on current states but are influenced by previous experiences and accumulated knowledge.

To address this gap, fractional-order modeling, particularly with the Caputo-Fabrizio derivative, offers a promising framework. This mathematical approach incorporates memory effects and past history into the modeling process, which is crucial for understanding the gradual evolution of gambling behavior over time. The Caputo-Fabrizio fractional derivative, unlike traditional derivatives, is designed to account for the influence of past states on current behaviors, making it a valuable tool for capturing the dynamics of professional gambling and addiction.

This research proposes a fractional logistic growth model to explore the interaction between professional gamblers and addicted individuals. By applying the Caputo-Fabrizio derivative, we aim to develop a more accurate model that reflects the complexities of gambling behavior, including the transition between professional gambling and addiction. Through this model, we seek to provide deeper insights into the dynamics of gambling behaviors, offering a foundation for more targeted and effective prevention and intervention strategies. The results of this research could potentially contribute to the development of new methods for managing and mitigating the negative impacts of gambling addiction, while also recognizing the distinct nature of professional gambling as a career.

This paper is organized as follows: Section 2 describes the development of the mathematical model. Section 3 confirms the well-posedness of the model, ensuring the existence and uniqueness of its solution. The computation of equilibrium points is presented in section 4, with their stability analyzed in section 5. Section 6 explores the sensitivity analysis for the different parameters. The paper concludes with numerical simulations and optimal control analysis in the final section, leading to the conclusions.

1.1. **Preliminaries.** Fractional calculus enhances traditional calculus by enabling the differentiation and integration of non-integer order. This advanced mathematical framework is especially useful for modeling systems with memory and hereditary properties [7–10]. Various techniques exist for defining these fractional operations, including the Riemann-Liouville fractional integral and derivative, the Caputo derivative, and the Caputo-Fabrizio derivative. In this analysis, Caputo-Fabrizio fractional-order operators are employed due to their practical advantages and effectiveness.

Caputo-Fabrizio Fractional Derivative. Let $f \in H^1(a,b), a < b, \eta \in (0,1)$ and $a \in [-\infty, t)$. Then the fractional derivative of order η in Caputo-Fabrizio sense is defined as [16]

$$ {}_{a}^{CF}D_t^{\eta} f(t) = \frac{d^{\eta} f(t)}{dt^{\eta}} = \frac{M(\eta)}{1-\eta} \int_a^t f'(\xi) exp\left[\frac{-\eta(t-\xi)}{1-\eta}\right] d\xi,\ t > a \geq 0, \tag{1.1} $$

where $M(\eta)$ is the normal function such that $M(0) = 1 = M(1)$ and $H^1(a,b) = \{f : f \in L^2(a,b)\ and\ f' \in L^2(a,b)\}$ where $L^2(a,b)$ is the space of square integrable functions on the interval (a,b).

The Caputo–Fabrizio fractional integral is defined with an exponential kernel, which ensures finite memory and avoids singular behavior at the lower limit. This property makes the CF derivative particularly suitable for modeling real-world processes with fading memory, where recent states influence the system more strongly than distant ones. For this reason, it is preferred in modeling behavioral and social phenomena such as gambling addiction, where historical dependence decays exponentially over time. The main advantages of this derivative are its non-singular kernel, the ability for Caputo-Fabrizio derivative-based fractional differential equations to maintain boundary conditions identical to those of integer-order differential equations, and the fact that the Caputo-Fabrizio derivative of a constant is zero.

2. Mathematical Model Development

This section introduces a fractional order model to represent the dynamics of gambling addiction propagation. A unique subset of the population, consisting of professional gamblers, must be distinguished. These individuals engage in gambling as part of their profession and exhibit controlled gambling behaviors, which set them apart from those with gambling addiction. To capture this complexity, we categorize individuals professionally involved in gambling as the professional compartment. Thus, the total population $X(t)$ is divided into five compartments based on their gambling behavior: non-gambler $N(t)$, those in an exposed phase $E(t)$, individuals addicted to gambling $A(t)$, professional gamblers $P(t)$, and those who have recovered from gambling addiction $R(t)$.

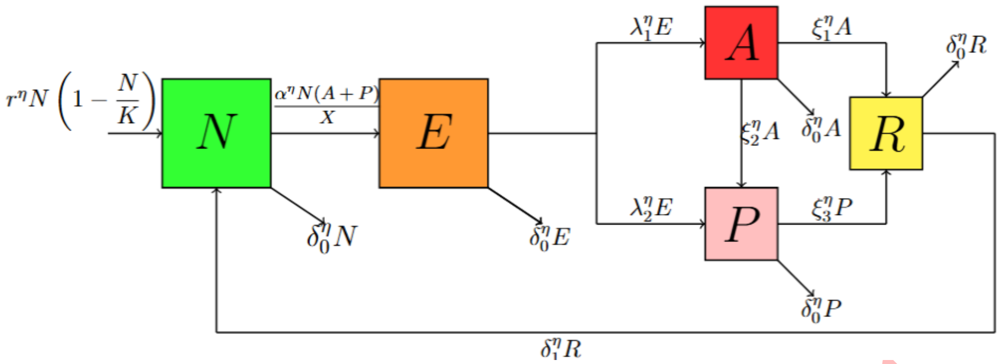

FIGURE 1. Flow diagram of the proposed model.

The first group, $N(t)$, represents non-gambler (susceptible) who are not yet addicted but are at risk of developing a gambling addiction. The second group, $E(t)$, consists of individuals who are occasionally exposed to gambling but do not progress to addiction. The third group, $A(t)$, represents addicted individuals who engage in gambling extensively and exhibit addictive behaviors. The fourth group, $P(t)$, represents professional gamblers who maintain controlled gambling habits. Lastly, the fifth group, $R(t)$, represents individuals who have ceased gambling and recovered from addiction.

Susceptible individuals enter the system at a rate of $r^\eta N$ and may begin gambling due to influences from others, such as contact rates α^η from addicted and professional gamblers, which can lead them to the exposed compartment. Exposed individuals can transition to addiction or become professional gamblers, entering the addicted and professional compartments at rates λ_1^η and λ_2^η, respectively. Addicted individuals can transition to the professional or recovered compartments through interventions such as education or treatment, at rates ξ_1^η and ξ_2^η. Some professional gamblers may leave the profession, moving to the recovered class at a rate of ξ_3^η. Additionally, recovered individuals may become susceptible again at a rate of δ_1^η. The entire population is subject to a general death rate δ_0^η. Based on these interactions and the flow diagram in Figure 1, the dynamics of gambling addiction in the population are represented by the following system of nonlinear fractional differential equations:

$$
\begin{aligned}
\frac{d^\eta N(t)}{dt^\eta} &= r^\eta N(t)\left(1-\frac{N(t)}{K}\right) - \frac{\alpha^\eta N(t)(A(t)+P(t))}{N(t)+E(t)+A(t)+P(t)+R(t)} - \delta_0^\eta N(t) + \delta_1^\eta R(t),\\
\frac{d^\eta E(t)}{dt^\eta} &= \frac{\alpha^\eta N(t)(A(t)+P(t))}{N(t)+E(t)+A(t)+P(t)+R(t)} - \delta_0^\eta E(t) - \lambda_1^\eta E(t) - \lambda_2^\eta E(t),\\
\frac{d^\eta A(t)}{dt^\eta} &= \lambda_1^\eta E(t) - \xi_1^\eta A(t) - \xi_2^\eta A(t) - \delta_0^\eta A(t),\\
\frac{d^\eta P(t)}{dt^\eta} &= \lambda_2^\eta E(t) + \xi_2^\eta A(t) - \xi_3^\eta P(t) - \delta_0^\eta P(t),\\
\frac{d^\eta R(t)}{dt^\eta} &= \xi_1^\eta A(t) + \xi_3^\eta P(t) - \delta_1^\eta R(t) - \delta_0^\eta R(t),
\end{aligned}
\tag{2.1}
$$

$$
N(0)=N_0,\ E(0)=E_0,\ A(0)=A_0,\ P(0)=P_0,\ R(0)=R_0. \tag{2.2}
$$

Here, $X(t)=N(t)+E(t)+A(t)+P(t)+R(t)\neq 0$, with $\eta\in(0,1]$ representing the order of the derivative. The parameters are raised to the power of η to maintain dimensional consistency throughout the system. Note that the notation $r^\eta, \alpha^\eta, \lambda_i^\eta, \xi_i^\eta, \delta_j^\eta$ is a formal rescaling device introduced to preserve dimensional consistency between the Caputo-Fabrizio derivative ${}^{CF}D_t^\eta x(t)$, which has units $[x]/[t^\eta]$, and the corresponding rate terms, which have units

Table 1. Parameter Description.

Parameter	Description
r	Birth or immigration rate of individuals
K	Carrying capacity
α	Transmission rate from non-gamblers to gamblers
λ_1	Rate of transition from exposed users to addicted users
λ_2	Rate of transition from exposed users to professional users
ξ_1	Rate of transition from addicted users to recovered class
ξ_2	Rate of transition from addicted users to professional users
ξ_3	Rate of transition from professional users to recovered class
δ_0	Natural death rate
δ_1	Recovered individuals return to the non-gambler population

$[x]/[t]$. This η scaling does not imply a literal physical modification of the parameter values. It is primarily used to facilitate analytical investigation of the influence of the fractional order η on the effective reproduction number $\mathcal{R}_0$, allowing the relationship between memory effects and persistence thresholds of gambling behavior.

3. Analysis of the Model

This section shows that the proposed model has a unique solution that is both non-negative and bounded.

3.1. **Positivity and Boundedness.** Positivity and boundedness of the proposed model is demonstrated by employing a generalized mean value theorem and a fractional comparison principle [39].

Theorem 3.1. *Consider initial conditions $(N(t_0), E(t_0), A(t_0), P(t_0), R(t_0))$ within $\mathbb{R}^5_+$. Let $(N(t), E(t), A(t), P(t), R(t))$ denote the solution of model (2.1) corresponding to these initial conditions. The set $\mathbb{R}^5_+$ is positively invariant under this model. Moreover, we have the following results:*

$$\begin{aligned}
\limsup_{t\to\infty} N(t) \le N_\infty =& \frac{(\lambda_1^\eta + \lambda_2^\eta + \delta_0^\eta)E_\infty}{\alpha^\eta(A_\infty + P_\infty)},\\
\limsup_{t\to\infty} E(t) \le E_\infty =& \frac{r^\eta N_\infty + \delta_1^\eta R_\infty}{\lambda_1^\eta + \lambda_2^\eta + \delta_0^\eta},\\
\limsup_{t\to\infty} A(t) \le A_\infty =& \frac{\lambda_1^\eta E_\infty}{\xi_1^\eta + \xi_2^\eta + \delta_0^\eta},\\
\limsup_{t\to\infty} P(t) \le P_\infty =& \frac{\lambda_2^\eta E_\infty + \xi_2^\eta A_\infty}{\xi_3^\eta + \delta_0^\eta},\\
\limsup_{t\to\infty} R(t) \le R_\infty =& \frac{\xi_1^\eta A_\infty + \xi_3^\eta P_\infty}{\delta_0^\eta + \delta_1^\eta}.
\end{aligned} \tag{3.1}$$

Proof. From model (2.1), we obtain the following results:

$$\begin{aligned}
\left.\frac{d^\eta N(t)}{dt^\eta}\right|_{N(t_0)=0} =& \delta_1^\eta R(t) \ge 0,\\
\left.\frac{d^\eta E(t)}{dt^\eta}\right|_{E(t_0)=0} =& \frac{\alpha^\eta N(t)(A(t)+P(t))}{N(t)+A(t)+P(t)+R(t)} \ge 0,\\
\left.\frac{d^\eta A(t)}{dt^\eta}\right|_{A(t_0)=0} =& \lambda_1^\eta E(t) \ge 0,
\end{aligned} \tag{3.2}$$

$$\left.\frac{d^\eta P(t)}{dt^\eta}\right|_{P(t_0)=0} = \lambda_2^\eta E(t) + \xi_2^\eta A(t) \geq 0,$$

$$\left.\frac{d^\eta R(t)}{dt^\eta}\right|_{R(t_0)=0} = \xi_2^\eta A(t) + \xi_3^\eta P(t) \geq 0.$$

For all $t \geq 0$, using the generalized mean value theorem [1] and system 3.2, we can conclude that $N(t), E(t), A(t), P(t), R(t) \geq 0$. Furthermore, the third equation of system 2.1 implies that:

$$\frac{d^\eta A(t)}{dt^\eta} \leq \lambda_1^\eta E(t) - \xi_1^\eta A(t) - \xi_2^\eta A(t) - \delta_0^\eta A(t).$$

By applying the fractional comparison principle, it follows that

$$\limsup_{t\to\infty} A(t) \leq A_\infty = \frac{\lambda_1^\eta E_\infty}{\xi_1^\eta + \xi_2^\eta + \delta_0^\eta}.$$

The fourth equation in system (2.1) indicates that:

$$\frac{d^\eta P(t)}{dt^\eta} \leq \lambda_2^\eta E(t) + \xi_2^\eta A(t) - \xi_3^\eta P(t) - \delta_0^\eta P(t).$$

By utilizing the fractional comparison principle, it follows that

$$\limsup_{t\to\infty} P(t) \leq P_\infty = \frac{\lambda_2^\eta E_\infty + \xi_2^\eta A_\infty}{\xi_3^\eta + \delta_0^\eta}.$$

The last equation in system (2.1) implies that:

$$\frac{d^\eta R(t)}{dt^\eta} \leq \xi_1^\eta A(t) + \xi_3^\eta P(t) - \delta_1^\eta R(t) - \delta_0^\eta R(t).$$

By employing the fractional comparison principle, it follows that

$$\limsup_{t\to\infty} R(t) \leq R_\infty = \frac{\xi_1^\eta A_\infty + \xi_3^\eta P_\infty}{\delta_1^\eta + \delta_0^\eta}.$$

The second equation in system (2.1) indicates that:

$$\frac{d^\eta E(t)}{dt^\eta} \leq \alpha^\eta N(t)(A(t) + P(t)) - \delta_0^\eta E(t) - \lambda_1^\eta E(t) - \lambda_2^\eta E(t),$$

which leads to:

$$\limsup_{t\to\infty} N(t) \leq N_\infty = \frac{(\lambda_1^\eta + \lambda_2^\eta + \delta_0^\eta)E_\infty}{\alpha^\eta (A_\infty + P_\infty)}.$$

This result provides the first estimate in (3.1). Additionally, the first equation in system (2.1) yields:

$$\frac{d^\eta (N(t) + E(t))}{dt^\eta} \leq r^\eta N(t) + \delta_1^\eta R(t) - (\lambda_1^\eta + \lambda_2^\eta + \delta_0^\eta)E(t),$$

which leads to:

$$\limsup_{t\to\infty} E(t) \leq E_\infty = \frac{r^\eta N_\infty + \delta_1^\eta R_\infty}{\lambda_1^\eta + \lambda_2^\eta + \delta_0^\eta}.$$

This result provides the second estimate in (3.1). $\square$

[1]**Remark 1** Consider $f(t) \in \mathbf{C}[0,b]$ and $\frac{d^\eta f(t)}{dt^\eta} \in \mathbf{C}[0,b]$ for $0 < \eta \leq 1$. Then, if

- $\frac{d^\eta f(t)}{dt^\eta} \geq 0 \ \forall \ t \in (0,b]$, then $f(t)$ is non-decreasing,
- $\frac{d^\eta f(t)}{dt^\eta} \leq 0 \ \forall \ t \in (0,b]$, then $f(t)$ is non-increasing [31].

3.2. **Existence and Uniqueness Criteria.**

Theorem 3.2. *A unique solution to the model (2.1) exists for non-negative initial values when $t \geq 0$. Furthermore, all solutions are bounded.*

Proof. The right-hand side (RHS) of the model (2.1) is both continuous and bounded, as shown in Equation (3.1). We now explicitly verify the local Lipschitz continuity of the system's right-hand side by estimating

$$\|F(X_1, t) - F(X_2, t)\| \leq L\|X_1 - X_2\|, \text{ where } F = (F_1, F_2, F_3, F_4, F_5),$$

corresponds to the right-hand sides of (2.1). The boundedness of all terms involving ratios such as

$$\frac{N(A+P)}{N+E+A+P+R},$$

ensures that F satisfies the Lipschitz condition in the positively invariant domain $\mathbb{R}^5_+$. Therefore, based on Remark 2[2], we conclude that, under appropriate initial conditions, the solution to the model (2.1) exists uniquely and remains bounded for all $t > 0$. □

4. Analysis of Equilibrium Points and Reproduction Number

To find the equilibrium points, we set the right-hand sides of the equations in the proposed model (2.1) equal to zero. As a result, the system yields two equilibrium points for the model (2.1).

- Gambler-free equilibrium exists when gambler populations become zero, i.e.

  $$E_1 = (N^*, 0, 0, 0, 0),$$

  where

  $$N^* = K\left(1 - \frac{\delta_0^\eta}{r^\eta}\right).$$

  Gambler-free equilibrium exists only if $r^\eta > \delta_0^\eta$; otherwise, it becomes negative, which is not practically true.
- **Reproduction Number:** The reproduction number, denoted as $\mathcal{R}_0$, is defined as "the average number of secondary cases produced by an infected individual in a fully susceptible population [37]." To determine $\mathcal{R}_0$, the next-generation matrix method is employed at the gambler-free equilibrium point. Let

  $$D^\eta X = f(X) - v(X),$$

  $$X = [N, E, A, P, R].$$

  The terms for entering and exiting a class are represented by $v(X)$, while $u(X)$ denotes the terms used to describe newly created gamblers.
  The reproduction number is the largest eigenvalue of UV^{-1}, where

  $$U = \left(\frac{\partial u}{\partial X}\right)_{E_1}, \; V = \left(\frac{\partial v}{\partial X}\right)_{E_1}.$$

  The Jacobian matrix of U at E_1

  $$U = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^\eta & \alpha^\eta & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

[2]**Remark 2** For a fractional differential equation with the Caputo-Fabrizio derivative $\frac{d^\eta f(t)}{dt^\eta} = y(t, x), t > t_0$, with $0 < \eta \leq 1$ and $y : [t_0, \infty) \times \Omega \to \mathbb{R}^n$, if $y(t, x)$ is continuous, bounded, and satisfies a Lipschitz condition with respect to x [29], there exists a unique solution on $[t_0, \infty) \times \Omega$.

The Jacobian matrix of V at E_1

$$V = \begin{bmatrix} r^\eta - \delta_0^\eta & 0 & \alpha^\eta & \alpha^\eta & -\delta_1^\eta \\ 0 & \delta_0^\eta + \lambda_1^\eta + \lambda_2^\eta & 0 & 0 & 0 \\ 0 & -\lambda^\eta & \xi_1^\eta + \xi_2^\eta + \delta_0^\eta & 0 & 0 \\ 0 & -\lambda_2^\eta & -\xi_2^\eta & \xi_3^\eta + \delta_0^\eta & 0 \\ 0 & 0 & -\xi_1^\eta & -\xi_3^\eta & \delta_0^\eta + \delta_1^\eta \end{bmatrix}.$$

The reproduction number is defined as follows:

$$\mathcal{R}_0 = \frac{\alpha^\eta(\xi_1^\eta\lambda_2^\eta + \xi_2^\eta\lambda_1^\eta + \xi_2^\eta\lambda_2^\eta + \xi_3^\eta\lambda_1^\eta + \delta_0^\eta\lambda_1^\eta + \delta_0^\eta\lambda_2^\eta)}{(\xi_3^\eta + \delta_0^\eta)(\xi_1^\eta + \xi_2^\eta + \delta_0^\eta)(\delta_0^\eta + \lambda_1^\eta + \lambda_2^\eta)}. \tag{4.1}$$

- A Gambler-Persistence Equilibrium Point exists when there are gamblers present, i.e.

$$E_2 = (\tilde{N}, \tilde{E}, \tilde{A}, \tilde{P}, \tilde{R}),$$

where

$$\tilde{N} = \frac{(C_2C_3C_4 + C_4C_5 + C_6)C_1\tilde{E}}{C_1C_2C_3C_4(\mathcal{R}_0 - 1)},$$

$$\tilde{A} = \frac{\lambda_1^\eta \tilde{E}}{C_3},\ \tilde{P} = \frac{(\lambda_2^\eta C_3 + \lambda_1^\eta \xi_2^\eta)\tilde{E}}{C_2C_3},\ \tilde{R} = \frac{C_6\tilde{E}}{C_2C_3C_4},$$

and

$$\tilde{E} = \frac{C_1^2C_2C_3C_4(\mathcal{R}_0 - 1)[C_2C_3C_4 + C_4C_5 + C_6]K(r^\eta - \delta_0^\eta) + KC_1^3C_2^2C_3^2C_4^2(\mathcal{R}_0 - 1)^2(\mathcal{R}_1 - 1)}{C_1^2 r^\eta (C_2C_3C_4 + C_4C_5 + C_6)^2},$$

where

$C_1 = \lambda_1^\eta + \lambda_2^\eta + \delta_0^\eta,$ $\quad C_2 = \xi_3^\eta + \delta_0^\eta,$

$C_3 = \xi_1^\eta + \xi_2^\eta + \delta_0^\eta,$ $\quad C_4 = \delta_0^\eta + \delta_1^\eta,$

$C_5 = C_2\lambda_1^\eta + C_3\lambda_2^\eta + \lambda_1^\eta\xi_2^\eta,$ $\quad C_6 = \xi_1^\eta\lambda_1^\eta C_2 + \xi_3^\eta\lambda_2^\eta C_3 + \xi_3^\eta\xi_2^\eta\lambda_1^\eta,$

and

$$\mathcal{R}_1 = \frac{\delta_1^\eta C_6}{C_1C_2C_3C_4},\ \mathcal{R}_0 = \frac{\alpha^\eta C_5}{C_1C_2C_3}.$$

Hence, Gambler-persistence equilibrium point exists if $\mathcal{R}_0 > 1$ and $\mathcal{R}_1 > 1$.

5. Stability Analysis

Theorem 5.1. *The gambler-free equilibrium point* $E_1 = (N^*, 0, 0, 0, 0) = \left(\left(1 - \frac{\delta_0^\eta}{r^\eta}\right)K, 0, 0, 0, 0\right)$ *of the proposed model* (2.1) *is locally asymptotically stable when* $\mathcal{R}_0 < 1$.

Proof. The Jacobian matrix at E_1 is

$$J_{E_1} = \begin{bmatrix} -r^\eta + \delta_0^\eta & 0 & -\alpha^\eta & -\alpha^\eta & \delta_1^\eta \\ 0 & -\lambda_1^\eta - \lambda_2^\eta - \delta_0^\eta & \alpha^\eta & \alpha^\eta & 0 \\ 0 & \lambda_1^\eta & -\xi_1^\eta - \xi_2^\eta - \delta_0^\eta & 0 & 0 \\ 0 & \lambda_2^\eta & \xi_2^\eta & -\xi_3^\eta - \delta_0^\eta & 0 \\ 0 & 0 & \xi_1^\eta & \xi_3^\eta & -\delta_0^\eta - \delta_1^\eta \end{bmatrix}.$$

The two eigenvalues, $-r^\eta + \delta_0^\eta$ and $-\delta_0^\eta - \delta_1^\eta$, are both strictly negative. Now, for the remaining eigenvalue:

$$J_{E_1}^* = \begin{bmatrix} -\lambda_1^\eta - \lambda_2^\eta - \delta_0^\eta & \alpha^\eta & \alpha^\eta \\ \lambda_1^\eta & -\xi_1^\eta - \xi_2^\eta - \delta_0^\eta & 0 \\ \lambda_2^\eta & \xi_2^\eta & -\xi_3^\eta - \delta_0^\eta \end{bmatrix}.$$

The characteristic equation for the matrix $J^*_{E_1}$ is expressed as follows:

$$\Lambda^3 + l_1\Lambda^2 + l_2\Lambda + l_3 = 0, \tag{5.1}$$

where

$$l_1 = C_1 + C_2 + C_3,\; l_2 = C_3C_2 + C_1C_2 + C_1C_3 - \alpha^\eta(\lambda_1^\eta + \lambda_2^\eta),\; l_3 = C_1C_2C_3(1 - \mathcal{R}_0),$$

and

$$l_1l_2 > l_3,$$

i.e.

$$(C_1C_2C_3)(\mathcal{R}_0 - 1) + (C_1 + C_2 + C_3)(C_3C_2 + C_1C_2 + C_1C_3) > (C_1 + C_2 + C_3)(\lambda_1^\eta + \lambda_2^\eta).$$

By applying the Matignon criterion[3], if l_1, l_3, and $l_1l_2 - l_3$ are all greater than zero, it can be concluded that the eigenvalues of the system will have negative real parts. Consequently, the proposed model (2.1) will exhibit stability when $\mathcal{R}_0 < 1$. □

Theorem 5.2. *The gambler-persistence equilibrium point $E_2 = (\tilde{N}, \tilde{E}, \tilde{A}, \tilde{P}, \tilde{R})$ in the proposed model (2.1) is locally asymptotically stable when $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix at E_2 is

$$J_{E_2} = \begin{bmatrix} r^\eta - \dfrac{2r^\eta\tilde{N}}{K} - \alpha^\eta(\tilde{A} + \tilde{P}) - \delta_0^\eta & 0 & -\alpha^\eta\tilde{N} & -\alpha^\eta\tilde{N} & \delta_1^\eta \\ \alpha^\eta(\tilde{A} + \tilde{P}) & -\lambda_1^\eta - \lambda_2^\eta - \delta_0^\eta & \alpha^\eta\tilde{N} & \alpha^\eta\tilde{N} & 0 \\ 0 & \lambda_1^\eta & -\xi_1^\eta - \xi_2^\eta - \delta_0^\eta & 0 & 0 \\ 0 & \lambda_2^\eta & \xi_2^\eta & -\xi_3^\eta - \delta_0^\eta & 0 \\ 0 & 0 & \xi_1^\eta & \xi_3^\eta & -\delta_0^\eta - \delta_1^\eta \end{bmatrix}.$$

The characteristic equation for the matrix J_{E_2} is expressed as follows:

$$\Lambda^5 + m_1\Lambda^4 + m_2\Lambda^3 + m_3\Lambda^2 + m_4\Lambda + m_5 = 0. \tag{5.2}$$

To ensure that the equilibrium point E_2 is locally asymptotically stable, the Matignon criteria must be satisfied, which requires that all Hurwitz determinants H_i (for $i = 1, 2, \ldots, 5$) be positive [1]. Therefore, the endemic equilibrium point will be stable if the following conditions hold:

- $m_i > 0$ for $i = 1, 2, 3, 4, 5$,
- $m_1m_2m_3 > m_3^2 + m_1^2m_4$,
- $(m_1m_4 - m_5)(m_1m_2m_3 - m_3^2 - m_1^2m_4) > m_4(m_1m_2 - m_3)^2$.

These conditions ensure that the eigenvalues of the system have negative real parts, thereby guaranteeing the local asymptotic stability of the equilibrium point. □

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[3] **Matignon Criteria**[1] Consider our model described by the following fractional differential equation:

$$\frac{d^\eta X(t)}{dt^\eta} = F(X(t)),$$

where $X(t) = (x_1(t); x_2(t); \ldots; x_n(t))^T$ and $F(X(t)) = (f_1; f_2; \ldots; f_n)^T$. The equilibrium point is denoted by $E^* = (x_1^*; x_2^*; \ldots; x_n^*)^T$. For this fractional-order system, the equilibrium points are considered asymptotically stable if all eigenvalues (λ_j) of the Jacobian matrix J evaluated at equilibrium E^* satisfy the criterion:

$$|\arg(\mathrm{eig}(J))| = |\arg(\lambda_j)| > \frac{\pi}{2}\eta,$$

where $j = 1, 2, \ldots, n$, J is the Jacobian matrix of the system at the equilibrium point E^* , and λ_j is eigenvalues of the Jacobian matrix.

6. Sensitivity Analysis

This study explores the influence of various parameters of the proposed model (2.1) on individual behavior patterns. A primary factor, the threshold parameter ($\mathcal{R}_0$), significantly shapes gambling behavior. To assess this, we calculate the partial derivatives of $\mathcal{R}_0$ referred to as elasticities with respect to each parameter. Table 2 and 2 summarizes the results of the sensitivity analysis for the threshold parameter.
The normalized sensitivity index of each parameter is defined as:

$$\zeta_q = \frac{q}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial q}.$$

Figure 3 illustrating the relationship of the threshold parameter $\mathcal{R}_0$ with two key factors: the order of the derivative

Table 2. Sensitivity analysis of $\mathcal{R}_0$ for all parameters.

Parameter	α	δ_0	λ_1	λ_2	ξ_1	ξ_2	ξ_3
Normalized Sensitivity Index	0.7	-0.3321	0.0460	0.0092	-0.1651	-0.0101	-0.2477

η and the rate of gambling involvement α. There is Sharp increases in $\mathcal{R}_0$ as η approaches 1, suggesting that higher derivative orders significantly elevate $\mathcal{R}_0$. This implies that as the order of the fractional derivative increases, the potential for sustained gambling behavior (as modeled by $\mathcal{R}_0$) intensifies. For each curve (order of derivative), as α increases, $\mathcal{R}_0$ initially rises sharply before stabilizing, with higher values of η consistently resulting in a higher $\mathcal{R}_0$ for any given α. This behavior suggests that both a higher rate of gambling involvement and a higher derivative order contribute to an elevated threshold, intensifying the modeled gambling behavior. Overall, these graphs highlight the sensitivity of $\mathcal{R}_0$ to both the order of the derivative and the gambling involvement rate, with both parameters playing a significant role in influencing the modeled threshold behavior.

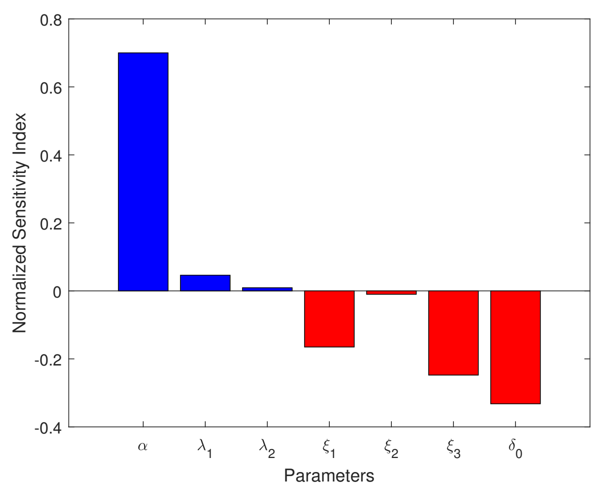

Figure 2. Graphical Representation of sensitivity indices of $\mathcal{R}_0$.

7. Numerical Simulation

This section presents numerical simulations to validate the theoretical results and emphasize the significance of the $\mathcal{R}_0$ threshold value. These simulations enhance understanding of the interactions among different groups of individuals, as well as the characteristics and dynamics of gambling at the population level. Table 4 lists the variables and parameters used for the evaluation [7, 17, 19, 25–27]. The model is solved using the Adams-Bashforth method [5].

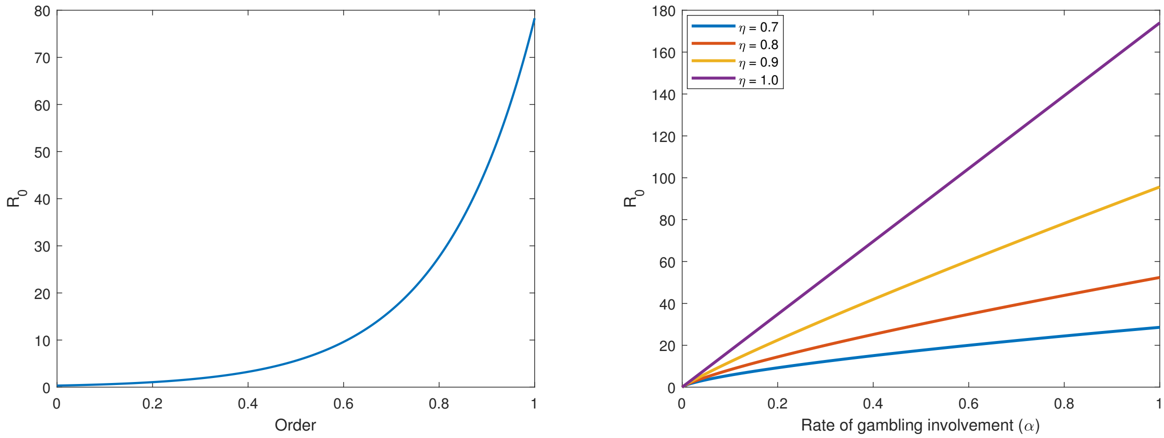

FIGURE 3. Relation of $\mathcal{R}_0$ with the order of derivative η and rate of gambling involvement (α).

7.1. **Error and Convergence Analysis of the Adams–Bashforth Fractional Scheme.** To validate the accuracy and convergence of the proposed Adams–Bashforth fractional numerical scheme for the Caputo–Fabrizio (CF) derivative, we consider a test problem with a known analytical solution. We consider the fractional differential equation

$$^{CF}D_t^{\eta} y(t) = -y(t), \quad y(0) = 1, \quad 0 < \eta \leq 1,$$

whose exact analytical solution is given by

$$y(t) = \exp\left(-\frac{\eta t}{1-\eta} \left(1 - e^{-\frac{1-\eta}{\eta} t} \right) \right).$$

Using the two-step Adams–Bashforth fractional scheme with the CF kernel, the discretized form is written as

$$y_{n+1} = y_n + \frac{1-\eta}{M(\eta)} \left[f(t_n, y_n) - f(t_{n-1}, y_{n-1}) \right] + \frac{\eta h}{M(\eta)} f(t_n, y_n),$$

where $M(\eta) = 1 - \eta + \eta e^{-\frac{1-\eta}{\eta} h}$, h is the time step size, and $f(t, y) = -y$. The numerical error is defined as

$$E(h) = \max_{0 \leq n \leq N} |y(t_n) - y_n|,$$

and the convergence order p is estimated using

$$p = \frac{\log\left(\frac{E(h_1)}{E(h_2)} \right)}{\log\left(\frac{h_1}{h_2} \right)}.$$

We evaluate the error for different step sizes $h = 0.1, 0.05, 0.025$, and 0.0125 with $\eta = 0.9$. The results are summarized in Table 3. The numerical results clearly show that the Adams-Bashforth scheme for the Caputo-Fabrizio fractional

TABLE 3. Error and convergence order for the Adams–Bashforth CF scheme.

Step size (h)	**Error $E(h)$**	**Estimated order (p)**
0.1	2.41×10^{-3}	–
0.05	6.08×10^{-4}	1.98
0.025	1.53×10^{-4}	1.99
0.0125	3.84×10^{-5}	1.99

derivative exhibits **second-order convergence**, consistent with the theoretical prediction for the classical Adams-Bashforth method. This validates the accuracy and reliability of the numerical simulations used in this study.
The variation in non-gambler (N), exposed (E), addicted (A), professional (P), and recovered populations (R) is analyzed for fractional derivative orders (η) of 0.7, 0.8, 0.9, and 1. Figure 4 reveals that non-gambler and recovered populations decreases as the order of the derivative increases, with the recovered population approaching zero for higher values of η. In contrast, the exposed population increases significantly with the increases in derivative order. Similarly, the addicted population exhibits a steady rise as the derivative order increases. The professional population also follows an increasing trend when η increases. As demonstrated, a lower value of the derivative order indicates a smaller addicted population, suggesting that fractional derivative models offer a more accurate representation of population dynamics compared to integer-order derivatives, which can be beneficial for society by better informing interventions and policies. Fractional orders enable more flexibility and accurate characterization of these population variations, capturing subtle changes that integer orders might miss. This highlights the importance of fractional calculus in modeling complex real-world behaviors.

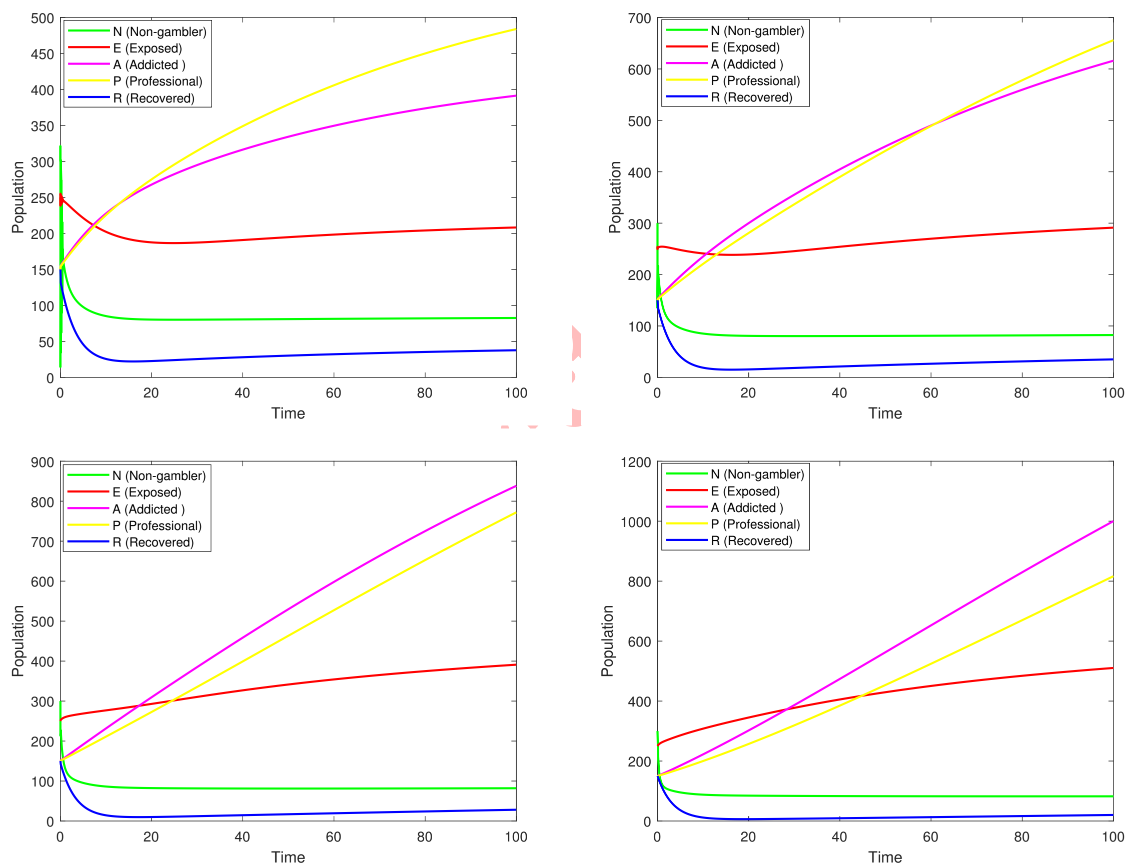

FIGURE 4. Variation in non-gambler (N), exposed (E), addicted (A), professional (P), and recovered populations (R) for derivative orders (η) of 0.7, 0.8, 0.9, and 1.

7.2. **Optimal Control Analysis.** Optimal control theory deals with finding a control strategy that optimizes a specific objective (such as minimizing costs or maximizing benefits) within a dynamic system governed by differential equations. For the proposed model, optimal control is used to identify the best possible interventions (or controls) to guide the system towards a desired state while adhering to constraints. Consider a generalized formulation of

Table 4. Parameter description.

Parameter	Description	Value
r	Birth or immigration rate of individuals	1.5
K	Carrying capacity	100
α	Transmission rate from non-gamblers to gamblers	0.45
λ_1	Rate of transition from exposed users to addicted users	0.03
λ_2	Rate of transition from exposed users to professional users	0.02
ξ_1	Rate of transition from addicted users to recovered class	0.003
ξ_2	Rate of transition from addicted users to professional users	0.002
ξ_3	Rate of transition from professional users to recovered class	0.004
δ_0	Natural death rate	0.002
δ_1	Recovered individuals return to the non-gambler population	0.3

Equation (2.1) by introducing a control variable, $u(t)$, which takes values in the range $[0,1]$. When $u(t)=0$, no control interventions are applied, whereas $u(t)=1$ represents an idealized, fully controlled scenario, which is rarely achievable in practice, particularly in cases such as gambling. Intermediate values of $u(t)$ within $(0,1)$ capture the varying intensities of intervention. The purpose of incorporating $u(t)$ is to reduce the population of exposed individuals by a factor of $(1-u)$. As a result, the model (2.1) is modified to (7.1) after introducing the control variable:

$$\begin{aligned}
\frac{d^\eta N(t)}{dt^\eta} =& r^\eta N(t)\left(1-\frac{N(t)}{K}\right)-\frac{(1-u)\alpha^\eta N(t)(A(t)+P(t))}{N(t)+E(t)+A(t)+P(t)+R(t)}-\delta_0^\eta N(t)+\delta_1^\eta R(t),\\
\frac{d^\eta E(t)}{dt^\eta} =& \frac{(1-u)\alpha^\eta N(t)(A(t)+P(t))}{N(t)+E(t)+A(t)+P(t)+R(t)}-\delta_0^\eta E(t)-\lambda_1^\eta E(t)-\lambda_2^\eta E(t),\\
\frac{d^\eta A(t)}{dt^\eta} =& \lambda_1^\eta E(t)-\xi_1^\eta A(t)-\xi_2^\eta A(t)-\delta_0^\eta A(t),\\
\frac{d^\eta P(t)}{dt^\eta} =& \lambda_2^\eta E(t)+\xi_2^\eta A(t)-\xi_3^\eta P(t)-\delta_0^\eta P(t),\\
\frac{d^\eta R(t)}{dt^\eta} =& \xi_1^\eta A(t)+\xi_3^\eta P(t)-\delta_1^\eta R(t)-\delta_0^\eta R(t),
\end{aligned} \tag{7.1}$$

and the primary objective is to determine the optimal control variable $u(t)$ that minimizes the following control objective function:

$$J(u)=\int_0^T [W_1E+D_1u^2(t)]dt. \tag{7.2}$$

where W_1 is a nonnegative weight representing the impact of gambling, and D_1 measures the relative cost of interventions over the interval $[0,T]$.

The objective is to minimize the number of exposed gamblers, while simultaneously minimizing the control cost associated with the function $u(t):[0,T]\to[0,1]$. To solve Equations (7.1) and (7.2), we derive the necessary optimality conditions for the problem by defining the scalar Hamiltonian function as follows:

$$H_1(x,u(t),\lambda_i,t)=W_1E+D_1u^2(t)+\lambda_N h_1(t)+\lambda_E h_2(t)+\lambda_A h_3(t)+\lambda_P h_4(t)+\lambda_R h_5(t), \tag{7.3}$$

where

$$\begin{aligned}
h_1(t) =& r^\eta N(t)\left(1-\frac{N(t)}{K}\right)-\frac{(1-u)\alpha^\eta N(t)(A(t)+P(t))}{N(t)+E(t)+A(t)+P(t)+R(t)}-\delta_0^\eta N(t)+\delta_1^\eta R(t),\\
h_2(t) =& \frac{(1-u)\alpha^\eta N(t)(A(t)+P(t))}{N(t)+E(t)+A(t)+P(t)+R(t)}-\delta_0^\eta E(t)-\lambda_1^\eta E(t)-\lambda_2^\eta E(t),\\
h_3(t) =& \lambda_1^\eta E(t)-\xi_1^\eta A(t)-\xi_2^\eta A(t)-\delta_0^\eta A(t),\\
h_4(t) =& \lambda_2^\eta E(t)+\xi_2^\eta A(t)-\xi_3^\eta P(t)-\delta_0^\eta P(t),
\end{aligned}$$

$$h_5(t) = \xi_1^\eta A(t) + \xi_3^\eta P(t) - \delta_1^\eta R(t) - \delta_0^\eta R(t),$$

and the adjoint variables are denoted as $\lambda_N, \lambda_E, \lambda_A, \lambda_P$, and λ_R.

$$\lambda_i = (\lambda_N, \lambda_E, \lambda_A, \lambda_P, \lambda_R),$$
$$x = (N, E, A, P, R).$$

In accordance with the approach presented in [6], we derive the necessary optimality conditions for the system of Equations (7.1) and (7.2), as expressed in Equations (7.4)–(7.6).

$$\frac{d^\eta N(t)}{dt^\eta} = \frac{\partial H_1}{\partial \lambda_N}(t), \frac{d^\eta E(t)}{dt^\eta} = \frac{\partial H_1}{\partial \lambda_E}(t), \frac{d^\eta A(t)}{dt^\eta} = \frac{\partial H_1}{\partial \lambda_A}(t), \frac{d^\eta P(t)}{dt^\eta} = \frac{\partial H_1}{\partial \lambda_P}(t), \frac{d^\eta R(t)}{dt^\eta} = \frac{\partial H_1}{\partial \lambda_R}(t), \tag{7.4}$$

$$\frac{d^\eta \lambda_N(t)}{dt^\eta} = -\frac{\partial H_1}{\partial N}(t), \frac{d^\eta \lambda_E(t)}{dt^\eta} = -\frac{\partial H_1}{\partial E}(t), \frac{d^\eta \lambda_A(t)}{dt^\eta} = -\frac{\partial H_1}{\partial A}(t), \frac{d^\eta \lambda_P(t)}{dt^\eta} = -\frac{\partial H_1}{\partial P}(t), \frac{d^\eta \lambda_R(t)}{dt^\eta} = -\frac{\partial H_1}{\partial R}(t), \tag{7.5}$$

$$\frac{\partial H_1}{\partial u}(t) = 0, \tag{7.6}$$

with transversality conditions

$$\lambda_i(T) = 0, \ i \in (\lambda_N, \lambda_E, \lambda_A, \lambda_P, \lambda_R). \tag{7.7}$$

Solving the system of Equations (7.4) is equivalent to solving Equation (7.1), while solving the system of Equations (7.5) corresponds to solving the adjoint system of equations given by:

$$\begin{aligned}
\frac{d^\eta \lambda_N(t)}{dt^\eta} =& \lambda_N \left[-r^\eta \left(1 - \frac{2N}{K}\right) + v(A+P)(E+A+P+R) + \delta_0^\eta\right] - \lambda_E(A+P)(E+A+P+R)v, \\
\frac{d^\eta \lambda_E(t)}{dt^\eta} =& - W_1 - \lambda_N N(A+P)v + \lambda_E\left[vN(A+P) + \delta_0^\eta + \lambda_1^\eta + \lambda_2^\eta\right] - \lambda_A \lambda_1^\eta - \lambda_P \lambda_2^\eta, \\
\frac{d^\eta \lambda_A(t)}{dt^\eta} =& \lambda_N N(N+E+R)v - \lambda_E N(N+E+R)v + \lambda_A(\xi_1^\eta + \xi_2^\eta + \delta_0^\eta) - \lambda_P \xi_2^\eta - \lambda_R \xi_1^\eta, \\
\frac{d^\eta \lambda_P(t)}{dt^\eta} =& \lambda_N N(N+E+R)v - \lambda_E N(N+E+R)v + \lambda_P(\xi_3^\eta + \delta_0^\eta) - \lambda_R \xi_3^\eta, \\
\frac{d^\eta \lambda_R(t)}{dt^\eta} =& \lambda_N[-vN(A+P) - \delta_1^\eta] + \lambda_E N(A+P)v + \lambda_R(\delta_0^\eta + \delta_1^\eta),
\end{aligned} \tag{7.8}$$

where

$$v = \frac{(1-u)\alpha^\eta}{(N+E+A+P+R)^2}.$$

Therefore, the optimal control $u^*(t)$ for the dynamic system (7.1), which minimizes the objective functional (7.2), is determined by:

$$u^*(t) = min\left[max\left(0, \frac{1}{2D_1}(\lambda_E - \lambda_N)\frac{\alpha^\eta N(A+P)}{N+E+A+P+R}\right), 1\right]. \tag{7.9}$$

Additionally, the fractional derivative state Equation (7.1), the fractional derivative adjoint Equation (7.8), along with the characterization of the optimal control (7.9) and the boundary conditions (7.7), together define the optimality system. These equations provide the analytical solution to the optimal control problem being addressed. Figure 5 illustrates the changes in the non-gambler (N) and exposed (E) populations for different derivative orders (η) of 0.7, 0.8, 0.9, and 1, in the absence of optimal control measures. The results reveal a consistent decline in the non-gambler population over time when no control strategies are applied, suggesting a gradual reduction in this group. In contrast, the exposed population shows a continuous increase under the same conditions, indicating that more individuals are becoming exposed as time progresses. These trends highlight the impact of the lack of control measures on the population dynamics. Furthermore, the variations across different derivative orders demonstrate the sensitivity of these population changes to the fractional-order parameter, emphasizing the complexity of modeling such systems. The findings underscore the importance of incorporating optimal control strategies to regulate the growth of the exposed group and maintain the stability of the non-gambler population.

Each graph of Figure 6 demonstrates the impact of optimal control strategies on the non-gambler (N) and exposed (E) populations across different fractional derivative orders (η) of 0.7, 0.8, 0.9, and 1. The analysis reveals that

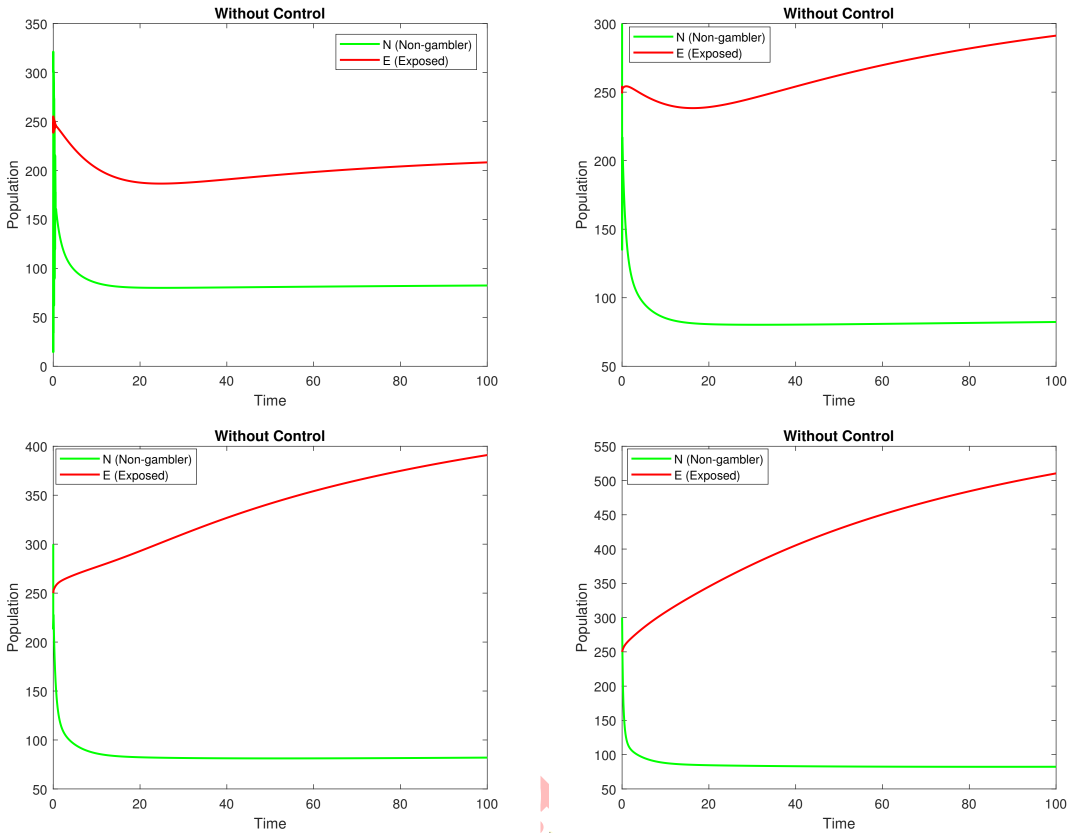

FIGURE 5. Variation in non-gambler (N) and exposed (E) populations without optimal control for derivative orders (η) of 0.7, 0.8, 0.9, and 1.

applying these controls results in a consistent increase in the non-gambler population, signifying the effectiveness of the intervention in promoting non-gambling behavior. Simultaneously, a marked reduction in the exposed population is observed, highlighting the ability of the controls to mitigate exposure effectively. These patterns underscore the significance of optimal control in shaping population dynamics and demonstrate the adaptability of the approach under varying derivative orders. The results provide valuable insights into how fractional calculus-based control strategies can be tailored to achieve specific outcomes, offering a robust framework for managing dynamic population systems.

Policy Suggestion.

- **Objective:** Reduce the rate at which non-gamblers become exposed to gambling.
- **Strategy:**
  (1) Launch public awareness programs through various channels (social media, TV, community outreach) highlighting the negative effects of gambling, such as addiction, financial loss, and psychological distress.
  (2) Provide counseling and early intervention programs for those showing initial signs of gambling behavior.
- **Model Impact:** This is to modeled by reducing the transmission rate, which controls how likely non-gamblers are to become exposed due to contact with addicted or professional gamblers.

Public awareness campaigns and advertisements should be carefully crafted to emphasize the detrimental effects of gambling, particularly its potential for addiction, financial hardship, and emotional turmoil. These campaigns must underscore the broader societal consequences of gambling, highlighting the negative impact on individuals, families,

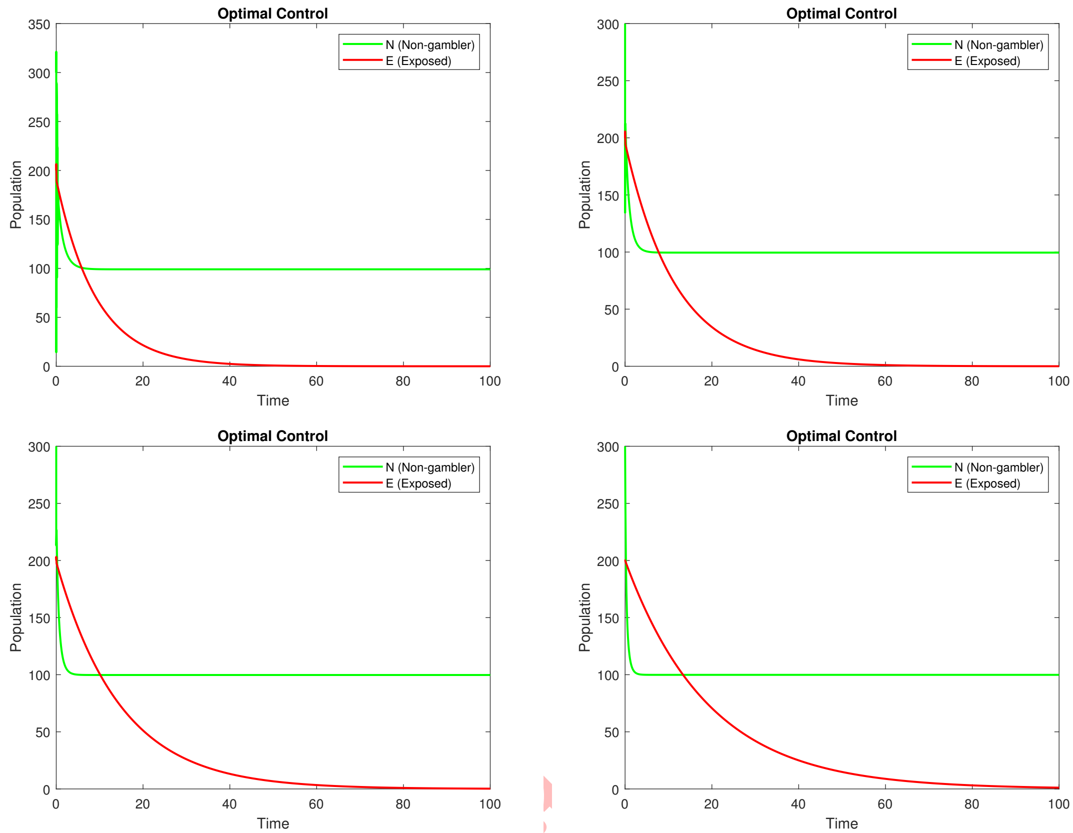

FIGURE 6. Variation in non-gambler (N) and exposed (E) populations under optimal control for derivative orders (η) of 0.7, 0.8, 0.9, and 1.

and communities. To maximize reach and engagement, media channels such as television, social media, and public service announcements should be leveraged, ensuring that the message resonates with a wide and diverse audience.

8. Conclusion

The increasing prevalence of gambling addiction in modern society necessitates comprehensive and effective strategies to mitigate its detrimental impact. This study underscores the significance of understanding various user groups, including non-gamblers, exposed individuals, addicted gamblers, professional gamblers, and those in recovery. By providing a nuanced perspective on the progression of gambling addiction, the research highlights not only the behavioral transitions but also the long-term effects and key factors influencing recovery processes. A fractional-order mathematical model incorporating memory effects through Caputo-Fabrizio derivatives is proposed, advancing our understanding of the complex dynamics of gambling addiction. The model distinguishes between two equilibrium states: the gambler-free equilibrium and the gambler-persistence equilibrium, with stability conditions governed by the gambler reproduction number. Through sensitivity and optimal control analyses, coupled with numerical simulations employing the Adams-Bashforth method, the study offers valuable insights into the dynamics of gambling behaviors and the effectiveness of intervention strategies. These findings emphasize the critical role of advanced mathematical methodologies in addressing gambling addiction. By facilitating the design of targeted prevention and intervention measures, this research contributes significantly to the development of more effective strategies for mitigating the adverse societal and individual impacts of gambling. Key findings of the study include:

- A nonlinear fractional-order model for gambling behavior with professional gambler is proposed.
- The reproduction number $\mathcal{R}_0$ rises with η, suggesting that number of problem gambler increases as η increases. This illustrates the superior efficiency of a fractional-order model compared to classical (integer-order) model.
- Optimal control theory is used to devise intervention strategies that minimize the number of exposed individuals and control costs, demonstrating their effectiveness in reducing gambling addiction.
- Simulations validate the theoretical findings and emphasize the importance of the threshold value, providing insights into the dynamics of gambling at the population level.
- The findings highlight the need for effective control measures and suggest that fractional-order models can better inform intervention strategies and policies.

Overall, the fractional-order model for analyzing gambling behavior is not only time-efficient but also asymptotically stable. This approach facilitates the establishment of a healthier gambling environment by integrating essential criteria. The insights gained from this research can inform the design of more effective prevention and recovery programs, ultimately contributing to the well-being of individuals and communities affected by gambling.

Future scope

The manuscript suggests expanding the model to include additional factors influencing gambling behavior and validating the results with empirical data. It also highlights the potential of applying these models within game theory frameworks to advance artificial intelligence in contexts involving competition and uncertainty.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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