



Quartic B-spline method for solving time fractional telegraph equation

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Abstract

In the present research, a quartic B-spline method is used for the solution of time-fractional telegraph equations. In the first step the Caputo fractional derivative is applied to the time-fractional telegraph equation combined with finite difference for time discretization, which reduces the problem to algebraic equations, which are solved. Finally, the numerical method proposed is demonstrated to be stable without any conditions by using the Fourier method. The error norms L_2 and L_∞ are calculated in order to test accuracy. The two tested examples exhibited the obtained results that the method is an effective numerical scheme to solve fractional equations.

Keywords. Quartic B-spline method, Von Neumann method, Finite difference techniques, Caputo time-fractional derivative, Diffusion telegraph equation.

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1. INTRODUCTION

It is considered the following model of the time fractional telegraph equation of the form [19]

$$\frac{\partial^\kappa y(\zeta, \eta)}{\partial \eta^\kappa} + \frac{\partial y(\zeta, \eta)}{\partial \eta} - \frac{\partial^2 y(\zeta, \eta)}{\partial \zeta^2} = g(\zeta, \eta), \quad (\zeta, \eta) \in [c, d] \times (0, T], \quad (1.1)$$

with initial conditions:

$$y(x, 0) = \xi_0(\zeta), \quad \frac{\partial y(x, 0)}{\partial \eta} = \xi_1(\zeta), \quad \zeta \in [c, d],$$

and boundary conditions:

$$y(c, \eta) = 0, \quad y(d, \eta) = 0,$$

where $g(\zeta, \eta) : [c, d] \times [0, T] \rightarrow \mathbb{R}$, $\xi_0(x), \xi_1(x) : [a, b] \rightarrow \mathbb{R}$, Actually, each of the aforementioned functions needs to be provided. We need to define the fractional derivative

$$\frac{\partial^\kappa y(\zeta, \eta)}{\partial \eta^\kappa} = \frac{1}{\Gamma(n - \kappa)} \int_c^\eta \frac{\partial^n y(\zeta, s)}{\partial s^n} (\eta - s)^{n - \kappa - 1} ds, \quad (1.2)$$

where $\Gamma(n - \kappa)$ is Gamma function which is defined as the following formula

$$\Gamma(n - \kappa) = \int_0^\infty \zeta^{n - \kappa - 1} e^{-\zeta} d\zeta, \quad (\Re(\kappa) > 0). \quad (1.3)$$

In recent decades, it has been observed that many scientists, researchers, and engineers are interested in the topic of fractional calculus due to the value it possesses, which has given it many applications in astrophysics, classical engineering, quantum mechanics, and other fields [15, 17, 18, 22]. Since fractional derivatives represent the memory and heredity features of various materials and processes, several writers have concluded thus far that fractional-order differential equations are more appropriate than integer-order ones. Fractional differential equations (FDEs) have

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recently attracted a lot of attention in a variety of scientific fields, including engineering, physics, chemistry, and others [5]. Given the importance of the Riemann-Liouville definition and the Caputo definition of the fractional derivative, we present an overview of the difference between them. In Caputo, the integral of the noninteger order is calculated after the derivative of the integer order has been calculated. Determine the derivative of integer order in the Riemann-Liouville equation after calculating the integral of noninteger order. It is crucial to note that the Caputo derivative can be applied to issues in which the function and its corresponding integer-order derivatives have beginning conditions. Fractional Calculus has evolved since its inaugural congress at the University of New Haven in 1974 [8], and several applications have surfaced in a wide range of scientific fields. As a consequence, different definitions of the fractional derivative are accessible in the literature, as are different methods for resolving derivative-related difficulties. Telegraph equations are partial differential equations and are classified as hyperbolic. They can be applied by simulating the process of reaction propagation. This simulation appears in the study of wave phenomena and in the study of the so-called random walk theory. It is also found in the propagation of waves of the electrical signal in the transmission medium [25]. Many numerical and analytical methods have been developed to address telegraph problems [21]. Overall, considered, the hyperbolic telegraph equation is essential for expanding knowledge and streamlining systems across a variety of scientific and technological fields [3]. Fractional diffusion telegraph equation is generated from the second-order partial differential equation by replacing the exponent of the second derivative of time with a real number κ where $\kappa \in (1, 2)$ [19]. Many researchers have achieved the fractional telegraph equation through several methods, including the solution of fractional telegraph problems with respect to time or space. Among those methods that were used is the method of generalized differentiation in terms of Mittag-Leffler functions [7, 11], such as homotopy analysis method [13], and the Tau method for the approximation of fractional telegraph equation [6] see also [12]. To find the solution to linear fractional partial differential model, B-spline method is used because this method has high flexibility that we can find the estimated solution at every node in the domain with great accuracy. Spline functions have shown themselves to be very useful tools for approximating curves because of their rich geometric features. This has led to the development of several numerical techniques for solving fractional differential equations that use B-spline interpolation. For example, in [2], application of cubic B-splines for second order fractional Sturm-Liouville problems, for more information, please see [16, 30] Thus, we will obtain a system contains range matrices due to numerical techniques that use B-spline method, which leads to ease of finding the solution through the computer [1, 23]. The first differential methodology-based approach is the Finite Difference Method. This method simulates both temporal and spatial fluctuations by differentiating the domain into a grid. Forward, backward, or central differential form simulations are all possible [28]. Using finite differences the fractional derivative of time is approximated which is defined using Caputo definition and the aim is to solve the fractional diffusion telegraph equation and approximate the solution and its derivatives with respect to ζ by the quartic B-spline (*QBS*) method. The study is organized as follows: Section 2 Brief overview of the numerical method used to approximate the solution for ζ . Section 3 describes how the suggested methodology is implemented numerically. Section 4 addresses the stability analysis, while section 5 analyzes numerical data using examples to provide context, Finally, we point out our comments on the results and what can be done in the future in section 6.

2. THE NUMERICAL METHOD

2.1. QBS method. To find the solution of TFTE, we apply a *QBS* method where, we impose the partition $\Delta : c = \zeta_0 < \zeta_1 < \zeta_2 < \dots < \zeta_{n-1} < \zeta_n = d$ on endowed interval $[c, d]$ such that $h = \frac{d-c}{n}$ and $\zeta_i = c + ih$ where $i = 0(1)n$ and n is mesh size of Δ . We may now define *QBS* In the following format [24]

$$\mathbb{B}_i^4(\zeta) = \frac{1}{24h^4} \begin{cases} (\zeta - \zeta_{i-2})^4, & \text{if } \zeta_{i-2} \leq \zeta \leq \zeta_{i-1}, \\ h^4 + 4h^3(\zeta - \zeta_{i-1}) + 6h^2(\zeta - \zeta_{i-1})^2 + 4h(\zeta - \zeta_{i-1})^3 - 4(\zeta - \zeta_{i-1})^4, & \text{if } \zeta_{i-1} \leq \zeta \leq \zeta_i, \\ 11h^4 + 12h^3(\zeta - \zeta_i) - 6h^2(\zeta - \zeta_i)^2 + 12h(\zeta - \zeta_i)^3 + 6(\zeta - \zeta_i)^4, & \text{if } \zeta_i \leq \zeta \leq \zeta_{i+1}, \\ h^4 + 4h^3(\zeta_{i+2} - \zeta) + 6h^2(\zeta_{i+2} - \zeta)^2 + 4h(\zeta_{i+2} - \zeta)^3 - 4(\zeta_{i+2} - \zeta)^4, & \text{if } \zeta_{i+1} \leq \zeta \leq \zeta_{i+2}, \\ (\zeta_{i+3} - \zeta)^4, & \text{if } \zeta_{i+2} \leq \zeta \leq \zeta_{i+3}, \\ 0, & \text{Otherwise.} \end{cases} \quad (2.1)$$



TABLE 1. The values of $\mathbb{B}_i(\zeta)$ and their derivative.

	ζ_{i-2}	ζ_{i-1}	ζ_i	ζ_{i+1}	ζ_{i+2}	ζ_{i+3}
\mathbb{B}_i	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
\mathbb{B}'_i	0	$\frac{1}{6h}$	$\frac{1}{2h}$	$-\frac{1}{2h}$	$-\frac{1}{6h}$	0
\mathbb{B}''_i	0	$\frac{1}{2h^2}$	$-\frac{1}{2h^2}$	$-\frac{1}{2h^2}$	$\frac{1}{2h^2}$	0
\mathbb{B}'''_i	0	$\frac{1}{h^3}$	$-\frac{3}{h^3}$	$\frac{3}{h^3}$	$-\frac{1}{h^3}$	0

We first begin by writing the approximate solution as follows

$$\hat{y}(\zeta, \eta) = \sum_{i=-2}^{n+1} c_i \mathbb{B}_i(\zeta), \tag{2.2}$$

where $\mathbb{B}_i(x)$ is *QBS* functions. $\hat{y}_i^j = \hat{y}(\zeta_i, \eta_j)$ is numerical solution, $j = 0(1)m$. Now, $\hat{y}_i^j, (\hat{y}_\zeta)_i^j, (\hat{y}_{\zeta\zeta})_i^j$, and $(\hat{y}_{\zeta\zeta\zeta})_i^j$ it can be expressed in the following formulas

$$\begin{cases} \hat{y}_i^j = \frac{c_{i-2}^j}{24} + \frac{11c_{i-1}^j}{24} + \frac{11c_i^j}{24} + \frac{c_{i+1}^j}{24}, \\ (\hat{y}_\zeta)_i^j = \frac{c_{i-2}^j}{6h} + \frac{c_{i-1}^j}{2h} - \frac{c_i^j}{2h} - \frac{c_{i+1}^j}{6h}, \\ (\hat{y}_{\zeta\zeta})_i^j = \frac{c_{i-2}^j}{2h^2} - \frac{c_{i-1}^j}{2h^2} - \frac{c_i^j}{2h^2} + \frac{c_{i+1}^j}{2h^2}, \\ (\hat{y}_{\zeta\zeta\zeta})_i^j = \frac{c_{i-2}^j}{h^3} - \frac{3c_{i-1}^j}{h^3} + \frac{3c_i^j}{h^3} - \frac{c_{i+1}^j}{h^3}. \end{cases} \tag{2.3}$$

Table 1 shows the solution values and their derivatives at the nodes using the *QBS* method. These values will be used to solve TFTE.

2.2. Properties of B-spline Basis Functions. The B-spline basis functions have the following properties [10, 26]:

- Positivity: $\mathbb{B}_i^r(\zeta) > 0, \zeta \in (\zeta_i, \zeta_{i+r+1})$.
- Local support: $\mathbb{B}_i^r(\zeta) = 0, \zeta \notin (\zeta_i, \zeta_{i+r+1})$.
- Piecewise polynomial: $\mathbb{B}_i^r(\zeta)$ are piecewise polynomial functions of degree r .
- Partition of unity: $\sum_{i=s-r}^s c_i \mathbb{B}_i^r(\zeta) = 1, \zeta \in (\zeta_s, \zeta_{s+1})$.
- Continuity: If the interior knot ζ_i has multiplicity p_i then $\mathbb{B}_i^r(\zeta)$ is C^{r-p_i} at $\zeta = \zeta_i$ and C^∞ elsewhere.

3. TIME DISCRETIZATION FORMULATION

It is known that central estimates are usually used to approximate second-order partial derivatives [28], and we mentioned earlier that the fractional derivative of time, defined by Caputo's definition, will be approximated using finite differences. Accordingly, the term $\frac{\partial^\kappa y(\zeta, \eta)}{\partial \eta^\kappa}$ will be approximated as follows:

$$\frac{\partial^\kappa \hat{y}(\zeta, \eta_{j+1})}{\partial \eta^\kappa} = \frac{1}{\Gamma(3 - \kappa)} \sum_{\nu=0}^j b_\nu \left[\frac{\hat{y}(\zeta, \eta_{j+1-\nu}) - 2\hat{y}(\zeta, \eta_{j-\nu}) + \hat{y}(\zeta, \eta_{j-1-\nu})}{\Delta \eta^\kappa} \right] + e_{\Delta \eta}^{j+1}, \tag{3.1}$$

where $e_{\Delta \eta}^{j+1} \leq \sigma(\Delta \eta^2)$, $\eta_j = j\Delta \eta, j = 0(1)m, \Delta \eta = \frac{T}{m}, b_\nu = (\nu + 1)^{2-\omega} - \nu^{2-\omega}$, and $e_{\Delta s}^{j+1}$ is the truncation error such that:

- $b_0 = 1$ and $b_\nu > 0, \nu = 1, 2, \dots, j$,
- $b_0 \geq b_1 \geq \dots \geq b_\nu, b_\nu \rightarrow 0$ as $\nu \rightarrow \infty$,
- $-b_\nu + (2b_\nu - b_{\nu-1}) + \sum_{\nu=0}^{j-1} (-b_{\nu-1} + 2b_\nu + b_{\nu+1}) + (2b_0 + b_1) = 1$.



Now, substituting (3.1) in (1.1) we obtain,

$$\frac{1}{\Gamma(3-\kappa)} \sum_{\nu=0}^j b_{\nu} \left[\frac{\hat{y}(\zeta, \eta_{j+1-\nu}) - 2\hat{y}(\zeta, \eta_{j-\nu}) + \hat{y}(\zeta, \eta_{j-1-\nu})}{\Delta t^{\kappa}} \right] + \left[\frac{\hat{y}(\zeta, \eta_{j+1}) - \hat{y}(\zeta, \eta_j)}{\Delta \eta} \right] - \frac{\partial^2 \hat{y}(\zeta, \eta_{j+1})}{\partial \zeta^2} = f(\zeta, \eta_{j+1}), \quad (3.2)$$

suppose that $\hat{y}^{j+1} = \hat{y}(\zeta, \eta_{j+1})$, $\alpha = \frac{1}{\Gamma(3-\kappa)\Delta \eta^{\kappa}}$ and $\beta = \frac{1}{\Delta \eta}$. The last equation can be written as follows

$$(\alpha + \beta)\hat{y}^{j+1} - (2\alpha + \beta)\hat{y}^j + \alpha\hat{y}^{j-1} + \alpha \sum_{\nu=0}^j b_{\nu} [\hat{y}(\zeta, \eta_{j+1-\nu}) - 2\hat{y}(\zeta, \eta_{j-\nu}) + \hat{y}(\zeta, \eta_{j-1-\nu})] - \frac{\partial^2 \hat{y}^{j+1}}{\partial \zeta^2} = f(\zeta, \eta_{j+1}). \quad (3.3)$$

where, $j = 0(1)m$. The term \hat{y}^{-1} appears when $j = 0$ or $\nu = j$, by using the initial condition we delete it and we get.

$$\begin{cases} \hat{y}_{\eta}^0 = \frac{\hat{y}^1 - \hat{y}^{-1}}{2\Delta \eta}, \\ \Rightarrow \hat{y}^{-1} = \hat{y}^1 - 2\Delta \zeta \xi_1(x). \end{cases} \quad (3.4)$$

By substituting (2.3) into (3.3), the following system can be found:

$$\begin{aligned} & \left[\frac{1}{24}(\alpha + \beta) - \frac{1}{2h^2} \right] c_{i-2}^{j+1} + \left[\frac{11}{24}(\alpha + \beta) + \frac{1}{2h^2} \right] c_{i-1}^{j+1} + \left[\frac{11}{24}(\alpha + \beta) + \frac{1}{2h^2} \right] c_i^{j+1} + \left[\frac{1}{24}(\alpha + \beta) - \frac{1}{2h^2} \right] c_{i+1}^{j+1} \\ & = (2\alpha + \beta) \left[\frac{c_{i-2}^j}{24} + \frac{11c_{i-1}^j}{24} + \frac{11c_i^j}{24} + \frac{c_{i+1}^j}{24} \right] - \alpha \left[\frac{c_{i-2}^{j-1}}{24} + \frac{11c_{i-1}^{j-1}}{24} + \frac{11c_i^{j-1}}{24} + \frac{c_{i+1}^{j-1}}{24} \right] \\ & - \alpha \sum_{\nu=1}^j b_{\nu} \left[\frac{c_{i-2}^{j+1-\nu}}{24} + \frac{11c_{i-1}^{j+1-\nu}}{24} + \frac{11c_i^{j+1-\nu}}{24} + \frac{c_{i+1}^{j+1-\nu}}{24} \right] - 2 \left[\frac{c_{i-2}^{j-\nu}}{24} + \frac{11c_{i-1}^{j-\nu}}{24} + \frac{11c_i^{j-\nu}}{24} + \frac{c_{i+1}^{j-\nu}}{24} \right] \\ & + \left[\frac{c_{i-2}^{j-1-\nu}}{24} + \frac{11c_{i-1}^{j-1-\nu}}{24} + \frac{11c_i^{j-1-\nu}}{24} + \frac{c_{i+1}^{j-1-\nu}}{24} \right] + f(\zeta_i, \eta_{j+1}). \end{aligned} \quad (3.5)$$

The above system consists of $(n+1)$ equations and $(n+4)$ unknowns. Therefore, we need to add two equations using the boundary conditions (1.1).

$$\begin{cases} \frac{c_{-2}^j}{24} + \frac{11c_{-1}^j}{24} + \frac{11c_0^j}{24} + \frac{c_1^j}{24} = 0 \\ \frac{c_{N-2}^j}{24} + \frac{11c_{N-1}^j}{24} + \frac{11c_N^j}{24} + \frac{c_{N+1}^j}{24} = 0. \end{cases} \quad (3.6)$$

The Pseudoinverse approach [9] will be used to solve the problem in this instance since there will be one more unknown than equations. We must determine the values of c_i^j when $j = 0$ before we can solve problem (3.5), and we will do this by applying the beginning condition (1.1).

$$\begin{cases} (\hat{y}_0^j)_{\zeta} = \frac{d}{d\zeta} \xi_0(\zeta_i), i = 0, \\ \hat{y}_i^0 = u(\zeta_i, 0) = \xi_0(\zeta_i), i = 0, 1, 2, \dots, n, \\ (\hat{y}_i^0)_{\zeta} = \frac{d}{d\zeta} \xi_0(\zeta_i), i = n. \end{cases} \quad (3.7)$$

The final result is a system that can be written as follows:

$$\mathbb{H}c_0 = \delta, \quad (3.8)$$

where \mathbb{H} , c_0 , and δ are follows:

$$\mathbb{H} = \begin{bmatrix} \frac{1}{6h} & \frac{1}{2h} & -\frac{1}{11h} & -\frac{1}{6h} & \cdots & 0 & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} \\ 0 & \cdots & \cdots & 0 & 0 & \frac{1}{6h} & \frac{1}{2h} & -\frac{1}{2h} & -\frac{1}{6h} \end{bmatrix},$$



$$\delta^T = \left[\frac{d}{d\zeta} \xi_0(\zeta_0) \quad \xi_0(\zeta_0) \quad \xi_0(\zeta_1) \quad \dots \quad \xi_0(\zeta_n) \quad \frac{d}{d\zeta} \xi_0(\zeta_n) \right]^T,$$

$$c_0^T = [c_{-2}^0 \quad c_{-1}^0 \quad c_0^0 \quad \dots \quad c_n^0 \quad c_{n+1}^0]^T,$$

where (3.8) has $(n + 4)$ of unknowns and $(n + 3)$ of equations.

4. STABILITY

To discuss stability in this section, we will use one of the famous methods called Von Neumann method [28, 29], which requires that we impose the error as follows:

$$\lambda_k^j = \varepsilon_k^j - \tilde{\varepsilon}_k^j, \quad k = 0, 1, 2, \dots, n, \quad j = 0, 1, 2, \dots, m, \tag{4.1}$$

where ε_k^j is the fourier mode's growth factor, and $\tilde{\varepsilon}_k^j$ is its approximation. When the error disappear as the computation progresses, the numerical scheme is stable. Let $g(\zeta, \eta) = 0$, and hence from (4.1) and (3.5) we obtain on roundoff error equation:

$$\begin{aligned} & \left[\frac{1}{24}(\alpha + \beta) - \frac{1}{2h^2} \right] \lambda_{i-2}^{j+1} + \left[\frac{11}{24}(\alpha + \beta) + \frac{1}{2h^2} \right] \lambda_{i-1}^{j+1} + \left[\frac{11}{24}(\alpha + \beta) + \frac{1}{2h^2} \right] \lambda_i^{j+1} + \left[\frac{1}{24}(\alpha + \beta) - \frac{1}{2h^2} \right] \lambda_{i+1}^{j+1} \\ & = (2\alpha + \beta) \left[\frac{\lambda_{i-2}^j}{24} + \frac{11\lambda_{i-1}^j}{24} + \frac{11\lambda_i^j}{24} + \frac{\lambda_{i+1}^j}{24} \right] - \alpha \left[\frac{\lambda_{i-2}^{j-1}}{24} + \frac{11\lambda_{i-1}^{j-1}}{24} + \frac{11\lambda_i^{j-1}}{24} + \frac{\lambda_{i+1}^{j-1}}{24} \right] \\ & - \alpha \sum_{\nu=1}^j b_\nu \left(\left[\frac{\lambda_{i-2}^{j+1-\nu}}{24} + \frac{11\lambda_{i-1}^{j+1-\nu}}{24} + \frac{11\lambda_i^{j+1-\nu}}{24} + \frac{\lambda_{i+1}^{j+1-\nu}}{24} \right] - 2 \left[\frac{\lambda_{i-2}^{j-\nu}}{24} + \frac{11\lambda_{i-1}^{j-\nu}}{24} + \frac{11\lambda_i^{j-\nu}}{24} + \frac{\lambda_{i+1}^{j-\nu}}{24} \right] \right. \\ & \left. + \left[\frac{\lambda_{i-2}^{j-1-\nu}}{24} + \frac{11\lambda_{i-1}^{j-1-\nu}}{24} + \frac{11\lambda_i^{j-1-\nu}}{24} + \frac{\lambda_{i+1}^{j-1-\nu}}{24} \right] \right). \end{aligned} \tag{4.2}$$

The boundary conditions of the Equation (4.2) are

$$\lambda_0^j = 0, \quad \lambda_n^j = 0, \quad j = 0, 1, 2, \dots, m, \tag{4.3}$$

and the initial conditions

$$\lambda_k^0 = h_0(\zeta_k), \quad (\lambda_\eta)_k^0 = h_1(\zeta_k), \quad k = 0(1)n. \tag{4.4}$$

Define the grid function

$$\lambda^j(\zeta) = \begin{cases} \lambda_k^j, & \zeta_k - \frac{h}{2} < \zeta \leq \zeta_k + \frac{h}{2}, \quad k = 0(1)n, \\ 0, & c < \zeta < \frac{h}{2} \quad \text{or} \quad d - \frac{h}{2} < \zeta < d. \end{cases} \tag{4.5}$$

The Fourier series of $\lambda^j(\zeta)$ can be form

$$\lambda^j(\zeta) = \sum_{\mu=-\infty}^{\infty} q^j(\mu) e^{\frac{i2\pi\mu\zeta}{(d-c)}}, \quad j = 0(1)m, \tag{4.6}$$

where

$$q^j(\mu) = \frac{1}{(d-c)} \int_c^d \lambda^j(\zeta) e^{\frac{-i2\pi\mu\zeta}{(d-c)}} d\zeta. \tag{4.7}$$

Let $\lambda^j = [\lambda_1^j, \lambda_2^j, \dots, \lambda_{n-1}^j]^T$, and introduce the norm [14]

$$\|\lambda^j\|_2 = \left(\sum_{k=1}^{m-1} h |\lambda_k^j|^2 \right)^{\frac{1}{2}} = \left[\int_c^d |\lambda_k^j|^2 d\zeta \right]^{\frac{1}{2}}. \tag{4.8}$$

By using Parseval's equality[4], it is clear that $\int_c^d |\lambda_k^j|^2 d\zeta = \sum_{\mu=-\infty}^{\infty} |q^j(\mu)|^2$ therefore the following relation is get:

$$\|\lambda^j\|_2^2 = \sum_{\mu=-\infty}^{\infty} |q^j(\mu)|^2. \tag{4.9}$$



Now, we assume that $\lambda_k^j = q^j e^{ipkh}$ is solution of the equations (4.2)-(4.4), where $i \in \mathbb{C}$ and $p \in \mathbb{R}$.

So the equation (4.2) can be written as follows

$$\begin{aligned} & [\frac{1}{24}(\alpha + \beta) - \frac{1}{2h^2}]q^{j+1}e^{ip(k-2)h} + [\frac{11}{24}(\alpha + \beta) + \frac{1}{2h^2}]q^{j+1}e^{ip(k-1)h} + [\frac{11}{24}(\alpha + \beta) + \frac{1}{2h^2}]q^{j+1}e^{ipkh} \\ & + [\frac{1}{24}(\alpha + \beta) - \frac{1}{2h^2}]q^{j+1}e^{ip(k+1)h} = (2\alpha + \beta)[\frac{q^j e^{ip(k-2)h}}{24} + \frac{11q^j e^{ip(k-1)h}}{24} + \frac{11q^j e^{ipkh}}{24} + \frac{q^j e^{ip(k+1)h}}{24}] \\ & - \alpha[\frac{q^{j-1} e^{ip(k-2)h}}{24} + \frac{11q^{j-1} e^{ip(k-1)h}}{24} + \frac{11q^{j-1} e^{ipkh}}{24} + \frac{q^{j-1} e^{ip(k+1)h}}{24}] - \alpha[\frac{e^{ip(k-2)h}}{24} + \frac{11e^{ip(k-1)h}}{24} \\ & + \frac{11e^{ipkh}}{24} + \frac{e^{ip(k+1)h}}{24}] \sum_{\nu=1}^j b_\nu (q^{j+1-\nu} - 2q^{j-\nu} + q^{j-1-\nu}), \end{aligned} \quad (4.10)$$

we divide (4.10) on e^{ipkh} , so it becomes as follows

$$q^{j+1} = 2\tau q^j - \omega q^{j-1} - \omega \sum_{\nu=1}^j b_\nu (q^{j+1-\nu} - 2q^{j-\nu} + q^{j-1-\nu}), \quad (4.11)$$

where

$$\begin{aligned} \tau &= \frac{[\frac{e^{-2iph}}{24} + \frac{11e^{-iph}}{24} + \frac{11}{24} + \frac{e^{iph}}{24}]}{\frac{1}{(\alpha + \frac{\beta}{2})} [r_1 \frac{e^{-2iph}}{24} + r_2 \frac{11e^{-iph}}{24} + r_2 + r_1 \frac{e^{iph}}{24}]}, \\ \omega &= \frac{[\frac{e^{-2iph}}{24} + \frac{11e^{-iph}}{24} + \frac{11}{24} + \frac{e^{iph}}{24}]}{\frac{1}{\alpha} [r_1 \frac{e^{-2iph}}{24} + r_2 \frac{11e^{-iph}}{24} + r_2 + r_1 \frac{e^{iph}}{24}]}, \end{aligned}$$

$$r_1 = [\frac{1}{24}(\alpha + \beta) - \frac{1}{2h^2}], \text{ and } r_2 = [\frac{1}{24}(\alpha + \beta) + \frac{1}{2h^2}].$$

Clearly $\tau \leq 1$ at $(\alpha + \frac{\beta}{2})^{-1}r_1 \geq \frac{1}{24}$ and $(\alpha + \frac{\beta}{2})^{-1}r_2 \geq \frac{11}{24}$, and also $\omega \leq 1$ at $(\alpha)^{-1}r_1 \geq \frac{1}{24}$ and $(\alpha)^{-1}r_2 \geq \frac{11}{24}$.

Lemma 4.1. *If q^j is a solution (4.11), then $|q^j| \leq 2|q^0|$, $j = 0, 1, \dots, m$.*

Proof. By induction we prove to result, For $j = 0$, the Equation (4.11) implies

$$q^j = 2\tau q^0,$$

since $\tau \leq 1$ then, $|q^j| \leq 2|q^0|$. Now, $|q^j| \leq 2|q^0|$, $j = 0, 1, 2, \dots, m-1$. Using Equation (4.11) one can have

$$|q^{j+1}| = 2\tau|q^j| - \omega|q^{j-1}| - \omega \sum_{\nu=1}^j b_\nu (|q^{j+1-\nu}| - 2|q^{j-\nu}| + |q^{j-1-\nu}|).$$

This implies

$$|q^{j+1}| \leq 2\tau|q^0| - 2\omega|q^0| - 2\omega \sum_{\nu=1}^j b_\nu (|q^0| - 2|q^0| + |q^0|).$$

Hence,

$$|q^{j+1}| \leq 2|q^0|.$$

□

Theorem 4.2. *System stability of (3.5) is unconditionally.*

Proof. Lemma (4.1) and form (4.9) allow us to proceed to $\|\lambda^j\|_2 \leq 2\|\lambda^0\|_2$, $j = 0(1)m$. This indicates that the system (3.5) is unconditionally stable. □



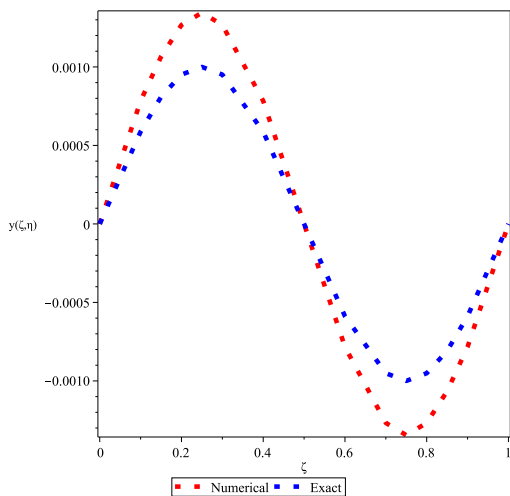


FIGURE 1. The exact solution and numerical solution of $\kappa = 1.5$ and $t=0.1$ for Example 5.1.

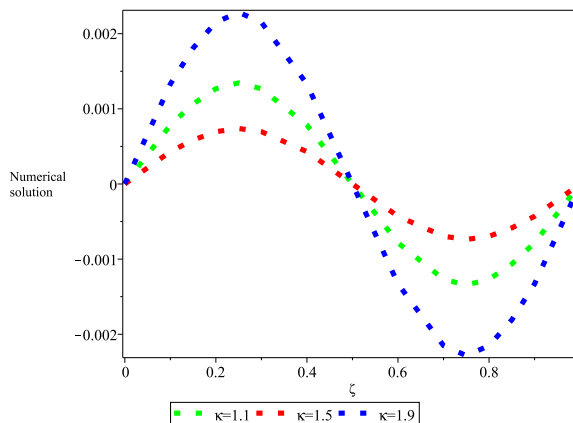


FIGURE 2. Approximate solutions of Example 5.1 with $n = 10$, $t = 0.1$ and different values of κ .

5. NUMERICAL EXAMPLES

In this section, we prove the accuracy of the numerical results, through which we know that the method adopted in our study is an accurate and correct method. For this purpose, examples with accurate solutions are solved and the numerical solutions are compared with the accurate ones, as well as compared with numerical solutions of other studies. It is worth noting that we used the Maple 15 program to find the numerical results. The error norms L_2 , and L_∞ are used to test the accuracy of the method that is being described which are computed in [27] as follows:

$$L_2 = \|y - \hat{y}_n\|_2 \simeq \sqrt{h \sum_{j=0}^n |y_j - (\hat{y}_n)_j|^2},$$

and

$$L_\infty = \|y - \hat{y}_n\|_\infty \simeq \max_j |y_j - (\hat{y}_n)_j|,$$

where \hat{y} and y stand for the approximate and exact solutions at the i th knot, respectively.

Example 5.1. We consider the following TFTE [19],

$$\begin{aligned} \frac{\partial^\kappa y(\zeta, \eta)}{\partial \eta^\kappa} + \frac{\partial y(\zeta, \eta)}{\partial \eta} - \frac{\partial^2 y(\zeta, \eta)}{\partial \zeta^2} &= g(\zeta, \eta), \\ y(\zeta, 0) = 0, \quad \frac{\partial y(\zeta, 0)}{\partial \eta} &= 0, \quad \zeta \in [0, 1], \\ y(0, \eta) = 0, \text{ and } y(1, \eta) &= 0, \quad \eta \in [0, 1]. \end{aligned} \tag{5.1}$$

The exact solution of problem (5.1) is $y(\zeta, \eta) = \eta^3 \sin(2\pi\zeta)$, and $g(\zeta, \eta) = \sin(2\pi\zeta) \left(\frac{6\eta^{3-\kappa}}{\Gamma(4-\kappa)} + \frac{6\eta^{4-\kappa}}{\Gamma(5-\kappa)} + 4\pi^2\eta^3 \right)$. Figure 1 represents the exact solution and numerical solution of $\kappa = 1.5$ and $\eta = 0.1$, Figure 2 shows the difference with values of κ and Figure 3 shows the effect of different t values on the absolute errors of Example 5.1. In Table 2, the numerical solution was compared with the exact solution, and in Table 3, L_2 and L_∞ values were compared for different η values when $n = 10$ and $\kappa = 1.3$, while in Table 4, a comparison was made between L_2 and L_∞ for different values of n and for different values of κ when $\Delta\eta = 0.1$.



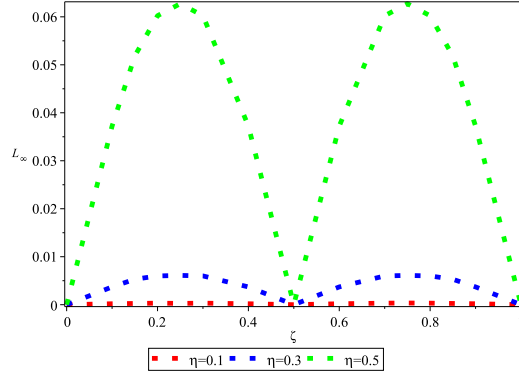


FIGURE 3. The effect of different values of η on the absolute error in the Example 5.1.

TABLE 2. Comparison between numerical solution and exact solution when $\kappa = 1.3$ and $n = 10$ for Example 5.1.

(ζ, η)	Numerical solution	Exact solution
(0.1,0.1)	5.85×10^{-4}	5.87×10^{-4}
(0.2,0.2)	5.66×10^{-3}	7.60×10^{-3}
(0.3,0.3)	1.59×10^{-2}	2.56×10^{-2}
(0.4,0.4)	2.08×10^{-2}	3.76×10^{-2}
(0.5,0.5)	-8.45×10^{-18}	0

TABLE 3. Comparison between L_2 and L_∞ when $\kappa = 1.3$ and $n = 10$ for Example 5.1.

η	L_2	L_∞
0.01	3.8214×10^{-7}	5.1398×10^{-7}
0.03	7.0564×10^{-6}	9.4909×10^{-6}
0.05	2.0749×10^{-5}	2.7907×10^{-5}
0.07	3.0264×10^{-5}	4.0705×10^{-5}

TABLE 4. Comparison between L_2 and L_∞ at $\Delta\eta = 0.1$ for Example 5.1.

n	L_2	L_∞	κ
10	1.0950×10^{-4}	1.4728×10^{-4}	$\kappa = 1.4$
20	7.7297×10^{-5}	1.4702×10^{-4}	
40	5.4651×10^{-5}	1.4701×10^{-4}	
10	6.9953×10^{-4}	9.4087×10^{-4}	$\kappa = 1.8$
20	4.9451×10^{-4}	9.4062×10^{-4}	
40	3.4966×10^{-4}	9.4060×10^{-4}	



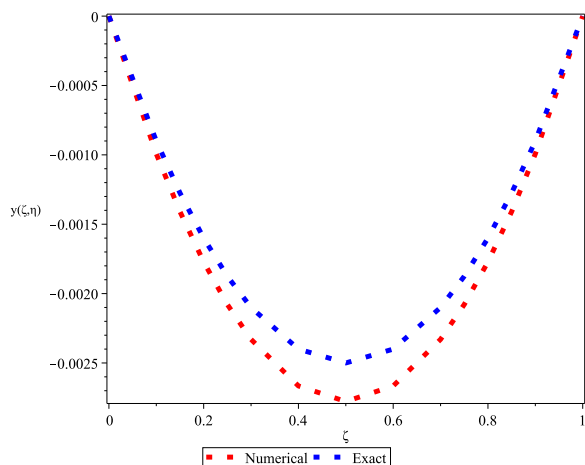


FIGURE 4. The exact solution and numerical solution of $\kappa = 1.5$ and $t=0.1$ for Example 5.2.

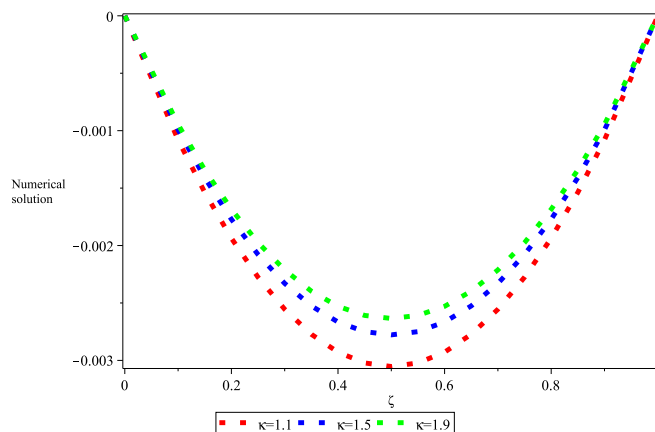


FIGURE 5. Approximate solutions of Example 5.2 with $n = 10$, $t = 0.1$ and different values of κ .

TABLE 5. the errors of our method with [20] are compared Example 5.2.

n	Proposed method	[20]	κ
20	3.45432×10^{-9}	3.6141×10^{-6}	$\kappa = 1.5$
40	3.45324×10^{-9}	1.2242×10^{-5}	
80	3.45324×10^{-9}	4.0311×10^{-5}	
20	4.560×10^{-10}	3.3327×10^{-3}	$\kappa = 1.8$
40	4.560×10^{-10}	4.2982×10^{-3}	
80	4.560×10^{-10}	2.0125×10^{-3}	

Example 5.2. We consider the following TFTE [20],

$$\begin{aligned} \frac{\partial^\kappa y(\zeta, \eta)}{\partial \eta^\kappa} + \frac{\partial y(\zeta, \eta)}{\partial \eta} - \frac{\partial^2 y(\zeta, \eta)}{\partial \zeta^2} &= g(\zeta, \eta), \\ y(\zeta, 0) = 0, \quad \frac{\partial y(\zeta, 0)}{\partial \eta} &= 0, \quad \zeta \in [0, 1], \\ y(0, \eta) = 0 \quad \text{and} \quad y(1, \eta) &= 0, \quad \eta \in [0, 1]. \end{aligned} \tag{5.2}$$

The exact solution of problem (5.2) is $y(\zeta, \eta) = \eta^2(\zeta^2 - \zeta)$, and $g(\zeta, \eta) = 2\eta(\zeta^2 - \zeta)\left(\frac{\Gamma(3-\kappa)-\eta^{1-\kappa}}{\Gamma(3-\kappa)}\right) - 2\eta^2$. Figure 4 represents the exact solution and numerical solution of $\kappa = 1.5$ and $\eta = 0.1$, Figure 5 Shows the difference with values of κ and Figure 6 shows the effect of different t values on the absolute errors of Example 5.2, In Table 5, a comparison is made between the solution of Example 5.2 using QBS and the source method [20] with respect the absolute error.

6. CONCLUSIONS

In this paper, B-spline quartic functions are applied to solve the telegraph equation. The method was tested on two problems, and the numerical results obtained in this study give acceptable and valuable results. The numerical solutions of the proposed method were compared with exact solutions and other methods to prove its effectiveness and accuracy. In addition, stability was derived using the Fourier method, and we proved that the method is stable without



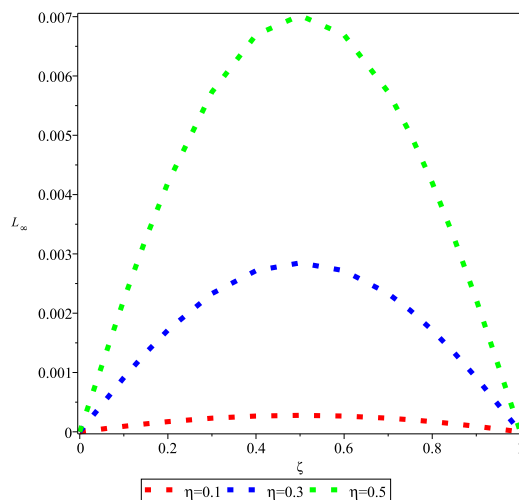


FIGURE 6. The effect of different values of η on the absolute error in Example 5.2.

any condition or restriction. We came to the conclusion that the suggested approach retains high accuracy when the calculation of error norms L_2 and L_∞ and demonstrated its accuracy. and that the numerical results converge to an exact solution, which makes it extremely encouraging for handling the solution of this kind of equation using fractions.

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APPENDIX

Algorithm

- (1) INITIALIZE: Set parameters and grid.
- (2) BUILD: Construct banded matrix \mathbb{H} and vector δ .
- (3) SOLVE: $C = \mathbb{H} \backslash \delta$ (linear system).
- (4) INTERP: $y(\zeta) =$ B-spline expansion using coefficients C .
- (5) RETURN: Piecewise polynomial solution $y(\zeta)$.

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