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# Fuzzy partial differential equations with fuzzy coefficients for $S$-linearly correlated fuzzy processes

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**Abstract**

In this paper, we introduce the concept of the distributive product of $S$-linearly correlated fuzzy processes. Within this framework, we study linear fuzzy partial differential equations with fuzzy coefficients for $S$-linearly correlated fuzzy processes. In particular, we study the fuzzy transport and heat equations with fuzzy coefficients and present their general solutions.

**Keywords.** Fuzzy partial differential equation, Fuzzy transport equation, Fuzzy heat equation, Fuzzy coefficients, S-linearly correlated fuzzy processes.

**2010 Mathematics Subject Classification.** 65L05, 34K06, 34K28.



## 1. Introduction

In many real-world phenomena, the available information is often imprecise or uncertain. The examination of fuzzy partial differential equations provides a powerful tool to handle these uncertainties, enabling more effective modeling and analysis of systems influenced by fuzziness [1, 17, 29]. In 1999, Buckley and Feuring [11] introduced the concept of fuzzy partial differential equations (FPDEs) and proposed methods for solving them. Since then, these equations have attracted considerable interest, with significant advancements and contributions reported in [2, 3, 5, 10, 19, 22]. In particular, the fuzzy transport and heat equations have been explored in [4, 18, 25, 28]. To investigate FPDEs, it is necessary to define fuzzy derivatives, such as $gH$-derivative, $\mathcal{D}^*$-derivative, strongly generalized derivative, and etc, see [7–9, 12, 20, 27]. A key challenge in many fuzzy derivative formulations is ensuring the existence of the associated differences. To address this issue, the authors in [13, 14] developed a calculus framework tailored to particular types of fuzzy functions, with values confined to a two-dimensional Banach space, $\mathbb{R}_{\mathcal{F}(A)}$ and, more broadly, in spaces of dimension $n$.

This paper focuses on FPDEs with fuzzy coefficients, particularly fuzzy transport equations—an area that has received relatively little attention despite its relevance to engineering, applied sciences, and biology. The limited investigation into fuzzy coefficients in the existing literature [5, 27] highlights a significant gap. To address this, in [26], we introduced a distributive product and the notion of partial differentiability for studying FPDEs and presented their solutions. Building on this foundation, we further extend the findings in [26] by considering $S$-linearly correlated fuzzy processes, thus offering a more comprehensive framework for analyzing these equations. Recently, Shen investigated solutions for the fuzzy heat equation and the fuzzy wave equation for $S$-linearly correlated fuzzy processes with fuzzy coefficients [27]. In this work, Shen considers the cross-product operation on strongly linearly correlated fuzzy numbers, which corresponds to a particular case of the distributive product introduced in this article. In general, solving FPDEs based on the cross product leads to a system of coupled classical partial differential equations

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(PDEs) that are very hard to solve analytically [27]. In contrast, we consider a different type of distributive product that yields more tractable FPDEs that can be solved analytically.

The remainder of the paper is structured as follows. Section 2 provides basic definitions and preliminaries, and extends some known results on $S$-linearly correlated fuzzy numbers. In section 3, we introduce a distributive product, which serves as a fundamental tool for the study of fuzzy partial differential equations with fuzzy coefficients. Section 4 investigates second-order FPDEs, particularly fuzzy transport and heat equations with fuzzy coefficients, and presents their solutions.

## 2. Preliminaries

This section provides an overview of the fundamental concepts and results essential for this paper. A fuzzy subset of $\mathbb{R}$ is a map $A:\mathbb{R}\to[0,1]$, such that $A(t)$ represents the degree of membership of $t\in\mathbb{R}$. For $\alpha\in(0,1]$, the $\alpha$-level set of $A$ is given by $[A]_\alpha=\{t\in\mathbb{R}|A(t)\geq\alpha\}$ and for $\alpha=0$ by the closure of the support: $[A]_0=cl\{t\in\mathbb{R}|A(t)>0\}$. A fuzzy number is a fuzzy subset of $\mathbb{R}$ such that its $\alpha$-level are non-empty, closed, and bounded intervals of $\mathbb{R}$ [6]. The set of all fuzzy numbers is denoted by $\mathbb{R}_F$. So, all fuzzy number $A$ can be described in terms of $\alpha$-levels by $[A]_\alpha=[A^-_\alpha,A^+_\alpha]$, for all $\alpha\in[0,1]$. Moreover, fuzzy numbers can be categorized into two types-symmetric and non-symmetric as symmetry plays a significant role in defining strongly linearly independent fuzzy numbers. A fuzzy number is referred as symmetric with respect to $t\in\mathbb{R}$ if $A(t+s)=A(t-s)$ for all $s\in\mathbb{R}$ and, in this case, is denoted by $(A\mid t)$. It is said to be asymmetric or non-symmetric if for any $t\in\mathbb{R}$ there exists an $s\in\mathbb{R}$ such that $A(t+s)\neq A(t-s)$ [13]. In the sequel, we recall the notion of an SLI set.

**Definition 2.1.** *[14] Let $A_1,\dots,A_n\in\mathbb{R}_F$. We define the subset $S(A_1,\dots,A_n)$ of $\mathbb{R}_F$ as the set given by*

$$S(A_1,\dots,A_n)=\{q_1A_1+\cdots+q_nA_n\mid q_1,\dots,q_n\in\mathbb{R}\},$$

*where "+" and $q_iA_i$ stand for the usual addition and scalar multiplication in $\mathbb{R}_F$. The set $\{A_1,\dots,A_n\}$ is called strongly linearly independent fuzzy numbers (SLI) if for every $A\in S(A_1,\dots,A_n)$ such that*

$$(A=q_1A_1+\cdots+q_nA_n\mid 0)\Rightarrow q_1=q_2=\cdots=q_n=0.$$

*Moreover, if $A\in S(A_1,\dots,A_n)$ such that $\{A_1,\dots,A_n\}$ is SLI, then $A$ is called $S$-linearly correlated fuzzy numbers.*

**Example 2.1.** *[14] Let $A$ be an asymmetric triangular fuzzy number. The set $\{1,A,A^2,\dots,A^n\}$ is SLI, where $A^i$ stands for the Zadeh extension of the function $f(t)=t^i$ at $A$, i.e., by Nguyen's theorem, $[A^i]_\alpha=\{t^i\mid\forall t\in[A]_\alpha\}$ for all $\alpha\in[0,1]$ and $i=0,1,\dots,n$.*

For an SLI set $\{A_1,\dots,A_n\}$, there is a bijection $\psi$ between $\mathbb{R}^n$ and $S(A_1,\dots,A_n)$ which is defined by

$$\psi(q_1,\dots,q_n)=q_1A_1+\cdots+q_nA_n.$$

This bijection is essential as it facilitates defining algebraic operations and norms in $S(A_1,\dots,A_n)$ based on those in $\mathbb{R}^n$. So, $(S(A_1,\dots,A_n),\oplus_\psi,\cdot_\psi,\|\cdot\|_\psi)$ is a Banach space such that

$$A\oplus_\psi B=\psi(\psi^{-1}(A)+\psi^{-1}(B)),$$
$$\|A\|_\psi=\left\|\psi^{-1}(A)\right\|_\infty,$$
$$\lambda\cdot_\psi A=\psi(\lambda\cdot\psi^{-1}(A)),$$

for all $A,B\in S(A_1,\dots,A_n)$ and $\lambda\in\mathbb{R}$ [14]. The subtraction can be defined by

$$A\ominus_\psi B=A\oplus_\psi(-1)\cdot_\psi B,$$

for any $A,B\in S(A_1,\dots,A_n)$.

It is worth noting that the operation $\cdot_\psi$ coincides with the usual notion of scalar multiplication, that is, $\lambda\cdot_\psi A=\lambda A$ [13]. Thus, for notational convenience, we denoted $\lambda\cdot_\psi A$ simply by $\lambda A$.

From now on, we assume that the set $\{A_1,\dots,A_n\}$, that generates the space $S(A_1,\dots,A_n)$, is SLI. For a given SLI set $\{A_1,\dots,A_n\}$, an $S$-linearly correlated fuzzy process is a fuzzy function $f:T\to S(A_1,\dots,A_n)$ defined by

$$f(t)=f_1(t)A_1+\cdots+f_n(t)A_n,\ \forall t\in T,$$

where $f_i : T \to \mathbb{R}, i = 1, \ldots, n$. It is worth noting that every $S$-linearly correlated fuzzy process is a well-defined fuzzy function, since the right side of the equation above is given in terms of the usual addition and scalar product on fuzzy numbers.

**Definition 2.2.** *[15] Let $\{A_1, \ldots, A_n\}$ be an SLI set. The function $f : (a, b) \to S(A_1, \ldots, A_n)$ is called $\psi$-differentiable at $t \in (a, b)$ if there exists a fuzzy number $f'(t) \in S(A_1, \ldots, A_n)$ such that*

$$f'(t) = \lim_{h \to 0} \frac{1}{h} \left( f(t + h) \ominus_\psi f(t) \right).$$

**Theorem 2.1.** *[15] Let $f : (a, b) \to S(A_1, \ldots, A_n)$ with $f(t) = f_1(t)A_1 + \cdots + f_n(t)A_n$ be $\psi$-differentiable at $t$ if and only if the function $f_i : (a, b) \to \mathbb{R}$ is differentiable at $t$, for all $i = 1, \ldots, n$. Furthermore, we have*

$$f'(t) = f_1'(t)A_1 + \cdots + f_n'(t)A_n.$$

**Corollary 2.1.** *[15] Let $f : (a, b) \to S(A_1, \ldots, A_n)$ with $f(t) = f_1(t)A_1 + \cdots + f_n(t)A_n$ be $\psi$-differentiable of order $k$ at $t$ if and only if the function $f_i : (a, b) \to \mathbb{R}$ is differentiable of order $k$ at $t$, for all $i = 1, \ldots, n$. In addition, we have*

$$f^{(k)}(t) = f_1^{(k)}(t)A_1 + \cdots + f_n^{(k)}(t)A_n.$$

The following presents the concepts and results for $S$-linearly correlated fuzzy processes as an extension of those in [26]. In the rest of the paper, we focus on the case where $X \times T \subseteq \mathbb{R}^2$, although higher dimensions could be considered without loss of generality.

**Definition 2.3.** *Let $f : X \times T \to S(A_1, \ldots, A_n)$. The function $f$ is called $\psi$-partial differentiable with respect to $x$ and $t$ at $(x_0, t_0) \in X \times T$ if there exist $\frac{\partial f}{\partial x}(x_0, t_0) \in S(A_1, \ldots, A_n)$ and $\frac{\partial f}{\partial t}(x_0, t_0) \in S(A_1, \ldots, A_n)$ satisfying*

$$\frac{\partial f}{\partial x}(x_0, t_0) = \lim_{h \to 0} \frac{1}{h} \left( f(x_0 + h, t_0) \ominus_\psi f(x_0, t_0) \right),$$
$$\frac{\partial f}{\partial t}(x_0, t_0) = \lim_{h \to 0} \frac{1}{h} \left( f(x_0, t_0 + h) \ominus_\psi f(x_0, t_0) \right).$$

**Theorem 2.2.** *Let $f : X \times T \subseteq \mathbb{R}^2 \to S(A_1, \ldots, A_n)$ given by $f(x, t) = f_1(x, t)A_1 + \cdots + f_n(x, t)A_n$. The function $f$ is $\psi$-partial differentiable with respect to $x$ or $t$ on $X \times T$ if and only if the functions $f_i$ for all $i = 1, \ldots, n$ are partial differentiable functions with respect to $x$ or $t$ on $X \times T$ and, in such cases, we get*

$$\frac{\partial f}{\partial x}(x, t) = \frac{\partial f_1}{\partial x}(x, t)A_1 + \cdots + \frac{\partial f_n}{\partial x}(x, t)A_n,$$
$$\frac{\partial f}{\partial t}(x, t) = \frac{\partial f_1}{\partial t}(x, t)A_1 + \cdots + \frac{\partial f_n}{\partial t}(x, t)A_n.$$

*Proof.* It follows immediately from Definition 2.3 and Theorem 2.1. □

Similarly to calculus on $\mathbb{R}^n$, the concept of $\psi$-partial derivative can be applied recursively to define higher order derivatives. However, in this paper, the study focuses on second-order $\psi$-partial derivatives for $S$-linearly correlated fuzzy processes. For simplicity, we use the following notation to represent the first and second-order $\psi$-partial derivatives of an $S$-linearly correlated fuzzy processes $f : X \times T \to S(A_1, \ldots, A_n)$, if they exist, as follows:

$$\frac{\partial}{\partial x} f = f_x, \quad \text{and} \quad \frac{\partial}{\partial t} f = f_t,$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial^2 f}{\partial x \partial t} = f_{xt},$$
$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial^2 f}{\partial t^2} = f_{tt}, \quad \text{and} \quad \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial t \partial x} = f_{tx}.$$

**Theorem 2.3.** *Let $f : X \times T \to S(A_1, \dots, A_n)$ given by $f(x,t) = f_1(x,t)A_1 + \cdots + f_n(x,t)A_n$. The function $f$ is second-order $\psi$-partial differentiable with respect to $x, t$, or both $x, t$ on $X \times T$ if and only if there exist $\frac{\partial^2 f_i}{\partial x^2}$ or $\frac{\partial^2 f_i}{\partial t^2}$, or $\frac{\partial^2 f_i}{\partial x \partial t}$, for $i = 1, 2$, and we have*

$$\frac{\partial^2 f}{\partial x^2}(x,t) = \frac{\partial^2 f_1}{\partial x^2}(x,t)A_1 + \cdots + \frac{\partial^2 f_n}{\partial x^2}(x,t)A_n,$$
$$\frac{\partial^2 f}{\partial t^2}(x,t) = \frac{\partial^2 f_1}{\partial t^2}(x,t)A_1 + \cdots + \frac{\partial^n f_n}{\partial t^2}(x,t)A_n,$$
$$\frac{\partial^2 f}{\partial x \partial t}(x,t) = \frac{\partial^2 f_1}{\partial x \partial t}(x,t)A_1 + \cdots + \frac{\partial^n f_n}{\partial x \partial t}(x,t)A_n,$$

*respectively.*

*Proof.* It is an immediate consequence of Theorem 2.2. □

## 3. Distributive product for $S$-linearly correlated fuzzy processes



In this section, we extend the distributive product in $\mathbb{R}_{\mathcal{F}(A)}$, introduced in [26], to $S(A_1, \dots, A_n)$ as follows.

**Definition 3.1.** *A binary operator $\odot_P$ on $S(A_1, \dots, A_n)$ is said to be a distributive product if the following three properties are satisfied for all $A, B, C \in S(A_1, \dots, A_n)$ and $x \in \mathbb{R}$:*

1. *Commutativity: $A \odot_P B = B \odot_P A$,*
2. *Distributivity with respect to addition:*
$$A \odot_P (B \oplus_\psi C) = (A \odot_P B) \oplus_\psi (A \odot_P C),$$
3. *Commute with respect to the scalar multiplication:*
$$(xA) \odot_P B = x(A \odot_P B).$$

The following theorem states that a distributive product can always be written as a weighted sum of the distributive products of elements of the SLI set $\{A_1, \dots, A_n\}$ that generates the space $S(A_1, \dots, A_n)$.

**Theorem 3.1.** *Let $\odot_P$ be a distributive product on $S(A_1, \dots, A_n)$ and let $P_{i,j} = A_i \odot_P A_j$ for all $i, j = 1, \dots, n$. For every $A, B \in S(A_1, \dots, A_n)$ such that $A = p_1 A_1 + \cdots + p_n A_n$ and $B = q_1 A_1 + \cdots + q_n A_n$, we have that*

$$A \odot_P B = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j P_{i,j}. \tag{3.1}$$

*Proof.* From the definition of the addition $\oplus_\psi$, one can observe that for any $A = p_1 A_1 + \cdots + p_n A_n = (p_1 A_1) \oplus_\psi \cdots \oplus_\psi (p_n A_n)$. Using this fact and the three properties above of $\odot_P$ as in Definition 3.1, we obtain

$$\begin{aligned}
A \odot_P B &= (p_1 A_1 + \cdots + p_n A_n) \odot_P (q_1 A_1 + \cdots + q_n A_n) \\
&= [(p_1 A_1) \oplus_\psi \cdots \oplus_\psi (p_n A_n)] \odot_P [(q_1 A_1) \oplus_\psi \dots \oplus_\psi (q_n A_n)] \\
&= \sum_{j=1}^{n} [(p_1 A_1) \oplus_\psi \cdots \oplus_\psi (p_n A_n)] \odot_P (q_j A_j) \\
&= \sum_{j=1}^{n} (q_j A_j) \odot_P [(p_1 A_1) \oplus_\psi \cdots \oplus_\psi (p_n A_n)] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} (q_j A_j) \odot_P (p_i A_i) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} q_j \, [A_j \odot_P (p_i A_i)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\sum_{i=1}^{n} q_j \left[(p_i A_i) \odot_P A_j\right] \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} q_j p_i \underbrace{(A_i \odot_P A_j)}_{P_{i,j}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j P_{i,j}. \qquad (3.2)
\end{aligned}
$$

□

Taking into account the arithmetic operations on Banach space $S(A_1,\dots,A_n)$, Equation (3.2) can be rewritten in the following bilinear matrix form $\mathbf{A}^t\mathbf{P}\mathbf{B}$, where $\mathbf{P} = [P_{i,j}]_{n\times n}$, $\mathbf{A} = (p_1,\dots,p_n)^t$, and $\mathbf{B} = (q_1,\dots,q_n)^t$. That is,

$$
A \odot_P B = \begin{bmatrix} p_1 & \dots & p_n \end{bmatrix} \begin{bmatrix} P_{1,1} & \dots & P_{1,n} \\ \vdots & \ddots & \vdots \\ P_{n,1} & \dots & P_{n,n} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}, \qquad (3.3)
$$

where $A = p_1A_1 + \dots + p_nA_n$ and $B = q_1A_1 + \dots + q_nA_n$.

Theorem 3.1 establishes that if there exists a distributive product $\odot_P$ on $S(A_1,\dots,A_n)$, then the distributive product of two arbitrary elements is given as in Equation (3.3), that is, the summation of products of scalars $p_iq_j$ and $P_{i,j}$, where $P_{i,j}$ is the distributive product of $A_i$ and $A_j$. Theorem 3.2 states that formula (3.3) for any given symmetric matrix $\mathbf{P}$ defines a distributive product on $S(A_1,\dots,A_n)$. This results ensures not only the existence of distributive products on $S(A_1,\dots,A_n)$, but also each distributive product depends on $0.5n(n+1)$ elements $P_{i,j}$ of $S(A_1,\dots,A_n)$ (since $P_{i,j} = P_{j,i}$ for all $i$ and $j$).

**Theorem 3.2.** *Given $P_{i,j} \in S(A_1,\dots,A_n)$ with $P_{i,j} = P_{j,i}$ for $i,j = 1,\dots,n$. The binary operation $\odot_P$ on $S(A_1,\dots,A_n)$ defined as follows is a distributive product:*

$$
A \odot_P B = \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j P_{i,j} = \mathbf{A}^t\mathbf{P}\mathbf{B},
$$

*where $A = p_1A_1 + \dots p_nA_n$ and $B = q_1A_1 + \dots + q_nA_n$. Moreover, we have that*

- *$P_{i,j} = A_i \odot_P A_j$ for all $i,j = 1,\dots,n$,*
- *if $A_1 \odot_P A_j = A_j$ for all $j = 1,\dots,n$, then $A_1 \odot_P A = A$ for all $A \in S(A_1,\dots,A_n)$.*

*Proof.* Let $A = p_1A_1 + \dots + p_nAn$, $B = q_1A_1 + \dots + q_nAn$, $C = r_1A_1 + \dots + r_nA_n$. To show that $\odot_P$ is a distributive product, we have to verify that three properties of Definition 3.1 are satisfied:

1. The commutativity is evident due to the symmetry of $\mathbf{P}$;
2. Using the fact that $B \oplus_\psi C = \psi(\psi^{-1}(B) + \psi^{-1}(C)) = (q_1 + r_1)A_1 + \dots + (q_n + r_n)A_n$, we have

$$
\begin{aligned}
A \odot_P (B \oplus_\psi C) &= A \odot_P \left[(q_1 + r_1)A_1 + \dots + (q_n + r_n)A_n\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i(q_j + r_j)P_{i,j} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j P_{i,j}\right) \oplus_\psi \left(\sum_{i=1}^{n}\sum_{j=1}^{n} p_i r_j P_{i,j}\right) \\
&= (A \odot_P B) \oplus_\psi (A \odot_P C).
\end{aligned}
$$

3. Since $xA = (xp_1)A_1 + \dots + (xp_n)A_n$, it follows that

$$
(xA) \odot_P B = ((xp_1)A_1 + \dots + (xp_n)A_n) \odot_P B
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(xp_i)q_jP_{i,j} \\
&= x\left(\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_jP_{i,j}\right) \\
&= x(A \odot_P B).
\end{aligned}
$$

Therefore, the operation $\odot_P$ is a distributive product.

Since $A_k = \sum_{i=1}^{n}\delta_{k,i}A_i$ for all $k = 1,\dots,n$ with $\delta_{i,i} = 1$ and $\delta_{k,i} = 0$ if $k \neq i$, it follows that

$$
A_i \odot_P A_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{k,i}\delta_{k,j}P_{i,j} = P_{i,j}.
$$

By hypothesis, we have that $P_{i,j} = P_{j,i}$ for all $i$ and $j$. If $P_{1,j} = A_1 \odot_P A_j = A_j$ for all $j = 1,\dots,n$, then $A_1 = \sum_{i=1}^{n}\delta_{1,i}A_i$ and

$$
\begin{aligned}
A_1 \odot_P A &= \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{1,i}p_jP_{i,j} = \sum_{j=1}^{n}p_jP_{1,j} \\
&= \sum_{j=1}^{n}p_jA_1 \odot_P A_j = \sum_{j=1}^{n}p_jA_j = A.
\end{aligned}
$$

□

**Remark 3.1.** *[24] A noteworthy special case arises when considering the SLI set $\{1, A, A^2, \dots, A^n\}$, where $A$ is an asymmetric triangular fuzzy number, instead of the general SLI set $\{A_1, \dots, A_n\}$. In this case, one can consider the following distributive product. Setting*

$$
P_{i,j} = A^i \odot_P A^j = \begin{cases} A^{i+j}, & \text{if } i+j \leq n, \\ 0, & \text{if } i+j > n, \end{cases} \tag{3.4}
$$

*for $i, j = 0, 1, \dots, n$. For $B, C \in S(1, A, \dots, A^n)$, we have*

$$
\begin{aligned}
B \odot_P C &= \begin{bmatrix} p_0 & \dots & p_n \end{bmatrix} \begin{bmatrix} 1 & A & \dots & A^n \\ A & A^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^n & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{bmatrix} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-i}p_iq_jA^{i+j} \qquad (3.5) \\
&= \sum_{i=0}^{n}\left(\sum_{j=0}^{i}p_{i-j}q_j\right)A^i,
\end{aligned}
$$

*where $B = (p_0 + p_1A + \ldots + p_nA^n)$ and $C = (q_0 + q_1A + \ldots + q_nA^n)$.*

It should be noted that Definition 3.1 extends the notion of product presented in [24, 26]. Recall that $\mathbb{R}_{\mathcal{F}(A)} = S(1, A)$ where $A$ is an asymmetric fuzzy number. If we consider $P_{1,1} = 1 \odot_P 1 = 1$, $P_{1,2} = P_{2,1} = 1 \odot_P A = A$, and $P_{2,2} = A \odot_P A$ for some $P_{2,2} \in \mathbb{R}_{\mathcal{F}(A)}$, then, by Equation (3.1), the distributive product of $B = (r + pA) \in \mathbb{R}_{\mathcal{F}(A)}$ and $C = (s + qA) \in \mathbb{R}_{\mathcal{F}(A)}$ is given by $B \odot_P C = rs + (rq + sp)A + pqP_{2,2}$. This coincides with the definition of distributive product on $\mathbb{R}_{\mathcal{F}(A)}$ provided in [26].

The cross-product on $S(A_1,\ldots,A_n)$ introduced in [21] is also a particular case of the distributive product. Let $\{A_1,\ldots,A_n\}$ be an SLI set such that $A_1=1$ and $[A_i]_1=\{a_i\}$ for all $i=2,\ldots,n$ and let $A,B\in S(A_1,\ldots,A_n)$. The cross-product of $A$ and $B$ is given by

$$A\otimes B = bA\oplus_\psi aB\ominus_\psi ab,$$

where $[A]_1=\{a\}$ and $[B]_1=\{b\}$. If we consider $P_{i,j}=A_i\otimes A_j$ for all $i,j=1,\ldots,n$, then one can easily verify that $A\odot_P B = A\otimes B$.

## 4. Fuzzy partial differential equations for $S$-linearly correlated fuzzy processes

In this section, we present fuzzy solutions to the second-order fuzzy linear partial differential equations with fuzzy coefficients for $S$-linearly correlated fuzzy processes, using the concept of $\psi$-partial differentiability and the distributive product of fuzzy processes. For the convenience of calculation and without loss of generality, throughout the remainder of the paper, we consider the SLI set $\{1,A,A^2\}$, where $A$ is asymmetric, and denote by $\odot_P$ the product introduced in Remark 3.1. In fact, all the results present in this section can naturally be extended for a general SLI set $\{1,A,A^2,\ldots,A^k\}$, with $k\geq 2$.

Consider the following linear FPDEs

$$e\odot_P\frac{\partial^2 v}{\partial x^2}+r\odot_P\frac{\partial^2 v}{\partial x\partial t}+y\odot_P\frac{\partial^2 v}{\partial t^2}+k\odot_P\frac{\partial v}{\partial x}+l\odot_P\frac{\partial v}{\partial t}+q\odot_P v+w=j, \tag{4.1}$$

where the coefficients $e,r,y,k,l,q\in S(1,A,A^2)$ and $v:\mathbb{R}^2\to S(1,A,A^2)$. Therefore, we assume that

$$\begin{aligned} v(x,t)&=\psi(v_0(x,t),v_1(x,t),v_2(x,t))\\ &=v_0(x,t)+v_1(x,t)A+v_2(x,t)A^2,\\ w(x,t)&=\psi(w_0(x,t),w_1(x,t),w_2(x,t))\\ &=w_0(x,t)+w_1(x,t)A+w_2(x,t)A^2. \end{aligned}$$

As a result, by Theorems 3.2 and 2.3, we obtain the following systems of classical partial differential equations

$$e_0\frac{\partial^2 v_0}{\partial x^2}+r_0\frac{\partial^2 v_0}{\partial x\partial t}+y_0\frac{\partial^2 v_0}{\partial t^2}+k_0\frac{\partial v_0}{\partial x}+l_0\frac{\partial v_0}{\partial t}+q_0v_0+w_0=j_0, \tag{4.2}$$

$$\begin{aligned} &e_0\frac{\partial^2 v_1}{\partial x^2}+e_1\frac{\partial^2 v_0}{\partial x^2}+r_0\frac{\partial^2 v_1}{\partial x\partial t}+r_1\frac{\partial^2 v_0}{\partial x\partial t}+y_0\frac{\partial^2 v_1}{\partial t^2}+y_1\frac{\partial^2 v_0}{\partial t^2}+k_0\frac{\partial u_1}{\partial x}+k_1\frac{\partial v_0}{\partial x}\\ &\qquad+l_0\frac{\partial v_1}{\partial t}+l_1\frac{\partial v_0}{\partial t}+q_0v_1+q_1v_0+w_1=j_1, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &e_0\frac{\partial^2 v_2}{\partial x^2}+e_1\frac{\partial^2 v_1}{\partial x^2}+e_2\frac{\partial^2 v_0}{\partial x^2}+r_0\frac{\partial^2 v_2}{\partial x\partial t}+r_1\frac{\partial^2 v_1}{\partial x\partial t}+r_2\frac{\partial^2 v_0}{\partial x\partial t}+y_0\frac{\partial^2 v_2}{\partial t^2}+y_1\frac{\partial^2 v_1}{\partial t^2}+y_2\frac{\partial^2 v_0}{\partial t^2}+k_0\frac{\partial v_2}{\partial x}+k_1\frac{\partial v_1}{\partial x}\\ &\qquad+k_2\frac{\partial v_0}{\partial x}+l_0\frac{\partial v_2}{\partial t}+l_1\frac{\partial v_1}{\partial t}+l_2\frac{\partial v_0}{\partial t}+q_0v_2+q_1v_1+q_2v_0+w_2=j_2, \end{aligned} \tag{4.4}$$

with $e=e_0+e_1A+e_2A^2,\ \ldots,\ q=q_0+q_1A+q_2A^2$. Hence, the fuzzy solution of (4.1) is given in terms of the real solutions of (4.2),(4.3), and (4.4).

### 4.1. Fuzzy linear transport equations.

Consider the following fuzzy transport equation

$$\begin{cases}\frac{\partial v}{\partial t}=k\odot_P\frac{\partial v}{\partial x}\oplus_\psi w(x,t),\\ v(x,0)=g(x),\end{cases} \tag{4.5}$$

where $w:\mathbb{R}\times(0,\infty)\to S(1,A,A^2)$ is continuous, $g:\mathbb{R}\to S(1,A,A^2)$ and $k\in S(1,A,A^2)$ such that $k=k_0+k_1A+k_2A^2$. A fuzzy function $v(x,t)$ is a solution of (4.5) on $\mathbb{R}\times(0,\infty)$ if it is $\psi$-partial differentiable with respect to $t$ and $x$ and satisfies Equation (4.5).

**Theorem 4.1.** *Let $g : \mathbb{R} \to S(1, A, A^2)$ be an $\psi$-differentiable function with $g \in C^2(\mathbb{R}, S(1, A, A^2))$ and let $k = k_0 + k_1 A + k_2 A^2$. If $w : \mathbb{R} \times (0, \infty) \to S(1, A, A^2)$ is $\psi$-partial differentiable with respect to $x$ such that $w(., t) \in C^2(\mathbb{R}, S(1, A, A^2))$, then the solution of Equation (4.5) is given by*

$$v(x, t) = v_0(x, t) + v_1(x, t)A + v_2(x, t)A^2,$$

*where*

$$\begin{aligned}
v_0(x,t) &= g_0(x+k_0t) + \int_0^t w_0(x+k_0(t-s),s)\,ds,\\
v_1(x,t) &= g_1(x+k_0t) + \int_0^t w_1(x+k_0(t-s),s) + k_1\frac{\partial v_0}{\partial x}(x+k_0(t-s),s)\,ds\\
&= g_1(x+k_0t) + k_1tg_0'(x+k_0t) + \int_0^t w_1(x+k_0(t-s),s)\,ds + k_1\int_0^t\int_0^s \frac{\partial w_0}{\partial x}\left(x+k_0(t-z),z\right)\,dz\,ds,\\
v_2(x,t) &= g_2(x+k_0t) + \int_0^t w_2(x+k_0(t-s),s)\,ds + \int_0^t k_2\frac{\partial v_0}{\partial x}(x+k_0(t-s),s)\,ds + \int_0^t k_1\frac{\partial v_1}{\partial x}(x+k_0(t-s),s)\,ds\\
&= g_2(x+k_0t) + k_2tg_0'(x+k_0t) + k_1tg_1'(x+k_0t)\\
&+ \frac{k_1^2t^2}{2}g_0''(x+k_0t) + \int_0^t w_2(x+k_0(t-s),s)\,ds\\
&+ k_2\int_0^t\int_0^s \frac{\partial w_0}{\partial x}\left(x+k_0(t-z),z\right)\,dz\,ds\\
&+ k_1\int_0^t\int_0^s \frac{\partial w_1}{\partial x}\left(x+k_0(t-z),z\right)\,dz\,ds\\
&+ k_1^2\int_0^t\int_0^s\int_0^z \frac{\partial^2 w_1}{\partial x^2}\left(x+k_0(t-p),p\right)\,dp\,dz\,ds\,.
\end{aligned}$$

*Proof.* Equation (4.5) can be converted into the following system of PDEs:

$$\begin{cases}
\frac{\partial v_0}{\partial t} = k_0\frac{\partial v_0}{\partial x} + w_0(x,t),\\
\frac{\partial v_1}{\partial t} = k_0\frac{\partial v_1}{\partial x} + k_1\frac{\partial v_0}{\partial x} + w_1(x,t),\\
\frac{\partial v_2}{\partial t} = k_0\frac{\partial v_2}{\partial x} + k_2\frac{\partial v_0}{\partial x} + k_1\frac{\partial v_1}{\partial x} + w_2(x,t),\\
v_0(x,0) = g_0(x),\\
v_1(x,0) = g_1(x),\\
v_2(x,0) = g_2(x).
\end{cases} \tag{4.6}$$

The function $v_0$ is obtained by considering the system

$$\begin{cases}
\frac{\partial v_0}{\partial t} = k_0\frac{\partial v_0}{\partial x} + w_0(x,t),\\
v_0(x,0) = g_0(x).
\end{cases}$$

Applying the Method of Characteristics [23], we obtain that $v_0$ is given by:

$$v_0(x,t) = g_0(x+k_0t) + \int_0^t w_0(x+k_0(t-s),s)ds.$$

Using the obtained function $v_0$, we can obtain $v_1$ by solving:

$$\begin{cases}
\frac{\partial v_1}{\partial t} = k_0\frac{\partial v_1}{\partial x} + k_1\frac{\partial v_0}{\partial x} + w_1(x,t),\\
v_1(x,0) = g_1(x).
\end{cases}$$

The term $k_1 \frac{\partial v_0}{\partial x} + w_1$ in the right side of the partial equation corresponds to the source/sink term of this equation because $v_0$ is already known. Thus, the solution $v_1$ is given by:

$$v_1(x,t) = g_1(x+k_0t) + \int_0^t w_1(x+k_0(t-s),s)ds + k_1 \int_0^t \frac{\partial v_0}{\partial x}(x+k_0(t-s),s)ds.$$

Note that, we have

$$\frac{\partial v_0}{\partial x}(x+k_0(t-s),s) = g_0'(x+k_0t) + \int_0^s \frac{\partial w_0}{\partial x}(x+k_0(t-z),z)dz.$$

Thus, combining two last equalities, we obtain

$$v_1(x,t) = g_1(x+k_0t) + k_1 t g_0'(x+k_0t) + \int_0^t w_1(x+k_0(t-s),s)\,ds + k_1 \int_0^t \int_0^s \frac{\partial w_0}{\partial x}\,(x+k_0(t-z),z)\;dz\,ds.$$

Using $v_0$ and $v_1$, we can obtain $v_2$ by solving the system:

$$\begin{cases} \frac{\partial v_2}{\partial t} = k_0 \frac{\partial v_2}{\partial x} + k_2 \frac{\partial v_0}{\partial x} + k_1 \frac{\partial v_1}{\partial x} + w_2(x,t), \\ v_2(x,0) = g_2(x). \end{cases}$$

Again, using the Method of Characteristics, we obtain that $v_2$ is given by:

$$v_2(x,t) = g_2(x+k_0t) + \int_0^t w_2(x+k_0(t-s),s)ds + k_2 \int_0^t \frac{\partial v_0}{\partial x}(x+k_0(t-s),s)ds + k_1 \int_0^t \frac{\partial v_1}{\partial x}(x+k_0(t-s),s)ds.$$

Using differentiation under the integral sign and the chain rule, we have

$$\begin{aligned} \frac{\partial v_1}{\partial x}(x+k_0(t-s),s) &= g_1'(x+k_0t) + k_1 s g_0''(x+k_0t) + \int_0^s \frac{\partial w_1}{\partial x}(x+k_0(t-z),z)dz \\ &\quad + k_1 \int_0^s \int_0^z \frac{\partial^2 w_0}{\partial x^2}(x+k_0(t-p),p)\,dp\,dz\,. \end{aligned}$$

Substituting the expressions for $\frac{\partial v_0}{\partial x}(x+k_0(t-s),s)$ and $\frac{\partial v_1}{\partial x}(x+k_0(t-s),s)$ above in the solution $v_2$, we conclude

$$\begin{aligned} v_2(x,t) &= g_2(x+k_0t) + k_2 t g_0'(x+k_0t) + k_1 t g_1'(x+k_0t) \\ &\quad + \frac{k_1^2 t^2}{2} g_0''(x+k_0t) + \int_0^t w_2(x+k_0(t-s),s)\,ds \\ &\quad + k_2 \int_0^t \int_0^s \frac{\partial w_0}{\partial x}\,(x+k_0(t-z),z)\;dz\,ds \\ &\quad + k_1 \int_0^t \int_0^s \frac{\partial w_1}{\partial x}\,(x+k_0(t-z),z)\;dz\,ds \\ &\quad + k_1^2 \int_0^t \int_0^s \int_0^z \frac{\partial^2 w_1}{\partial x^2}\,(x+k_0(t-p),p)\;dp\,dz\,ds\,. \end{aligned}$$

□

Transport equation models the advection that is the movement of some quantity or substance due to a bulk flow. When the initial condition, advection velocity, and source term are uncertain, the equation describes the advection not only of the physical quantity itself but also of the associated uncertainty, which is transported through space without dissipation. Example 4.1 aims to illustrate this phenomenon along with the result presented in Theorem 4.1.

**Example 4.1.** *Consider the following fuzzy transport equation:*

$$\begin{cases} v_t = (1+3A+tA^2) \odot_P v_x \oplus_\psi \left(t + (t^2x)A + (e^x \sin(t)))A^2\right), \\ v(x,0) = x + (5e^x)A + (x^2+5)A^2. \end{cases} \tag{4.7}$$

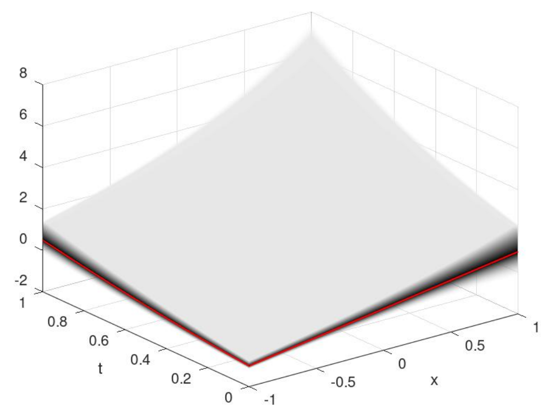


FIGURE 1. The 3D view of the fuzzy solution $v$ of Equation (4.7) for all $(x,t) \in [0,1] \times [-1,1]$, with the triangular fuzzy number $A = (-0.09; 0; 0.1)$. The gray-scale surfaces correspond to the endpoints of the $\alpha$-levels of the solution $v$, where the higher $\alpha$, the darker the surface. The red line depicts the 1-level set of the fuzzy solution $v$.



*The solution of Equation* (4.7) *is the fuzzy function* $v(x,t) = v_0(x,t) + v_1(x,t)A + v_2(x,t)A^2$ *such that* $v_0, v_1, v_2$ *are solutions of the following system of classical partial differential equations:*

$$\begin{cases} \frac{\partial v_0}{\partial t} = \frac{\partial v_0}{\partial x} + t, \\ \frac{\partial v_1}{\partial t} = \frac{\partial v_1}{\partial x} + 3\frac{\partial v_0}{\partial x} + t^2 x, \\ \frac{\partial v_2}{\partial t} = \frac{\partial v_2}{\partial x} + t\frac{\partial v_0}{\partial x} + 3\frac{\partial v_1}{\partial x} + e^x \sin(t), \\ v_0(x,0) = x, \\ v_1(x,0) = 5e^x, \\ v_2(x,0) = x^2 + 5. \end{cases} \tag{4.8}$$

*By Theorem 4.1, we have*

$$v_0(x,t) = x + \frac{t^2}{2} + t,$$

$$v_1(x,t) = 3t + \frac{t^4}{12} + \frac{t^3 x}{3} + 5e^{x+t},$$

*and*

$$v_2(x,t) = (x+t)^2 + 5 + \frac{t^2}{2} + \frac{t^4}{4} + e^{x+t}\left(15t + \frac{1 - e^{-t}(\cos t + \sin t)}{2}\right).$$

*Let* $A = (-0.09; 0; 0.1)$. *Figure 1 illustrates the fuzzy solution $v$ for the FPDE* (4.7) *for all* $(x,t) \in [0,1] \times [-1,1]$. *Figure 2 displays the solution $v$ at the time instances* $t = 0.05,\ 0.35,\ 0.65,\ 0.8$.

4.2. **Fuzzy heat equations.** In this subsection, let us turn our attention to fuzzy partial equations with fuzzy boundary and initial conditions of the form

$$\begin{cases} \frac{\partial v}{\partial t}(x,t) = K \odot_P \frac{\partial^2 v}{\partial x^2}(x,t) \oplus_\psi s(x,t), \qquad 0 < x < L, \qquad 0 < t < \infty, \\ v(0,t) = B, \\ v(L,t) = C, \\ v(x,0) = \varphi(x), \qquad 0 \le x \le L, \end{cases} \tag{4.9}$$

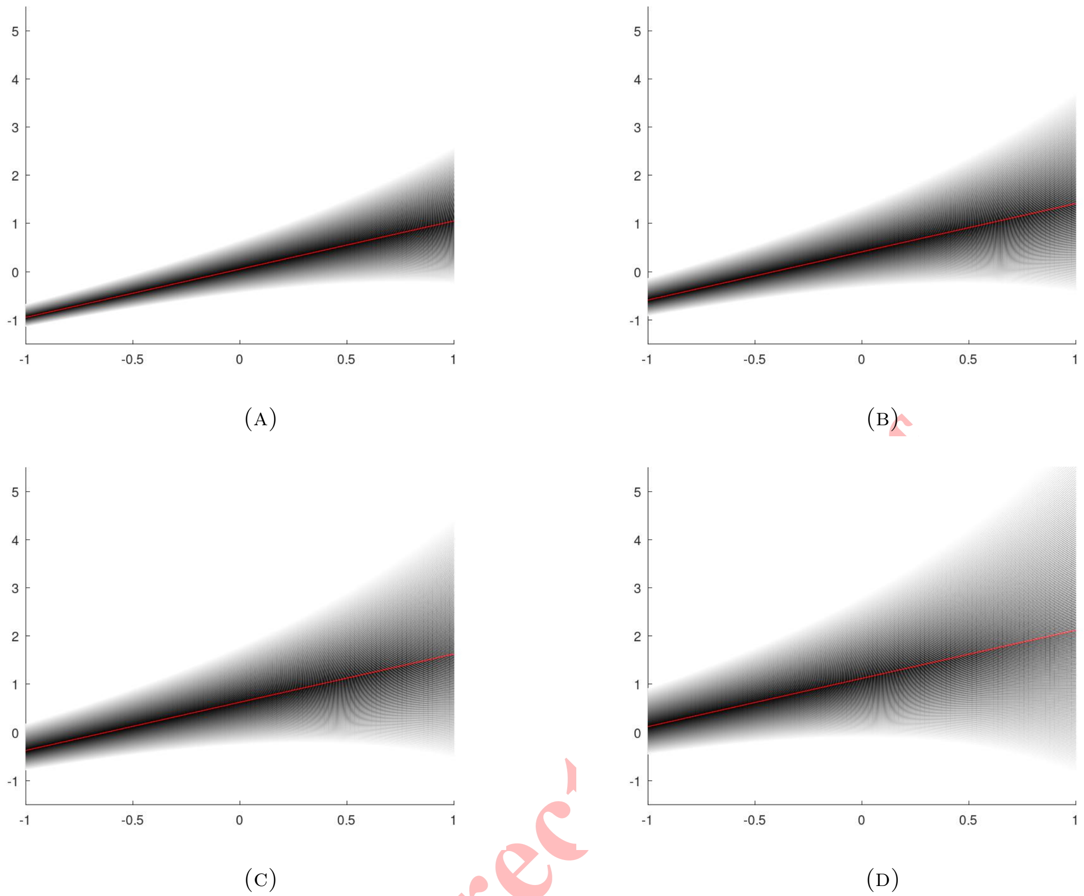


FIGURE 2. Let $A = (-0.09; 0; 0.1)$. Each graph from (A) to (D) presents a top view of the fuzzy solution $v(x, t_i)$ from Example 4.1 for $x \in [-1, 1]$ at the time instances $t_i = 0.05,\ 0.35,\ 0.65,\ 0.8$, respectively. The gray-scale curves correspond to the endpoints of the $\alpha$-levels of the solution $v$, where the higher $\alpha$, the darker the surface. The red curves depict the 1-level sets of the fuzzy solution $v(\cdot, t_i)$ at each fixed time $t_i$.



where $B = b_0 + b_1 A + b_2 A^2$ and $C = c_0 + c_1 A + c_2 A^2$ correspond to the boundary conditions (BC), $\varphi(x) = \varphi_0(x) + \varphi_1 A + \varphi_2 A^2$ corresponds to the fuzzy initial condition (IC) at $t = 0$ for $0 \le x \le L$, $s(x,t) = s_0(x,t) + s_1(x,t)A + s_2(x,t)A^2$ represents the fuzzy source (or sink) term of the FPDE for $t \in (0, \infty)$ and $x \in (0, L)$, $K = k_0 + k_1 A + k_2 A^2$ with $k_0 > 0$ stands for the fuzzy diffusion coefficient, and $v(x,t) = v_0(x,t) + v_1(x,t)A + v_2(x,t)A^2$ is the fuzzy state variable of interest, which depending on the context is the heat of a rod (or the concentration of a population) at position $x$ and instant $t$.

From Equation (3.5) and Theorem 2.3, solving Equation (4.9) is equivalent to solving the following classical FPDEs:

$$\begin{cases} \frac{\partial v_i}{\partial t} = k_0 \frac{\partial^2 v_i}{\partial x^2} + f_i, \\ v_i(0,t) = b_i, \\ v_i(L,t) = c_i, \\ v_i(x,0) = \varphi_i(x), \end{cases} \tag{4.10}$$

with

$$f_i = s_i + \sum_{j=0}^{i-1} k_{i-j} \frac{\partial^2 v_j}{\partial x^2}, \tag{4.11}$$

for $i = 0, 1, 2$. Note that the following PDEs must be solved sequentially. In fact, $v_1$ can only be found after determining $v_0$, while $v_2$ can be found just after $v_1$ a $v_0$ are known. Moreover, if we consider an SLI set $\{1, A, A^2, \dots, A^k\}$ with $k \geq 2$, then we would have $k$ PDEs which can be solved sequentially in a similar way.

In order to apply the eigenfunction expansion method, we need first to transform Equation (4.10) to a system of PDEs with homogeneous boundary conditions and also consider the additional hypotheses that $s_i, \varphi_i \in L^2$ for $i = 0, 1, 2$. Taking $v_i(x,t) = u_i(x,t) + \phi_i(x)$ and $\phi_i(x) = b_i + \frac{x}{L}(c_i - b_i)$ for all $x \in [0, L]$, $0 \leq t < \infty$ and $i = 0, 1, 2$, we obtain $\frac{\partial u_i}{\partial t} = \frac{\partial v_i}{\partial t}$ and $\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 v_i}{\partial x^2}$ and, consequently, the following PDEs with homogeneous BC:

$$\begin{cases} \frac{\partial u_i}{\partial t} = k_0 \frac{\partial^2 u_i}{\partial x^2} + f_i \\ u_i(0,t) = 0 \\ u_i(L,t) = 0 \\ u_i(x,0) = \varphi_i(x) - \phi_i(x) = \varphi_i(x) - b_i - \frac{x}{L}(c_i - b_i). \end{cases} \tag{4.12}$$

Now, Equation (4.12) can be solved using eigenfunction expansion method. To this end, let us consider the following associated Sturm-Liouville problem given by:

$$\begin{cases} X''(x) + \frac{\lambda^2}{k_0} X(x) = 0 \\ X(0) = 0 \\ X(L) = 0, \end{cases}$$

which produces the eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi}{L} x\right), \tag{4.13}$$

for all $n = 1, 2, 3, \dots$. For every $n$, consider the operator $S_n$ defined for every continuous function $g : [0, L] \to \mathbb{R}$ as follows:

$$S_n[g] = \frac{2}{L} \int_0^L g(\tau) X_n(\tau) d\tau.$$

One can easily verify that

$$S_n[X_m] = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m. \end{cases}$$

Hence, the set $\{X_n\}$ is an orthogonal set of functions. The values $S_n[g]$, $n = 1, 2, \dots$, correspond to the components of the Finite sine transform of the function $g$ and satisfy the following properties [16]:

(a) For every continuous function $g : [0, L] \to \mathbb{R}$, we have

$$g(x) = \sum_{i=1}^{\infty} S_n[g] X_n(x), \qquad \forall x \in [0, L];$$

(b) The operator $S_n$ is linear;

(c) If $g : [0, L] \to \mathbb{R}$ is twice continuously differentiable, then

$$S_n[g''] = -\left(\frac{n\pi}{L}\right)^2 S_n[g] + \frac{2n\pi}{L^2}\left(g(0) + (-1)^{n+1} g(L)\right).$$

Using property (a) above, for every $t \in (0, \infty)$, the functions $f_i$ and $(\varphi_i - \phi_i)$ can be written as

$$f_i(x,t) = \sum_{n=1}^{\infty} S_n[f_i(\cdot, t)] X_n(x), \qquad \forall x \in [0, L],$$

and

$$(\varphi_i - \phi_i)(x) = \varphi_i(x) - \phi_i(x) = \sum_{n=1}^{\infty} S_n[\varphi_i - \phi_i]X_n(x), \qquad \forall x \in [0, L].$$

Suppose that the solution of (4.12) has the form $u_i(x,t) = \sum_{n=1}^{\infty} T_{i_n}(t)X_n(x)$, with $T_{i_n}$ continuously differentiable. Using the fact that $X''(x) = -\left(\frac{n\pi}{L}\right)^2 X_n(x)$ for all $x \in (0, L)$, Equation (4.12) can be written as

$$\begin{cases} \sum_{n=1}^{\infty} T'_{i_n}(t)X_n(x) = -\left(\frac{n\pi}{L}\right)^2 k_0 \sum_{n=1}^{\infty} T_{i_n}(t)X_n(x) + \sum_{n=1}^{\infty} S_n[f_i(\cdot,t)]X_n(x), \\ u_i(0,t) = \sum_{n=1}^{\infty} T_{i_n}(t)X_n(0) = 0, \\ u_i(L,t) = \sum_{n=1}^{\infty} T_{i_n}(t)X_n(L) = 0, \\ u_i(x,0) = \sum_{n=1}^{\infty} T_{i_n}(0)X_n(x) = \sum_{n=1}^{\infty} S_n[\varphi_i - \phi_i]X_n(x). \end{cases} \tag{4.14}$$

Note that the boundary conditions above do not say anything about the functions $T_{i_n}$ since $X_n(0) = X_n(L) = 0$. However, since $\{X_n\}$ are orthogonal functions, for every $n$, we obtain the initial value problem (IVP):

$$\begin{cases} T'_{i_n}(t) + \left(\frac{n\pi}{L}\right)^2 k_0 T_{i_n} = S_n[f_i(\cdot,t)], \\ T_{i_n}(0) = S_n[\varphi_i - \phi_i]. \end{cases}$$

Using integrating factor, we obtain that the solution of this IVP is

$$T_{i_n}(t) = S_n[\varphi_i - \phi_i]e^{-\left(\frac{n\pi}{L}\right)^2 k_0 t} + \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 k_0 (t-\tau)} S_n[f_i(\cdot,\tau)]d\tau. \tag{4.15}$$

From Equation (4.11) and the properties of the operator $S_n$, we have

$$\begin{aligned} S_n[f_i(\cdot,\tau)] &= S_n\left[s_i(\cdot,\tau) + \sum_{j=0}^{i-1} k_{i-j}\frac{\partial^2 v_j}{\partial x^2}(\cdot,\tau)\right] \\ &= S_n\left[s_i(\cdot,\tau)\right] + \sum_{j=0}^{i-1} k_{i-j} S_n\left[\frac{\partial^2 v_j}{\partial x^2}(\cdot,\tau)\right] \\ &= S_n\left[s_i(\cdot,\tau)\right] + \sum_{j=0}^{i-1} -k_{i-j}\left(\frac{n\pi}{L}\right)^2 S_n\left[v_j(\cdot,\tau)\right] \\ &+ \sum_{j=0}^{i-1} k_{i-j}\frac{2n\pi}{L^2}\left(v_j(0,\tau) + (-1)^{n+1}v_j(L,\tau)\right). \end{aligned}$$

Using this last equality, we obtain

$$\begin{aligned} T_{i_n}(t) = {} & \frac{2}{L}e^{-\left(\frac{n\pi}{L}\right)^2 k_0 t}\int_0^L(\varphi_i(\tau) - \phi_i(\tau))X_n(\tau)d\tau + \frac{2}{L}\int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 k_0(t-\tau)}\int_0^L s_i(z,\tau)X_n(\tau)dzd\tau \\ & + \sum_{j=0}^{i-1} -k_{i-j}\left(\frac{n\pi}{L}\right)^2\int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 k_0(t-\tau)}\int_0^L v_j(z,\tau)X_n(\tau)dzd\tau \\ & + \sum_{j=0}^{i-1} k_{i-j}\frac{2n\pi}{L^2}\int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 k_0(t-\tau)}\left(v_j(0,\tau) + (-1)^{n+1}v_j(L,\tau)\right)d\tau. \end{aligned} \tag{4.16}$$

Thus, for $i = 0, 1, 2,$ the solution of Equation (4.10) is given by

$$\begin{aligned} v_i(x,t) &= \phi_i(x) + u_i(x,t) \\ &= b_i + \frac{x}{L}(c_i - b_i) + \sum_{n=1}^{\infty} X_n(x)T_{i_n}(t) \end{aligned} \tag{4.17}$$

$$= b_i + \frac{x}{L}(c_i - b_i) + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) T_{i_n}(t),$$

with $T_{i_n}$ given as in Equation (4.16). This proves the following theorem.

**Theorem 4.2.** *Consider the fuzzy PDE (4.9), with $k_0 > 0$ and the functions $s_i$ and $\varphi$ continuous for $i = 0, 1, 2$. The function $v(x,t) = v_0(x,t) + v_1(x,t)A + v_2(x,t)A^2$, with $v_i$ given by Equation (4.17) for $i = 0, 1, 2$, is the solution of the fuzzy PDE (4.9).*

The fuzzy heat equation models the diffusion of a concentration and its associated uncertainty, which tends to disperse and smooth out over time and space, indicating a dissipative process. This behavior is illustrated in Example 4.2.

**Example 4.2.** *Consider the following fuzzy heat equation*

$$\begin{cases} \frac{\partial v}{\partial t} = \left((1 + 0.5A + 0.25A^2) \odot_P \frac{\partial^2 v}{\partial x^2}\right) \oplus_\psi e^{-t} \left(\sin(\pi x) + 0.5 \sin(2\pi x) A\right), \\ v(0,t) = 0, \\ v(L,t) = 0, \\ v(x,0) = \sin(\pi x) + 2\sin(2\pi x)A. \end{cases} \tag{4.18}$$

*for $0 < x < 1$ and $t > 0$.*

*According to Theorem 4.2, the fuzzy solution of the FPDE (4.18) is given by*

$$v(x,t) = v_0(x,t) + v_1(x,t)A + v_2(x,t)A^2, \quad \forall (x,t) \in [0,1] \times [0,\infty).$$

*The function $v_0$ is obtained solving the following PDE:*

$$\begin{cases} \frac{\partial v_0}{\partial t} = \frac{\partial^2 v_0}{\partial x^2} + e^{-t}\sin(\pi x), \\ v_0(0,t) = v_0(1,t) = 0, \\ v_0(x,0) = \sin(\pi x). \end{cases} \tag{4.19}$$

*The function $v_0$ can be obtained using the eigenfunction expansion method. To this end, we are seeking for functions $v_0$ of the form*

$$v_0(x,t) = \sum_{n=1}^{\infty} X_n(x) T_{0_n}(t),$$

*for all $(x,t) \in [0,1] \times [0,\infty)$. The functions $X_n$ correspond to the solution of the following associated Sturm-Liouville problem*

$$\begin{cases} X - \lambda^2 X'' = 0, \\ X(0) = X(1) = 0. \end{cases}$$

*The solutions of this initial value problem (IVP) is the eigenfunction $X_n(x) = \sin(\lambda_n x)$ with $\lambda_n = n\pi$ for $n = 1, 2, 3, \ldots$. The function $T_{0_1}$ is the solution of the IVP:*

$$\begin{cases} T_{0_1} + \pi^2 T'_{0_1} = e^{-t}, \\ T_{0,1}(0) = 1. \end{cases}$$

*Using integrating factor, we obtain*

$$\begin{aligned} T_{0_1}(t) &= e^{-t\pi^2} \int_0^t e^{s(\pi^2-1)} ds + e^{-t\pi^2} \\ &= \frac{e^{-t}}{\pi^2 - 1} - \frac{e^{-\pi^2 t}}{\pi^2 - 1} + e^{-\pi^2 t}. \end{aligned}$$

*For all* $n \geq 2$*, we have* $T_{0_n}(t) = 0$ *which is the solution of* $T_{0_n} + \pi^2 T'_{0_n} = 0$ *with the initial condition* $T_{0_n}(0) = 0$*. Thus, the solution of PDE* (4.19) *is*

$$v_0(x,t) = X_1(x)T_{0_1}(t) = \sin(\pi x)\left(\frac{e^{-t}}{\pi^2-1} - \frac{e^{-\pi^2 t}}{\pi^2-1} + e^{-t\pi^2}\right).$$

*With the function* $v_0$*, we can find the function* $v_1$ *that corresponds to the solution of the following PDE:*

$$\begin{cases} \frac{\partial v_1}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} + 0.5\frac{\partial^2 v_0}{\partial x^2} + 0.5e^{-t}\sin(2\pi x), \\ v_1(0,t) = v_1(1,t) = 0, \\ v_1(x,0) = 2\sin(2\pi x). \end{cases} \tag{4.20}$$

*Note that*

$$\frac{\partial^2 v_0}{\partial x^2}(x,t) = -\pi^2 \sin(\pi x)\left(\frac{e^{-t}}{\pi^2-1} - \frac{e^{-\pi^2 t}}{\pi^2-1} + e^{-t\pi^2}\right).$$

*Using eigenfunction expansion method, the solution of* (4.20) *is of the form*

$$v_1(x,t) = \sum_{n=1}^{\infty} X_n(x)T_{1_n}(t),$$

*where* $X_n(x) = \sin(n\pi x)$ *is the solution of* $X - \lambda^2 X'' = 0$*, with* $X(0) = X(1) = 0$*, for* $n = 1, 2, \ldots$*. The function* $T_{1_1}$ *corresponds to the solution of*

$$\begin{cases} T'_{1_1} + \pi^2 T_{1_1} = -0.5\pi^2\left(\frac{e^{-t}}{\pi^2-1} - \frac{e^{-\pi^2 t}}{\pi^2-1} + e^{-t\pi^2}\right), \\ T_{1_1}(0) = 0, \end{cases}$$

*which is given by*

$$T_{1_1}(t) = -0.5\pi^2\left(\frac{\pi^2-2}{\pi^2-1}\right)te^{-\pi^2 t} - \frac{0.5\pi^2}{(\pi^2-1)^2}e^{-t} + \frac{0.5\pi^2}{(\pi^2-1)^2}e^{-\pi^2 t}.$$

*The function*

$$T_{1_2}(t) = \frac{0.5e^{-t}}{\pi^2-1} - \frac{0.5e^{-\pi^2 t}}{\pi^2-1} + 2e^{-\pi^2 t},$$

*is the solution of the following PDE:*

$$T'_{1_2} + \pi^2 T_{1_2} = 0.5e^{-t}, \quad \text{with } T_{1_2}(0) = 2.$$

*For all* $n \geq 3$*, we have* $T_{1_n}(t) = 0$ *since it is the solution of* $T_{1_n} + \pi^2 T'_{1_n} = 0$ *with the initial condition* $T_{1_n}(0) = 0$*. Thus, we obtain that the solution of* (4.20) *is*

$$v_1(x,t) = X_1(x)T_{1_1}(t) + X_2(x)T_{1_2}(t).$$

*Hence, the solution of PDE* (4.20) *is*

$$\begin{aligned} v_1(x,t) = \sin(\pi x)&\left(-0.5\pi^2\left(\frac{\pi^2-2}{\pi^2-1}\right)te^{-\pi^2 t} - \frac{0.5\pi^2}{(\pi^2-1)^2}e^{-t} + \frac{0.5\pi^2}{(\pi^2-1)^2}e^{-\pi^2 t}\right) \\ &+ \sin(2\pi x)\left(\frac{0.5}{\pi^2-1}e^{-t} - \frac{0.5}{\pi^2-1}e^{-\pi^2 t} + 2e^{-\pi^2 t}\right). \end{aligned}$$

*Now, we are ready to obtain* $v_2$*. To this end, we consider the PDE:*

$$\begin{cases} \frac{\partial v_2}{\partial t} = \frac{\partial^2 v_2}{\partial x^2} + 0.25\frac{\partial^2 v_0}{\partial x^2} + 0.5\frac{\partial^2 v_1}{\partial x^2}, \\ v_2(0,t) = v_2(1,t) = 0, \\ v_2(x,0) = 0. \end{cases} \tag{4.21}$$

*Again, the eigenfunction are the functions $X_n(x) = \sin(n\pi x)$ which correspond to the solutions of $X - \lambda^2 X'' = 0$ with $X(0) = X(1) = 0$. Note that $T_{2_n}(t) = 0 \quad \forall n \geq 3$, since $T_{2_n}$ is the solution of $T'_{2_n} + \pi^2 T_{2_n} = 0$ with $T_{2n}(0) = 0$. The function*

$$T_{2_1}(t) = \frac{-\pi^2 0.25}{(\pi^2-1)^2}e^{-t} + \frac{0.25\pi^2 e^{-\pi^2 t}}{(\pi^2-1)^2} - \pi^2 0.25\left(\frac{\pi^2-2}{\pi^2-1}\right)te^{-\pi^2 t} + \frac{0.5\pi^4}{4}\left(\frac{\pi^2-2}{\pi^2-1}\right)t^2 e^{-\pi^2 t} - \frac{0.5\pi^4}{2(\pi^2-1)^2}te^{-\pi^2 t} + \frac{0.5\pi^4}{2(\pi^2-1)^3}e^{-t} - \frac{0.5\pi^4}{2(\pi^2-1)^3}e^{-\pi^2 t},$$

*is the solution of*

$$\begin{cases} T'_{2_1} + \pi^2 T_{2_1} = -\pi^2 0.25\left(\frac{e^{-t}}{\pi^2-1} + \left(\frac{\pi^2-2}{\pi^2-1}\right)e^{-\pi^2 t}\right) - \pi^2 0.5\left(-\frac{\pi^2}{2}\left(\frac{\pi^2-2}{\pi^2-1}\right)te^{-\pi^2 t} - \frac{\pi^2}{2(\pi^2-1)^2}e^{-t} + \frac{\pi^2}{2(\pi^2-1)^2}e^{-\pi^2 t}\right), \\ T_{2_1}(0) = 0. \end{cases}$$

*Moreover, the solution of*

$$\begin{cases} T'_{2_2} + \pi^2 T_{2_2} = -4\pi^2 0.5\left(\frac{e^{t}}{2(\pi^2-1)} - \frac{e^{-\pi^2 t}}{2(\pi^2-1)} + 2e^{-\pi^2 t}\right), \\ T_{2_2}(0) = 0, \end{cases}$$

*is given by*

$$T_{2_2}(t) = \frac{-\pi^2}{(\pi^2-1)^2}e^{-t} + \frac{\pi^2}{(\pi^2-1)^2}e^{-\pi^2 t} + \pi^2\left(\frac{-4\pi^2+5}{\pi^2-1}\right)te^{-\pi^2 t}.$$

*Thus, the solution of PDE (4.21) is*

$$\begin{aligned} v_2(x,t) = \sin(\pi x)&\left(\frac{-\pi^2 e^{-t}}{4(\pi^2-1)^2} + \frac{\pi^2 e^{-\pi^2 t}}{4(\pi^2-1)^2} - \frac{\pi^2}{4}\left(\frac{\pi^2-2}{\pi^2-1}\right)te^{-\pi^2 t}\right. \\ &\left.+\frac{\pi^4}{8}\left(\frac{\pi^2-2}{\pi^2-1}\right)t^2 e^{-\pi^2 t} - \frac{\pi^4}{4(\pi^2-1)^2}te^{-\pi^2 t} + \frac{\pi^4}{4(\pi^2-1)^3}e^{-t} - \frac{\pi^4 e^{-\pi^2 t}}{4(\pi^2-1)^3}\right) \\ &+\sin(2\pi x)\left(\frac{-\pi^2 e^{-t}}{(\pi^2-1)^2} + \frac{\pi^2}{(\pi^2-1)^2}e^{-\pi^2 t} + \pi^2\left(\frac{-4\pi^2+5}{\pi^2-1}\right)te^{-\pi^2 t}\right). \end{aligned}$$

*Therefore, the solution of FPDE (4.18) is*

$$v(x,t) = v_0(x,t) + v_1(x,t)A + v_2(x,t)A^2$$

*with $v_0$, $v_1$, and $v_2$ given by Equations (4.19), (4.20), and (4.21), for all $(x,t) \in [0,1]\times[0,\infty)$, respectively. Figure 3 illustrates the fuzzy solution $v$ for the FPDE (4.18) in $[0,2]\times[0,0.5]$, with the triangular fuzzy number $A = (-0.09; 0; 0.1)$. Figure 4 displays the solution $v$ at the time instances $t = 0.05,\ 0.15,\ 0.2,\ 0.3$.*

## 5. Conclusions

In this study, we have proposed the concept of the distributive product and established its properties for $S$-linearly correlated fuzzy processes. Within this framework, we have investigated second-order fuzzy linear partial differential equations under the concept of $\psi$-partial differentiability, with fuzzy coefficients for $S$-linearly correlated fuzzy processes. The proposed product has provided a consistent way to transform FPDEs into classical PDEs, offering a clear analytical link between fuzzy and classical formulations. This approach differs from existing fuzzy theories by introducing a mathematically rigorous mechanism to handle interactions between correlated fuzzy quantities. Specifically, we have examined fuzzy transport and heat equations and shown that the obtained fuzzy solutions effectively describe transport and heat diffusion phenomena under uncertainty, demonstrating the applicability and significance of the proposed framework.

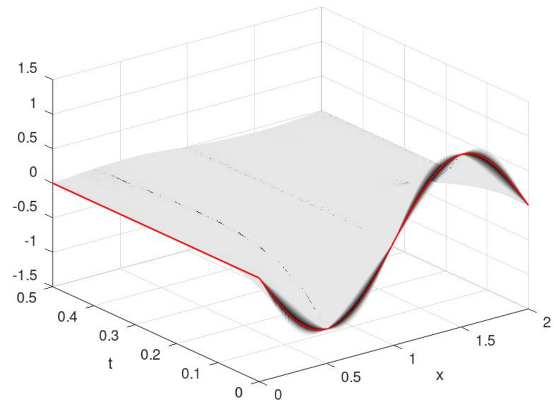


FIGURE 3. The 3D view of the fuzzy solution $v$ of Equation (4.18) for all $(x, t) \in [0, 2] \times [0, 0.5]$, with the triangular fuzzy number $A = (-0.09; 0; 0.1)$. The gray-scale surfaces correspond to the endpoints of the $\alpha$-levels of the solution $v$, where the higher $\alpha$, the darker the surface. The red line depicts the 1-level set of the fuzzy solution $v$.

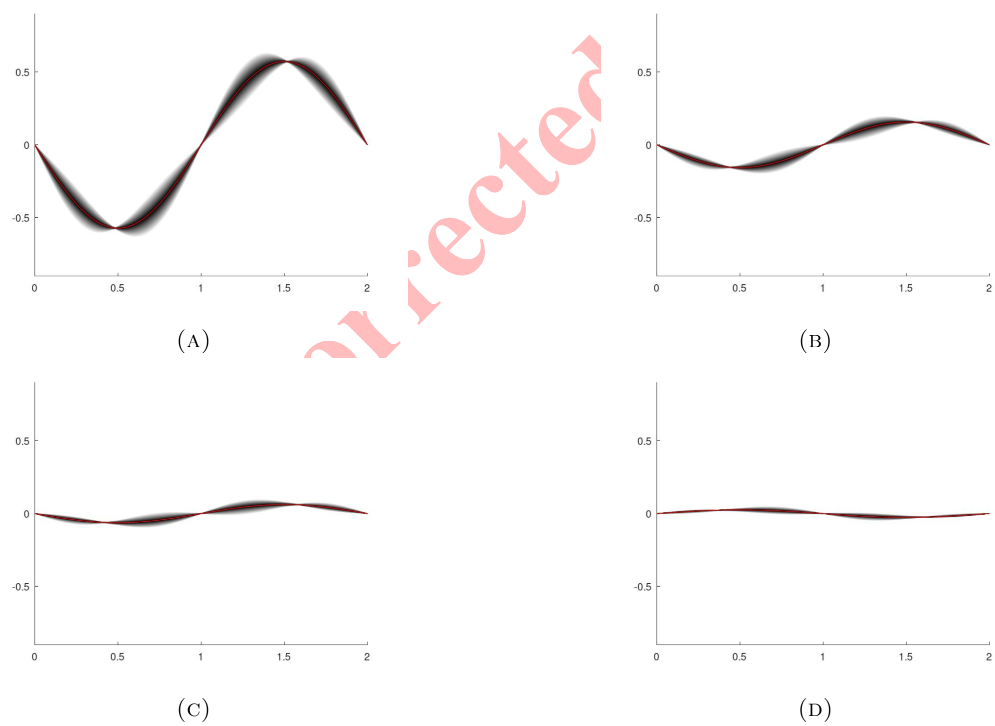


FIGURE 4. Let $A = (-0.09; 0; 0.1)$. Each graph from (A) to (D) presents a top view of the fuzzy solution $v(x, t_i)$ from Example 4.2 for $x \in [0, 2]$ at the time instances $t_i = 0.05,\ 0.15,\ 0.2,\ 0.3$, respectively. The gray-scale curves correspond to the endpoints of the $\alpha$-levels of the solution $v$, where the higher $\alpha$, the darker the surface. The red curves depict the 1-level sets of the fuzzy solution $v(\cdot, t_i)$ at each fixed time $t_i$.

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