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Optical soliton solutions of Gilson pickering equation and modulation instability in optical fiber

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Abstract

In this paper, we use the powerful and strong methods to get the solution of Gilson pickering equation known as (H+$\frac{G'(\eta)}{G(\eta)}$) expansion method and ($\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$) expansion method. The performance of these methods is useful and provides us with a lot of new general exact solutions for solving (NPDES). These method is used to solve many problems that occur in physics, fluid physics and optical fiber. Types of solutions are discussed singular, bright and rational. Modulation instability in higher order nonlinear partial differential equations is investigated. Modulation instability is a phenomenon observed in certain types of nonlinear systems, such as in optical fibers or plasma waves. By using linear technique, we establish the modulation instability and show the influence of higher order nonlinear components on modulation instability. Finally, we introduce figures in 2D and 3D. These graphs are very important and useful for describing the behavior of solutions.

Keywords. Soliton solutions, optical fiber, Gilson pickering equation, Nonlinear partial differential equation, (H+$\frac{G'(\eta)}{G(\eta)}$) expansion method, $\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$ expansion method Modulation instability.
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1. Introduction

It is known that the partial differential equations (PDEs) is used to discuss the complicated processes in various fields such as physics, quantum field theory, geochemistry, fluid physics, wave propagation, engineering, optical fibers, electricity, quantum mechanics and so on. Many scientists focused on the new results for NPDES, such as traveling wave equations, trigonometric, complex functions, and so on. Examples include Schrödinger equation [13, 16], Kudryashov-Sinelshchikov equation [24], Biswas-Milovic equation [31], Lakshmanan-Porsezian-Daniel model [7, 47], Chen-Lee-Liu equation[30, 34], Radhakrishnan-Kundu-Lakshmanan equation[35], Sasa-Satsuma equation[20], fractional Biswas-Arshed Model[11, 17, 38], Navier-Stokes equation [19], Korteweg-De Vries equation [10], Burgers' equation [48], Hirota equations[26]. Several methods are considered powerful tools to find new solutions for models of NPDES , such as *tanh* method [4], Sin-Gordon expansion method [14], kudrayashov method [15, 36], new kudryashov method [37], the first integral method [18], improved sardar -sub equation method [5], Sinh-Gordon expansion method [32], p^6 -model expation method [6], modified mapping method [43], generalized $tanh - coth$ method [21], generalized $tanh - coth$ method generalized $\frac{G'(\eta)}{G(\eta)}$) [29], Hirota bilinear method [2], improve $tanh(\phi(\xi)/2)$ [27].

In this work, we solve Gilson pickering equation [25, 28] by using (H+$\frac{G'(\eta)}{G(\eta)}$) expansion method, ($\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$) expansion.

$$2kq_x + q_t - q\sigma q_x - \beta q_x q_{xx} - \gamma q_{xxt} - qq_{xxx} = 0, \tag{1.1}$$

where the parameters σ, β, γ and k are arbitrary constants. The Gilso-Pickering equation is a nonlinear partial differential equation that arises in the study of dispersive wave phenomena in various physical contexts. It is an

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extension of the classical Korteweg-de Vries type models, incorporating additional nonlinear and dispersive terms to better capture the dynamics of certain nonlinear media. This equation contains cubic nonlinearities and higher-order derivatives, making it suitable for modeling wave propagation in systems where both nonlinear convection and higher-order dispersion are significant.The Gilson Pickering equation exhibits a wide range of nonlinear behaviors, including solitary wave solutions, shock-like structures, and complex interaction dynamics. This makes it a valuable model for both theoretical analysis and practical applications in nonlinear science.

The Gilson pickering equation has been investigated in Painlevé analysis and is giving by traveling wave equation. This equation includes many nonlinear equations, such as Camassa Holm equation, Rosenau-Hyman equation, and Fornberg-Whitham model. Gilson pickering equation is called Fornberg-Whitham equation when $k = \frac{1}{2}$, $\sigma = -1$, $\beta = 3$, $\gamma = 1$. If parameters $\gamma = 0$, $\sigma = 1$,$k = 0$, $\beta = 3$, then, Equation (1.1) is called Rosenau-Hyman equation. Consider the constants are giving by $\gamma = 1$, $\sigma = -3$, $\beta = 2$ then, (1.1) becomes Fornberg-Whitham model.

This equation solved by various methods, such as in 2020 by A. Yokus et.al., studied the solution of Gilson pickering by using $(\frac{G'(\eta)}{G(\eta)}, \frac{1}{G(\eta)})$ expansion method and *Sinh* Jorden method [50]. Additionally, in 2020 H. Rezazadeh et al. solved the Gilson pickering equation (1.1) using the modified kudryashov method [41]. In 2021 A. Rani et al. discussed the solution of Equation (1.1) using *tanh-coth* method [40]. A. Saha el al., in 2019 solved Equation (1.1) by using Collection method [9]. Also H. Rezazadeh et al., studied the bifurcation and solved Equation (1.1) using Jacobi elliptic function method and Exponential rational function method [42] in 2021. By using Bernouli sub-equation method [12] Gilson pickering equation solved by H. M. Baskonus in 2019. I. Samir et al. solved Equation (1.1) in 2022 using modified extended mapping method [45]. S. A. Allahyani et.al., solved this model in 2022 using Sardar's Sub-Equation method [8]. Extended simple equation method and the generalized Riccati equation mapping method used to solve (1.1) by H. U. Rehman et.al., in 2022 [44]. Exp-function method, multi-exp function method and multi hyperbolic tangent method, multi-exp function method and the multi hyperbolic tangent method used by O'Regan et al., to solve (1.1) in 2024 [33].

We also discuss the modulation instability and the influence of the nonlinearity in modulation instability. Modulation instability refers to a phenomenon that occurs in nonlinear optical systems particularly, in optical fibers. It is a process where small perturbations in the intensity of an optical signal can grow rapidly as it propagates through the fiber [1, 3, 22, 23, 39, 46, 49, 51]. Modulation instability interest in different fields such as plasma physics and nonlinear optics, the growth of modulation instability also discusses.

This paper is organized as following: Section 2 the descriptions of the methods known as (H+$\frac{G'(\eta)}{G(\eta)}$) expansion method and $(\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)})$ expansion method are presented. In section 3, application of proposed methods to find the solutions of models. Modulation instability analysis [MI] is also discussed in section 4. The graphical illustration in 2D and 3D to show the behavior of solutions are also provided in section 5. Finally, conclusion of our work is present in last section.

2. Descriptions of the methods

In this section, we show the steps of the methods to understand how to apply these methods on the previous equation. Consider the partial differential equation.

$$D\bigg(q, q_t, q_x, q_{tt}, q_{xx}, ...\bigg) = 0, \tag{2.1}$$

where D represents a polynomial comprising the unknown function $q = q(x, t)$ as well as its different partial derivatives.

By using the transformation

$$\eta = x - \nu t, q(x, t) = u(\eta), \tag{2.2}$$

where ν is arbitrary constant, substituting from (2.2) into (2.1) then, (2.1) becomes (ODE) as following:

$$S\bigg(u', u'', u''', ...\bigg) = 0, \tag{2.3}$$

where S is a polynomial in $u(\eta)$ and its derivatives.

2.1. **$(H+\frac{G'(\eta)}{G(\eta)})$ expansion method.** The basic steps of (H+$\frac{G'(\eta)}{G(\eta)}$) expansion method are showed as following:
step 1: The solution of (2.3) is obtained as:

$$u(\eta)=\sum_{i=-N}^{N} b_i(H+Q(\eta))^i, \tag{2.4}$$

where $Q(\eta)=\frac{G'(\eta)}{G(\eta)}$, $b_i(1=0,\pm1,\pm2,.......\pm N)$ and H constant can be determined later and $G=G(\eta)$ which satisfy nonlinear (ODE)

$$Q'(\eta)=A+BQ(\eta)^2+CQ(\eta), \tag{2.5}$$

where A,B,C are real constants and (2.5) have solutions as following:

When $(A(C-1)\not\equiv 0),(B(C-1)\not\equiv 0),\left(A^2-4BC+4B>0\right),\left(\Delta=A^2-4BC+4B\right)$,

$$Q(\eta)=\left(-\frac{A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}\right), \tag{2.6}$$

$$Q(\eta)=\left(-\frac{A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}\right), \tag{2.7}$$

$$Q(\eta)=\left(-\frac{A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\eta\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right), \tag{2.8}$$

$$Q(\eta)=\left(-\frac{A+\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\eta\right)\pm \operatorname{csch}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right), \tag{2.9}$$

$$Q(\eta)=\left(-\frac{2A+\sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}\eta}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\eta}{4}\right)\right)}{4(C-1)}\right), \tag{2.10}$$

$$Q(\eta)=\left(\frac{\frac{\pm\sqrt{\Delta(F^2+K^2)}-\sqrt{\Delta}F\cosh(\sqrt{\Delta}\eta)}{B+F\sin(\sqrt{\Delta}\eta)}-A}{2(C-1)}\right), \tag{2.11}$$

$$Q(\eta)=\left(\frac{\frac{\pm\sqrt{\Delta(F^2+K^2)}+\sqrt{\Delta}F\cosh(\sqrt{\Delta}\eta)}{B+F\sin(\sqrt{\Delta}\eta)}-A}{2(C-1)}\right), \tag{2.12}$$

where F,K are real constants and $F^2+K^2>0$.

$$Q(\eta)=\left(\frac{2B\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H\right), \tag{2.13}$$

$$Q(\eta)=\left(\frac{2B\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}\right), \tag{2.14}$$

$$Q(\eta)=\left(\frac{2B\cosh\left(\sqrt{\Delta}\eta\right)}{-A\cosh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\eta\right)\pm i\sqrt{\Delta}}\right), \tag{2.15}$$

$$Q(\eta)=\left(\frac{2B\sinh\left(\sqrt{\Delta}\eta\right)}{-A\sinh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\eta\right)\pm \sqrt{\Delta}}\right), \tag{2.16}$$

When $\left(A^2-4BC+4B<0\right),(A(C-1)\not\equiv 0),(B(C-1)\not\equiv 0)$.

$$Q(\eta)=\left(\frac{A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}\right), \tag{2.17}$$

$$Q(\eta)=\left(-\frac{A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}\right), \tag{2.18}$$

$$Q(\eta)=\left(-\frac{A+\sqrt{-\Delta}\left(\tanh\left(\sqrt{-\Delta}\eta\right)\pm \operatorname{sech}\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}\right), \tag{2.19}$$

$$Q(\eta)=\left(-\frac{A+\sqrt{-\Delta}\left(\tanh\left(\sqrt{-\Delta}\eta\right)\pm \operatorname{sech}\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}\right), \tag{2.20}$$

$$Q(\eta)=\left(-\frac{A+\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\eta\right)\pm \operatorname{csch}\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}\right), \tag{2.21}$$

$$Q(\eta)=\left(-\frac{2A+\sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}\eta}{4}\right)+\cot\left(\frac{\sqrt{-\Delta}\eta}{4}\right)\right)}{4(C-1)}\right), \tag{2.22}$$

$$Q(\eta)=\left(\frac{\pm\frac{\sqrt{-\Delta(F^2+K^2)}-\sqrt{-\Delta}F\cos\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{-\Delta}\eta\right)}-A}{2(C-1)}+\right), \tag{2.23}$$

$$Q(\eta)=\left(\frac{\pm\frac{\sqrt{-\Delta(F^2+K^2)}+\sqrt{-\Delta}F\cos\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{-\Delta}\eta\right)}-A}{2(C-1)}\right), \tag{2.24}$$

where F,K are constants such that $F^2-K^2>0$.

$$Q(\eta)=\left(\frac{2B\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}\right), \tag{2.25}$$

$$Q(\eta)=\left(\frac{2B\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}\right), \tag{2.26}$$

$$Q(\eta)=\left(\left(-\frac{2B\cos\left(\sqrt{\Delta}\eta\right)}{-A\cos\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sin\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}}\right)^2, \tag{2.27}$$

$$Q(\eta)=\left(\frac{2B\sin\left(\sqrt{-\Delta}\eta\right)}{-A\sin\left(\sqrt{-\Delta}\eta\right)+\sqrt{-\Delta}\cos\left(\sqrt{\Delta}\eta\right)\pm\sqrt{-\Delta}}\right). \tag{2.28}$$

When $B=0, A(C-1)\not\equiv 0$.

$$Q(\eta)=\left(-\frac{As}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}\right), \tag{2.29}$$

$$Q(\eta)=b\left(H-\frac{A(\sinh(A\eta)+\cosh(A\eta))}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}\right), \tag{2.30}$$

where s is constant.

When $A=B=0, (C-1)\not\equiv 0$.

$$Q(\eta)=\left(H-\frac{1}{(C-1)\eta+s}\right). \tag{2.31}$$

where

$$\eta=x-\nu t. \tag{2.32}$$

step 2: The value of positive integer of N can be determined by balancing Equation (2.3).

step3: Substituting Equations (2.4) and (2.5) into Equation (2.3), we get the polynomials of $(H+Q(\eta))^i$ and $(H+Q(\eta))^{-i}$ where $(i=0,1,2,3,.....)$, then collect all coefficient of the resulted polynomials and equal to zero, we get the set of algebraic equations for $b_i(1=0,\pm1,\pm2,.......\pm N)$ and the value of another constants by solving them by wolfram mathematica then, we get the solution of wave Equations (1.1).

2.2. $\left(\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}\right)$ **expansion method.** The steps of $\left(\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}\right)$ expansion method are obtained as following:
Step 1: The solution (2.3) is getting as:

$$u(\eta)=\sum_{j=0}^{N} g_j w(\eta)^j, \tag{2.33}$$

where $w(\eta)=\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$, g_j, a, b are arbitrary constants and $G=G(\eta)$ is solution of the auxiliary ODE as following:

$$G''(\eta)=-\frac{a\mu}{b^2}-\frac{G\mu}{b^2}-\frac{\lambda G'(\eta)}{b}, \tag{2.34}$$

where μ,λ are arbitrary real number. We can find the following condition

$$w'(\eta)=-\frac{\mu}{b^2}+\frac{(2\mu-\lambda)w(\eta)}{b}+(\lambda-\mu-1)w(\eta)^2, \tag{2.35}$$

where (2.33) have the following solutions:
When $\Delta=\lambda^2-4\mu>0$.
we have $G=-a+v_1\exp\left(\frac{\eta\left(-\sqrt{\Delta}-\lambda\right)}{2b}\right)+v_2\exp\left(\frac{\eta\left(-\lambda+\sqrt{\Delta}\right)}{2b}\right)$ where a, v_1, v_2 are constants that satisfy $a^2+v_1^2+v_2^2\neq 0$, as in case 2, thus

$$\begin{aligned} w_1 &= \frac{v_2\left(\lambda-\sqrt{\Delta}\right)\exp\left(\frac{\sqrt{\Delta}\eta}{2b}\right)+v_1\left(\sqrt{\Delta}+\lambda\right)}{bv_2\left(-\sqrt{\Delta}+\lambda-2\right)\exp\left(\frac{\sqrt{\Delta}\eta}{2b}\right)+bv_1\left(\sqrt{\Delta}+\lambda-2\right)} \\ &= \frac{\left[\lambda\left(v_2-v_1\right)-\sqrt{\Delta}\left(v_1+v_2\right)\right]\sinh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)+\left[\lambda\left(v_1+v_2\right)-\sqrt{\Delta}\left(v_2-v_1\right)\right]\cosh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)}{\sinh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)b\left(\left(\lambda-2\right)\left(v_2-v_1\right)-\sqrt{\Delta}\left(v_1+v_2\right)\right)+\cosh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)b\left(\left(\lambda-2\right)\left(v_1+v_2\right)-\sqrt{\Delta}\left(v_2-v_1\right)\right)}, \end{aligned} \tag{2.36}$$

$$w_1=\begin{cases} w_{1.1}=\frac{\lambda-2\mu}{2b(\lambda-\mu-1)}-\frac{\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)}{2b(\lambda-\mu-1)},(\lambda-2)\left(v_2-v_1\right)-\sqrt{\Delta}\left(v_1+v_2\right)=0,\\ \frac{\lambda-2\mu}{2b(\lambda-\mu-1)}-\frac{\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\eta}{2b}\right)}{2b(\lambda-\mu-1)},(\lambda-2)\left(v_1+v_2\right)-\sqrt{\Delta}\left(v_2-v_1\right)=0.\end{cases} \tag{2.37}$$

when $\lambda^2-\mu<0$.

where $G=\exp\left(-\frac{\eta\lambda}{2b}\right)\left(v_2\sin\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)+v_1\cos\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)\right)-a$

$$w_2=\frac{\left(\sqrt{-\Delta}v_1+\lambda v_2\right)\sin\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)+\left(\lambda v_1-\sqrt{-\Delta}v_2\right)\cos\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)}{b\left(\sqrt{-\Delta}v_1+(\lambda-2)v_2\right)\sin\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)+b\left((\lambda-2)v_1-\sqrt{-\Delta}v_2\right)\cos\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)}, \tag{2.38}$$

$$w_2=\begin{cases} w_{2.1}=\frac{\lambda-2\mu}{2b(\lambda-\mu-1)}-\frac{\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2b}\right)}{2b(\lambda-\mu-1)},(\lambda-2)v_2-\sqrt{-\Delta}v_1=0\\ w_{2.2}=\frac{\lambda-2\mu}{2b(\lambda-\mu-1)}-\frac{\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2b}\right)}{2b(\lambda-\mu-1)},(\lambda-2)v_1-\sqrt{-\Delta}v_2=0.\end{cases} \tag{2.39}$$

Step 2: Substituting Equation (2.33) and (2.35) into Equation (2.3), then collect the coefficient of w_j to zero and solving the system to get the solution of Equation (1.1).

3. Applications of methods

By inserting (2.2) into (1.1)then, Equation (1.1) becomes:

$$u'(\eta)\left(2k-\nu-\beta u''(\eta)-\sigma u(\eta)\right)+u^{(3)}(\eta)(\gamma\nu-u(\eta))=0, \tag{3.1}$$

Integrating Equation (3.1) once with respect to η and put the constant of integration to zero

$$2ku(\eta)+\gamma\nu u''(\eta)-u(\eta)u''(\eta)-\frac{1}{2}\beta u'(\eta)^2+\frac{1}{2}u'(\eta)^2-\nu u(\eta)-\frac{1}{2}\sigma u(\eta)^2. \tag{3.2}$$

By balancing $u''(\eta)$,$u(\eta)^2$ into (3.2) then $N=2$.

3.1. ($H+\frac{G'(\eta)}{G(\eta)}$) expansion method.

Putting $N=2$ in Equation (2.4) then (2.4) becomes:

$$u(\eta)=b_2(H+Q(\eta))^2+b_1(H+Q(\eta))+\frac{b_{-1}}{(H+Q(\eta))}+\frac{b_{-2}}{(H+Q(\eta))^2}+b_0. \tag{3.3}$$

Inserting (2.5) and (3.3) into (3.2) then, collect the coefficients of $(H+Q(\eta))$ where $Q(\eta)=\frac{G'(\eta)}{G(\eta)}$, we get the following system:

$$\begin{aligned}
&\frac{1}{2}B^2b_1^2H^4-\frac{1}{2}B^2\beta b_1^2H^4+2B^2\gamma\nu b_2H^4-12B^2b_{-1}b_2H^4-2B^2b_0b_2H^4-BCb_1^2H^3+BC\beta b_1^2H^3\\
&-2B^2\gamma\nu b_1H^3+2B^2b_0b_1H^3+32BC\gamma\nu b_2H^3+14B^2b_{-1}b_2H^3-8B^2\beta b_{-1}b_2H^3+4BCb_0b_2H^3\\
&+\frac{1}{2}C^2b_1^2H^2+ABb_1^2H^2-\frac{1}{2}C^2\beta b_1^2H^2-AB\beta b_1^2H^2+3BC\gamma\nu b_1H^2-18B^2b_{-1}b_1H^2+6B^2\beta b_{-1}b_1H^2\\
&-3BCb_0b_1H^2+2C^2\gamma\nu b_2H^2+4AB\gamma\nu b_2H^2-72B^2b_{-2}b_2H^2+24B^2\beta b_{-2}b_2H^2-39BCb_{-1}b_2H^2\\
&+12BC\beta b_{-1}b_2H^2-2C^2b_0b_2H^2-4ABb_0b_2H^2-ACb_1^2H+AC\beta b_1^2H-2B^2\gamma\nu b_{-1}H+2B^2b_{-1}b_0H\\
&-C^2\gamma\nu b_1H-2AB\gamma\nu b_1H+26B^2b_{-2}b_1H-8B^2\beta b_{-2}b_1H+18BCb_{-1}b_1H-6BC\beta b_{-1}b_1H+C^2b_0b_1H\\
&-4AC\gamma\nu b_2H+72BCb_{-2}b_2H-24BC\beta b_{-2}b_2H+13C^2b_{-1}b_2H+26ABb_{-1}b_2H-4C^2\beta b_{-1}b_2H\\
&-8AB\beta b_{-1}b_2H+4ACb_0b_2H+\frac{1}{2}B^2b_{-1}^2-\frac{1}{2}B^2\beta b_{-1}^2+\frac{1}{2}A^2b_1^2-\frac{1}{2}A^2\beta b_1^2+2B^2\gamma\nu b_{-2}+BC\gamma\nu b_{-1}\\
&+2kb_0-\nu b_0-2B^2b_{-2}b_0-BCb_{-1}b_0+AC\gamma\nu b_1-13BCb_{-2}b_1+4BC\beta b_{-2}b_1-3C^2b_{-1}b_1-6ABb_{-1}b_1\\
&+C^2\beta b_{-1}b_1+2AB\beta b_{-1}b_1-\sigma b_{-1}b_1-ACb_0b_1+2A^2\gamma\nu b_2-12C^2b_{-2}b_2-24ABb_{-2}b_2+4C^2\beta b_{-2}b_2
\end{aligned}$$

$$
\begin{aligned}
&+8AB\beta b_{-2}b_2-\sigma b_{-2}b_2-13ACb_{-1}b_2+4AC\beta b_{-1}b_2-2A^2b_0b_2-\frac{\sigma b_0^2}{2}=0,\\
&-2A^2\beta b_{-2}^2+4A^2b_{-2}^2-4A\beta b_{-2}^2BH^2-8Ab_{-2}^2BH^2+4A\beta b_{-2}^2CH+8Ab_{-2}^2CH-2\beta b_{-2}^2B^2H^4\\
&-4b_{-2}^2B^2H^4+4\beta b_{-2}^2BCH^3+8b_{-2}^2BCH^3-2\beta b_{-2}^2C^2H^2-4b_{-2}^2C^2H^2=0,\\
&-2A^2\beta b_{-2}b_{-1}-6A^2b_{-2}b_{-1}-4A\beta b_{-2}b_{-1}BH^2-12Ab_{-2}b_{-1}BH^2+8A\beta b_{-2}^2BH+12Ab_{-2}^2BH\\
&-4A\beta b_{-2}^2C+4A\beta b_{-2}b_{-1}CH+12Ab_{-2}b_{-1}CH-6Ab_{-2}^2C-2\beta b_{-2}b_{-1}B^2H^4-6b_{-2}b_{-1}B^2H^4\\
&+8\beta b_{-2}^2B^2H^3+12b_{-2}^2B^2H^3+4\beta b_{-2}b_{-1}BCH^3+12b_{-2}b_{-1}BCH^3-12\beta b_{-2}^2BCH^2\\
&-18b_{-2}^2BCH^2-2\beta b_{-2}b_{-1}C^2H^2-6b_{-2}b_{-1}C^2H^2+4\beta b_{-2}^2C^2H+6b_{-2}^2C^2H=0,\\
&-\frac{1}{2}A^2\beta b_{-1}^2+6A^2b_{-2}\gamma\nu-\frac{3}{2}A^2b_{-1}^2-6A^2b_{-2}b_0-4A\beta b_{-2}^2B-A\beta b_{-1}^2BH^2+12Ab_{-2}B\gamma H^2\nu\\
&-3Ab_{-1}^2BH^2-12Ab_{-2}b_0BH^2+8A\beta b_{-2}b_{-1}BH+18Ab_{-2}b_{-1}BH-4Ab_{-2}^2B-4A\beta b_{-2}b_{-1}C\\
&+A\beta b_{-1}^2CH-12Ab_{-2}\gamma CH\nu+3Ab_{-1}^2CH+12Ab_{-2}b_0CH-9Ab_{-2}b_{-1}C-\frac{1}{2}\beta b_{-1}^2B^2H^4\\
&+6b_{-2}B^2\gamma H^4\nu-\frac{1}{2}3b_{-1}^2B^2H^4-6b_{-2}b_0B^2H^4+8\beta b_{-2}b_{-1}B^2H^3+18b_{-2}b_{-1}B^2H^3\\
&-12\beta b_{-2}^2B^2H^2-12b_{-2}^2B^2H^2+\beta b_{-1}^2BCH^3-12b_{-2}B\gamma CH^3\nu+3b_{-1}^2BCH^3+12b_{-2}b_0BCH^3\\
&-12\beta b_{-2}b_{-1}BCH^2-27b_{-2}b_{-1}BCH^2+12\beta b_{-2}^2BCH+12b_{-2}^2BCH-2\beta b_{-2}^2C^2-\frac{1}{2}\beta b_{-1}^2C^2H^2\\
&+6b_{-2}\gamma C^2H^2\nu-\frac{3}{2}b_{-1}^2C^2H^2-6b_{-2}b_0C^2H^2+4\beta b_{-2}b_{-1}C^2H+9b_{-2}b_{-1}C^2H-2b_{-2}^2C^2-\frac{1}{2}b_{-2}^2\sigma=0,\\
&2A^2\beta b_{-2}b_1+2A^2b_{-1}\gamma\nu-2A^2b_{-1}b_0-8A^2b_{-2}b_1-4A\beta b_{-2}b_{-1}B+4A\beta b_{-2}b_1BH^2+4Ab_{-1}B\gamma H^2\nu\\
&-4Ab_{-1}b_0BH^2-16Ab_{-2}b_1BH^2+2A\beta b_{-1}^2BH-20Ab_{-2}B\gamma H\nu+4Ab_{-1}^2BH+20Ab_{-2}b_0BH\\
&-6Ab_{-2}b_{-1}B-A\beta b_{-1}^2C+10Ab_{-2}\gamma C\nu-32A\beta b_{-2}b_2CH^2-4A\beta b_{-2}b_1CH-4Ab_{-1}\gamma CH\nu\\
&+4Ab_{-1}b_0CH+16Ab_{-2}b_1CH-2Ab_{-1}^2C-10Ab_{-2}b_0C+2\beta b_{-2}b_1B^2H^4+2b_{-1}B^2\gamma H^4\nu\\
&-2b_{-1}b_0B^2H^4-8b_{-2}b_1B^2H^4+2\beta b_{-1}^2B^2H^3-20b_{-2}B^2\gamma H^3\nu+4b_{-1}^2B^2H^3+20b_{-2}b_0B^2H^3\\
&-12\beta b_{-2}b_{-1}B^2H^2-18b_{-2}b_{-1}B^2H^2+8\beta b_{-2}^2B^2H+4b_{-2}^2B^2H-4\beta b_{-2}^2BC+16b_{-2}b_2BCH^4\\
&-4\beta b_{-2}b_1BCH^3-4b_{-1}B\gamma CH^3\nu+4b_{-1}b_0BCH^3+16b_{-2}b_1BCH^3-3\beta b_{-1}^2BCH^2+30b_{-2}B\gamma CH^2\nu\\
&-6b_{-1}^2BCH^2-30b_{-2}b_0BCH^2+12\beta b_{-2}b_{-1}BCH+18b_{-2}b_{-1}BCH-2b_{-2}^2BC-2\beta b_{-2}b_{-1}C^2\\
&+2\beta b_{-2}b_1C^2H^2+2b_{-1}\gamma C^2H^2\nu-2b_{-1}b_0C^2H^2-8b_{-2}b_1C^2H^2+\beta b_{-1}^2C^2H-10b_{-2}\gamma C^2H\nu\\
&+2b_{-1}^2C^2H+10b_{-2}b_0C^2H-3b_{-2}b_{-1}C^2-b_{-2}b_{-1}\sigma=0,\\
&-3B^2b_{-1}b_1H^4+B^2\beta b_{-1}b_1H^4-12B^2b_{-2}b_2H^4+4B^2\beta b_{-2}b_2H^4-6B^2\gamma\nu b_{-1}H^3+6B^2b_{-1}b_0H^3\\
&+30B^2b_{-2}b_1H^3-8B^2\beta b_{-2}b_1H^3+6BCb_{-1}b_1H^3-2BC\beta b_{-1}b_1H^3+16BCb_{-2}b_2H^3-8BC\beta b_{-2}b_2H^3\\
&-3B^2b_{-1}^2H^2-3B^2\beta b_{-1}^2H^2+24B^2\gamma\nu b_{-2}H^2+9BC\gamma\nu b_{-1}H^2-24B^2b_{-2}b_0H^2-9BCb_{-1}b_0H^2\\
&-45BCb_{-2}b_1H^2+12BC\beta b_{-2}b_1H^2-3C^2b_{-1}b_1H^2-6ABb_{-1}b_1H^2+C^2\beta b_{-1}b_1H^2\\
&+2AB\beta b_{-1}b_1H^2-12C^2b_{-2}b_2H^2-24ABb_{-2}b_2H^2+4C^2\beta b_{-2}b_2H^2+8AB\beta b_{-2}b_2H^2\\
&+3BCb_{-1}^2H+3BC\beta b_{-1}^2H-24BC\gamma\nu b_{-2}H-3C^2\gamma\nu b_{-1}H-6AB\gamma\nu b_{-1}H+6B^2b_{-2}b_{-1}H\\
&+8B^2\beta b_{-2}b_{-1}H+24BCb_{-2}b_0H+3C^2b_{-1}b_0H+6ABb_{-1}b_0H+15C^2b_{-2}b_1H+30ABb_{-2}b_1H\\
&-4C^2\beta b_{-2}b_1H-8AB\beta b_{-2}b_1H+6ACb_{-1}b_1H-2AC\beta b_{-1}b_1H+24ACb_{-2}b_2H-8AC\beta b_{-2}b_2H\\
&-2B^2\beta b_{-2}^2-\frac{1}{2}C^2b_{-1}^2-ABb_{-1}^2-\frac{1}{2}C^2\beta b_{-1}^2-AB\beta b_{-1}^2-\frac{1}{2}\sigma b_{-1}^2+2kb_{-2}+4C^2\gamma\nu b_{-2}\\
&+8AB\gamma\nu b_{-2}-\nu b_{-2}+3AC\gamma\nu b_{-1}-3BCb_{-2}b_{-1}-4BC\beta b_{-2}b_{-1}-4C^2b_{-2}b_0-8ABb_{-2}b_0\\
&-\sigma b_{-2}b_0-3ACb_{-1}b_0-15ACb_{-2}b_1+4AC\beta b_{-2}b_1-3A^2b_{-1}b_1+A^2\beta b_{-1}b_1
\end{aligned}
$$

$$
\begin{aligned}
&-12A^2b_{-2}b_2+4A^2\beta b_{-2}b_2=0,\\
&2A^2\beta b_{-1}b_2-6A^2b_{-1}b_2+4A\beta b_{-2}b_1B+2Ab_{-1}B\gamma\nu+4A\beta b_{-1}b_2BH^2-12Ab_{-1}b_2BH^2\\
&-4A\beta b_{-1}b_1BH-16A\beta b_{-2}b_2BH+12Ab_{-1}b_1BH+48Ab_{-2}b_2BH-2Ab_{-1}b_0B-14Ab_{-2}b_1B\\
&+2A\beta b_{-1}b_1C+8A\beta b_{-2}b_2C-4A\beta b_{-1}b_2CH+12Ab_{-1}b_2CH-6Ab_{-1}b_1C-24Ab_{-2}b_2C\\
&-2\beta b_{-2}b_{-1}B^2+2\beta b_{-1}b_2B^2H^4+6b_{-1}b_2B^2H^4-4\beta b_{-1}b_1B^2H^3-16\beta b_{-2}b_2B^2H^3+12b_{-1}b_1B^2H^3\\
&+48b_{-2}b_2B^2H^3+12b_{-1}b_2B^2H^3+12\beta b_{-2}b_1B^2H^2+6b_{-1}B^2\gamma H^2\nu-6b_{-1}b_0B^2H^2\\
&-42b_{-2}b_1B^2H^2+2\beta b_{-1}^2B^2H-12b_{-2}B^2\gamma H\nu+12b_{-2}b_0B^2H\\
&-\beta b_{-1}^2BC+6b_{-2}B\gamma C\nu-4\beta b_{-1}b_2BCH^3+12b_{-1}b_2BCH^3+6\beta b_{-1}b_1BCH^2+24\beta b_{-2}b_2BCH^2\\
&-18b_{-1}b_1BCH^2-72b_{-2}b_2BCH^2-12\beta b_{-2}b_1BCH-6b_{-1}B\gamma CH\nu+6b_{-1}b_0BCH+42b_{-2}b_1BCH\\
&-6b_{-2}b_0BC+2\beta b_{-2}b_1C^2+b_{-1}\gamma C^2\nu+2\beta b_{-1}b_2C^2H^2-6b_{-1}b_2C^2H^2-2\beta b_{-1}b_1C^2H-8\beta b_{-2}b_2C^2H\\
&+6b_{-1}b_1C^2H+24b_{-2}b_2C^2H-b_{-1}b_0C^2-7b_{-2}b_1C^2+2b_{-1}k-b_{-1}\nu-b_{-1}b_0\sigma-b_{-2}b_1\sigma=0,\\
&-2A^2\beta b_1b_2+4A\beta b_{-1}b_2B+2Ab_1B\gamma\nu-4A\beta b_1b_2BH^2+2A\beta b_1^2BH-12Ab_2B\gamma H\nu+12Ab_0b_2BH\\
&-2Ab_0b_1B-14Ab_{-1}b_2B-A\beta b_1^2C+6Ab_2\gamma C\nu+4A\beta b_1b_2CH-6Ab_0b_2C+2\beta b_{-2}b_1B^2\\
&-2\beta b_1b_2B^2H^4+2\beta b_1^2B^2H^3-12b_2B^2\gamma H^3\nu+12b_0b_2B^2H^3+12\beta b_{-1}b_2B^2H^2+6b_1B^2\gamma H^2\nu\\
&-6b_0b_1B^2H^2-42b_{-1}b_2B^2H^2-4\beta b_{-1}b_1B^2H-16\beta b_{-2}b_2B^2H+12b_{-1}b_1B^2H+48b_{-2}b_2B^2H\\
&-6b_{-2}b_1B^2+2\beta b_{-1}b_1BC+8\beta b_{-2}b_2BC+4\beta b_1b_2BCH^3-3\beta b_1^2BCH^2+18b_2B\gamma CH^2\nu\\
&-18b_0b_2BCH^2-12\beta b_{-1}b_2BCH-6b_1B\gamma CH\nu+6b_0b_1BCH+42b_{-1}b_2BCH\\
&-6b_{-1}b_1BC-24b_{-2}b_2BC+2\beta b_{-1}b_2C^2+b_1\gamma C^2\nu-2\beta b_1b_2C^2H^2+\beta b_1^2C^2H\\
&-6b_2\gamma C^2H\nu+6b_0b_2C^2H-b_0b_1C^2-7b_{-1}b_2C^2+2b_1k-b_1\nu-b_0b_1\sigma-b_{-1}b_2\sigma=0,\\
&-2\beta b_2^2B^2-4b_2^2B^2=0,\\
&-2\beta b_1b_2B^2+8\beta b_2^2B^2H+12b_2^2B^2H-6b_1b_2B^2-4\beta b_2^2BC-6b_2^2BC=0,\\
&-4A\beta b_1b_2B+8A\beta b_2^2BH+4Ab_2^2BH-6Ab_1b_2B-4A\beta b_2^2C-2Ab_2^2C+2\beta b_{-1}b_2B^2\\
&+2b_1B^2\gamma\nu+8\beta b_2^2B^2H^3+4b_2^2B^2H^3-12\beta b_1b_2B^2H^2-18b_1b_2B^2H^2+2\beta b_1^2B^2H-20b_2B^2\gamma H\nu\\
&+4b_1^2B^2H+20b_0b_2B^2H-2b_0b_1B^2-8b_{-1}b_2B^2-\beta b_1^2BC+10b_2B\gamma C\nu-12\beta b_2^2BCH^2\\
&-6b_2^2BCH^2+12\beta b_1b_2BCH+18b_1b_2BCH-2b_1^2BC-10b_0b_2BC-2\beta b_1b_2C^2+4\beta b_2^2C^2H\\
&+2b_2^2C^2H-3b_1b_2C^2-b_1b_2\sigma=0,\\
&-2A^2\beta b_2^2-A\beta b_1^2B+8Ab_2B\gamma\nu-4A\beta b_2^2BH^2+8A\beta b_1b_2BH+6Ab_1b_2BH-Ab_1^2B\\
&-8Ab_0b_2B-4A\beta b_1b_2C+4A\beta b_2^2CH-3Ab_1b_2C+\beta b_{-1}b_1B^2+4\beta b_{-2}b_2B^2\\
&-2\beta b_2^2B^2H^4+8\beta b_1b_2B^2H^3+6b_1b_2B^2H^3-3\beta b_1^2B^2H^2+24b_2B^2\gamma H^2\nu-3b_1^2B^2H^2\\
&-24b_0b_2B^2H^2-8\beta b_{-1}b_2B^2H-6b_1B^2\gamma H\nu+6b_0b_1B^2H+30b_{-1}b_2B^2H\\
&-3b_{-1}b_1B^2-12b_{-2}b_2B^2+4\beta b_{-1}b_2BC+3b_1B\gamma C\nu+4\beta b_2^2BCH^3-12\beta b_1b_2BCH^2\\
&-9b_1b_2BCH^2+3\beta b_1^2BCH-24b_2B\gamma CH\nu+3b_1^2BCH+24b_0b_2BCH-3b_0b_1BC\\
&-15b_{-1}b_2BC-\frac{1}{2}\beta b_1^2C^2+4b_2\gamma C^2\nu-2\beta b_2^2C^2H^2+4\beta b_1b_2C^2H+3b_1b_2C^2H\\
&-\frac{1}{2}b_1^2C^2-4b_0b_2C^2+2b_2k-b_2\nu-b_0b_2\sigma-\frac{b_1^2\sigma}{2}=0,\\
&-4A\beta b_2^2B-4Ab_2^2B-\frac{1}{2}\beta b_1^2B^2+6b_2B^2\gamma\nu-12\beta b_2^2B^2H^2-12b_2^2B^2H^2+8\beta b_1b_2B^2H\\
&-18b_1b_2B^2H+\frac{1}{2}(-3)b_1^2B^2-6b_0b_2B^2-4\beta b_1b_2BC+12\beta b_2^2BCH+12b_2^2BCH
\end{aligned}
$$

$$-9b_1b_2BC-2\beta b_2^2C^2-2b_2^2C^2-\frac{b_2^2\sigma}{2}=0.$$

Solving the above system of equations, then we have the following cases:

Case 1.

$$\begin{aligned}&b_1=0,\sigma=\frac{12(2k-\nu)(A+H(BH-c))^2}{b_{-2}\left(4AB-c^2\right)}-4AB+c^2,\gamma=\frac{2k-\nu}{4AB\nu-c^2\nu},\beta=-2,b_2=0,\\&b_0=\frac{b_{-2}B}{A+H(BH-c)},b_{-1}=\frac{b_{-2}(c-2BH)}{A+H(BH-c)}.\end{aligned}\tag{3.4}$$

Case 2.

$$\begin{aligned}&b_1=0,\sigma=-\frac{12(2k-\nu)(A+H(BH-C))^2}{b_{-2}\left(4AB-C^2\right)}-4AB+C^2,\\&\gamma=-\frac{3(2k-\nu)\left(b_{-2}\left(C^2-4AB\right)^2+4(2k-\nu)(A+H(BH-C))^2\right)}{\nu\left(4AB-C^2\right)\left(b_{-2}\left(C^2-4AB\right)^2+12(2k-\nu)(A+H(BH-C))^2\right)},\beta=-2,b_2=0,\\&b_0=\frac{b_{-2}\left(b_{-2}B\left(C^2-4AB\right)^2+2(2k-\nu)(A+H(BH-C))\left(2B(A+3H(BH-C))+C^2\right)\right)}{(A+H(BH-C))\left(b_{-2}\left(C^2-4AB\right)^2+12(2k-\nu)(A+H(BH-C))^2\right)},\\&b_{-1}=\frac{b_{-2}(C-2BH)}{A+H(BH-C)}.\end{aligned}\tag{3.5}$$

Case 3.

$$\begin{aligned}&b_1=0,b_0=\frac{b_{-1}B}{C-2BH},\gamma=\frac{b_{-1}\left(4AB-C^2+\sigma\right)}{12\nu(C-2BH)(A+H(BH-C))},b_2=0,\beta=-2,\\&b_{-2}=\frac{b_{-1}(A+H(BH-C))}{C-2BH},k=\frac{b_{-1}\left(4AB-C^2\right)\left(4AB-C^2+\sigma\right)}{24(C-2BH)(A+H(BH-C))}+\frac{\nu}{2}.\end{aligned}\tag{3.6}$$

Case 4.

$$\begin{aligned}&b_1=0,\sigma=3\left(C^2-4AB\right),\gamma=\frac{b_{-2}\left(C^2-4AB\right)}{b_{-2}\left(C^2-4AB\right)^2+12k(A+H(BH-C))^2},\beta=-2,b_2=0,\\&b_0=\frac{b_{-2}B}{A+H(BH-C)},b_{-1}=\frac{b_{-2}(C-2BH)}{A+H(BH-C)},\nu=\frac{b_{-2}\left(C^2-4AB\right)^2}{6(A+H(BH-C))^2}+2k.\end{aligned}\tag{3.7}$$

Case 5.

$$\begin{aligned}&b_1=0,\sigma=2C^2-8AB,\gamma=\frac{b_{-2}\left(C^2-4AB\right)}{b_{-2}\left(C^2-4AB\right)^2+24k(A+H(BH-C))^2},\beta=-2,b_2=0,\\&b_0=\frac{b_{-2}B}{A+H(BH-C)},b_{-1}=\frac{b_{-2}(C-2BH)}{A+H(BH-C)},\nu=\frac{b_{-2}\left(C^2-4AB\right)^2}{12(A+H(BH-C))^2}+2k.\end{aligned}\tag{3.8}$$

From (2.6)–(2.31) and (3.3) into Equation (2.2) then, we have the following solution:
When $(A(C-1)\not\equiv 0),(B(C-1)\not\equiv 0),\left(A^2-4BC+4B>0\right),\left(\Delta=A^2-4BC+4B\right)$,

$$q(x,t)=\frac{b_{-2}}{\left(H-\frac{A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}\right)^2}+b_2\left(H-\frac{A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}\right)^2\tag{3.9}$$

$$+b_1\left(H-\frac{A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}\right)+\frac{b_{-1}}{H-\frac{A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}}+b_0, \tag{3.10}$$

$$q(x,t)=\frac{b_{-2}}{\left(H-\frac{A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}\right)^2}+b_2\left(H-\frac{A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}\right)^2$$
$$+b_1\left(H-\frac{A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}\right)+\frac{b_{-1}}{H-\frac{A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{2(C-1)}}+b_0, \tag{3.11}$$

$$q(x,t)=\frac{b_{-2}}{\left(H-\frac{A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\eta\right)+\pm i\operatorname{sech}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right)^2}$$
$$+b_2\left(H-\frac{A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\eta\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right)^2 \tag{3.12}$$
$$+b_1\left(H-\frac{A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\eta\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right)+\frac{b_{-1}}{H-\frac{A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\eta\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}}+b_0,$$

$$q(x,t)=\frac{b_{-2}}{\left(H-\frac{A+\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\eta\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right)^2}$$
$$+b_2\left(H-\frac{A+\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\eta\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right)^2 \tag{3.13}$$
$$+b_1\left(H-\frac{A+\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\eta\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right)+\frac{b_{-1}}{H-\frac{A+\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\eta\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}}+b_0,$$

$$q(x,t)=\frac{b_{-2}}{\left(H-\frac{2A+\sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}\eta}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\eta}{4}\right)\right)}{4(C-1)}\right)^2}+b_2\left(H-\frac{2A+\sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}\eta}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\eta}{4}\right)\right)}{4(C-1)}\right)^2$$
$$+b_1\left(H-\frac{2A+\sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}\eta}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\eta}{4}\right)\right)}{4(C-1)}\right)+\frac{b_{-1}}{H-\frac{2A+\sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}\eta}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\eta}{4}\right)\right)}{4(C-1)}}+b_0, \tag{3.14}$$

$$q(x,t)=\frac{b_{-2}}{\left(\frac{\frac{\sqrt{\Delta\left(F^2+K^2\right)}-\sqrt{\Delta}F\cosh\left(\sqrt{\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H\right)^2}+b_2\left(\frac{\frac{\pm\sqrt{\Delta\left(F^2+K^2\right)}-\sqrt{\Delta}F\cosh\left(\sqrt{\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H\right)^2$$
$$+b_1\left(\frac{\pm\frac{\sqrt{\Delta\left(F^2+K^2\right)}-\sqrt{\Delta}F\cosh\left(\sqrt{\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H\right)+\frac{b_{-1}}{\frac{\pm\frac{\sqrt{\Delta\left(F^2+K^2\right)}-\sqrt{\Delta}F\cosh\left(\sqrt{\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H}+b_0, \tag{3.15}$$

$$q(x,t)=\frac{b_{-2}}{\left(\frac{\frac{\sqrt{\Delta\left(F^2+K^2\right)}+\sqrt{\Delta}F\cosh\left(\sqrt{\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H\right)^2}+b_2\left(\frac{\pm\frac{\sqrt{\Delta\left(F^2+K^2\right)}+\sqrt{\Delta}F\cosh\left(\sqrt{\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H\right)^2$$
$$+b_1\left(\frac{\pm\frac{\sqrt{\Delta\left(F^2+K^2\right)}+\sqrt{\Delta}F\cosh\left(\sqrt{\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H\right)+\frac{b_{-1}}{\frac{\pm\frac{\sqrt{\Delta\left(F^2+K^2\right)}+\sqrt{\Delta}F\cosh\left(\sqrt{\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H}+b_0, \tag{3.16}$$

where F,K are real constants.

$$q(x,t)=\frac{b_{-2}}{\left(\frac{2B\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H\right)^2}+b_2\left(\frac{2B\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H\right)^2$$
$$+b_1\left(\frac{2B\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H\right)+\frac{b_{-1}}{\frac{2B\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H}+b_0, \tag{3.17}$$

$$q(x,t)=\frac{b_{-2}}{\left(\frac{2B\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H\right)^2}+b_2\left(\frac{2B\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H\right)^2$$
$$+b_1\left(\frac{2B\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H\right)+\frac{b_{-1}}{\frac{2B\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}{\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\eta}{2}\right)-A\sinh\left(\frac{\sqrt{\Delta}\eta}{2}\right)}+H}+b_0, \tag{3.18}$$

$$q(x,t)=\frac{b_{-2}}{\left(H+\frac{2B\cosh\left(\sqrt{\Delta}\eta\right)}{-A\cosh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\eta\right)\pm i\sqrt{\Delta}}\right)^2}\frac{b_{-1}}{H+\frac{2B\cosh\left(\sqrt{\Delta}\eta\right)}{-A\cosh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\eta\right)\pm i\sqrt{\Delta}}}$$
$$+b_0+b_1\left(H+\frac{2B\cosh\left(\sqrt{\Delta}\eta\right)}{-A\cosh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\eta\right)\pm i\sqrt{\Delta}}\right) \tag{3.19}$$
$$+b_2\Bigg(\Bigg(H+\frac{2B\cosh\left(\sqrt{\Delta}\eta\right)}{-A\cosh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\eta\right)\pm i\sqrt{\Delta}}\Bigg)^2, \tag{3.20}$$

$$q(x,t)=b_2\left(\frac{2B\sinh\left(\sqrt{\Delta}\eta\right)}{-A\sinh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}}+H\right)^2 \tag{3.21}$$
$$+b_1\left(\frac{2B\sinh\left(\sqrt{\Delta}\eta\right)}{-A\sinh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}}+H\right)$$
$$+\frac{b_{-1}}{\frac{2B\sinh\left(\sqrt{\Delta}\eta\right)}{-A\sinh\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}}+H}+\frac{b_{-2}}{\left(\frac{2B\sinh\left(\sqrt{\Delta}\eta\right)}{-A\sinh\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}}+H\right)^2}+b_0. \tag{3.22}$$

When $\left(A^2-4Bc+4B<0\right),(A(C-1)\not\equiv 0),(B(C-1)\not\equiv 0)$.

$$
\begin{aligned}
q(\mathrm{x},\mathrm{t}) &= b_2\left(\frac{A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}+H\right)^2+b_1\left(\frac{A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}+H\right)\\
&+\frac{b_{-1}}{\frac{A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}+H}+\frac{b_{-2}}{\left(\frac{A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}+H\right)^2}+b_0,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
q(x,t) &= b_1\left(H-\frac{A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}\right)+b_2\left(H-\frac{A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}\right)^2\\
&+\frac{b_{-1}}{H-\frac{A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}}+b_0+\frac{b_{-2}}{\left(H-\frac{A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{2(C-1)}\right)^2},
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
q(x,t) &= \frac{b_{-2}}{\left(H-\frac{A+\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\eta\right)+\sec\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}\right)^2}+b_2\left(H-\frac{A+\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\eta\right)\pm\sec\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}\right)^2\\
&+b_1\left(H-\frac{A+\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\eta\right)\pm\sec\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}\right)+\frac{b_{-1}}{H-\frac{A+\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\eta\right)\pm\sec\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}}+b_0,
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
q(x,t) &= \frac{b_{-2}}{\left(H-\frac{A+\sqrt{\Delta}\left(\cot\left(\sqrt{-\Delta}\eta\right)\pm\operatorname{csch}\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}\right)^2}+b_2\left(H-\frac{A+\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\eta\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\eta\right)\right)}{2(C-1)}\right)^2\\
&+b_1\left(H-\frac{A+\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\eta\right)\pm\operatorname{csch}\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}\right)+\frac{b_{-1}}{H-\frac{A+\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\eta\right)\pm\operatorname{csch}\left(\sqrt{-\Delta}\eta\right)\right)}{2(C-1)}}+b_0,
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
q(x,t) &= \frac{b_{-2}}{\left(H-\frac{2A+\sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}\eta}{4}\right)+\cot\left(\frac{\sqrt{-\Delta}\eta}{4}\right)\right)}{4(C-1)}\right)^2}+b_2\left(H-\frac{2A+\sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}\eta}{4}\right)+\cot\left(\frac{\sqrt{-\Delta}\eta}{4}\right)\right)}{4(C-1)}\right)^2\\
&+b_1\left(H-\frac{2A+\sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}\eta}{4}\right)+\cot\left(\frac{\sqrt{-\Delta}\eta}{4}\right)\right)}{4(C-1)}\right)+\frac{b_{-1}}{H-\frac{2A+\sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}\eta}{4}\right)+\cot\left(\frac{\sqrt{-\Delta}\eta}{4}\right)\right)}{4(C-1)}}+b_0,
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
q(x,t) &= \frac{b_{-2}}{\left(\frac{\frac{\sqrt{-\Delta\left(F^2+K^2\right)}-\sqrt{-\Delta}F\cos\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{-\Delta}\eta\right)}-A}{2(C-1)}+H\right)^2}+b_2\left(\frac{\pm\frac{\sqrt{-\Delta\left(F^2+K^2\right)}-\sqrt{-\Delta}F\cosh\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H\right)^2\\
&+b_1\left(\frac{\pm\frac{\sqrt{-\Delta\left(F^2+K^2\right)}-\sqrt{-\Delta}F\cos\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{-\Delta}\eta\right)}-A}{2(C-1)}+H\right)+\frac{b_{-1}}{\frac{\pm\frac{\sqrt{-\Delta\left(F^2+K^2\right)}-\sqrt{-\Delta}F\cosh\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{-\Delta}\eta\right)}-A}{2(C-1)}+H}+b_0,
\end{aligned}
\tag{3.28}
$$

$$q(x,t)=\frac{b_{-2}}{\left(\frac{\frac{\sqrt{-\Delta\left(F^2+K^2\right)}+\sqrt{-\Delta}F\cos\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{-\Delta}\eta\right)}-A}{2(C-1)}+H\right)^2}+b_2\left(\frac{\frac{\pm\sqrt{-\Delta\left(F^2+K^2\right)}+\sqrt{-\Delta}F\cos\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{\Delta}\eta\right)}-A}{2(C-1)}+H\right)^2$$
$$+b_1\left(\frac{\pm\frac{\sqrt{-\Delta\left(F^2+K^2\right)}+\sqrt{-\Delta}F\cos\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{-\Delta}\eta\right)}-A}{2(C-1)}+H\right)+\frac{b_{-1}}{\frac{\pm\frac{\sqrt{-\Delta\left(F^2+K^2\right)}+\sqrt{-\Delta}F\cos\left(\sqrt{-\Delta}\eta\right)}{B+F\sin\left(\sqrt{-\Delta}\eta\right)}-A}{2(C-1)}+H}+b_0, \tag{3.29}$$

where F, K are real constant where $F^2-K^2>0$.

$$q(x,t)=\frac{b_{-2}}{\left(\frac{2B\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}+H\right)^2}+b_2\left(\frac{2B\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}+H\right)^2$$
$$+b_1\left(\frac{2B\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\sinh\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}+H\right)+\frac{b_{-1}}{\frac{2B\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{\Delta}\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}+H}+b_0, \tag{3.30}$$

$$q(x,t)=\frac{b_{-2}}{\left(\frac{2B\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}+H\right)^2}+b_2\left(\frac{2B\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}+H\right)^2$$
$$+b_1\left(\frac{2B\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}+H\right)+\frac{b_{-1}}{\frac{2B\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}{\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\eta}{2}\right)-A\sin\left(\frac{\sqrt{-\Delta}\eta}{2}\right)}+H}+b_0, \tag{3.31}$$

$$q(x,t)=\frac{b_{-2}}{\left(H-\frac{2B\cos\left(\sqrt{-\Delta}\eta\right)}{-A\cos\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sin\left(\sqrt{-\Delta}\eta\right)\pm\sqrt{-\Delta}}\right)^2}+\frac{b_{-1}}{H-\frac{2B\cos\left(\sqrt{-\Delta}\eta\right)}{-A\cos\left(\sqrt{\Delta}\eta\right)+\sqrt{-\Delta}\sin\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}}}+b_0$$
$$+b_1\left(H-\frac{2B\cos\left(\sqrt{\Delta}\eta\right)}{-A\cos\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sin\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}}\right) \tag{3.32}$$
$$+b_2\left(\left(H-\frac{2B\cos\left(\sqrt{\Delta}\eta\right)}{-A\cos\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}\sin\left(\sqrt{\Delta}\eta\right)\pm\sqrt{\Delta}}\right)^2,\right.$$

$$q(x,t)=b_2\left(\frac{2B\sin\left(\sqrt{-\Delta}\eta\right)}{-A\sin\left(\sqrt{-\Delta}\eta\right)+\sqrt{-\Delta}\cos\left(\sqrt{\Delta}\eta\right)\pm\sqrt{-\Delta}}+H\right)^2$$
$$+b_1\left(\frac{2B\sin\left(\sqrt{-\Delta}\eta\right)}{-A\sin\left(\sqrt{-\Delta}\eta\right)+\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\eta\right)\pm\sqrt{-\Delta}}+H\right) \tag{3.33}$$
$$+\frac{b_{-1}}{\frac{2B\sin\left(\sqrt{-\Delta}\eta\right)}{-A\sin\left(\sqrt{-\Delta}\eta\right)\pm\sqrt{-\Delta}\cos\left(\sqrt{\Delta}\eta\right)+\sqrt{\Delta}}+H}+\frac{b_{-2}}{\left(\frac{2B\sin\left(\sqrt{-\Delta}\eta\right)}{-A\sin\left(\sqrt{-\Delta}\eta\right)\pm\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\eta\right)\pm\sqrt{-\Delta}}+H\right)^2}+b_0.$$

When $B=0, A(C-1)\not\equiv 0$.

$$q(x,t)=\frac{b_{-2}}{\left(H-\frac{As}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}\right)^2}+\frac{b_{-1}}{H-\frac{As}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}}+b_0$$

$$+b_1\left(H-\frac{As}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}\right)+b_2\left(H-\frac{As}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}\right)^2, \quad (3.34)$$

$$q(x,t)=b_2\left(H-\frac{A(\sinh(A\eta)+\cosh(A\eta))}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}\right)^2+b_1\left(H-\frac{A(\sinh(A\eta)+\cosh(A\eta))}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}\right)$$
$$+\frac{b_{-1}}{H-\frac{A(\sinh(A\eta)+\cosh(A\eta))}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}}+\frac{b_{-2}}{\left(H-\frac{A(\sinh(A\eta)+\cosh(A\eta))}{(C-1)\{\sinh(A\eta)-\cosh(A\eta)+s\}}\right)^2}+b_0. \quad (3.35)$$

When $A=B=0,(C-1)\not\equiv 0$.

$$q(x,t)=b_2\left(H-\frac{1}{(C-1)\eta+s}\right)^2+b_1\left(H-\frac{1}{(C-1)\eta+s}\right)$$
$$+\frac{b_{-1}}{H-\frac{1}{(C-1)\eta+s}}+\frac{b_{-2}}{\left(H-\frac{1}{(C-1)\eta+s}\right)^2}+b_0. \quad (3.36)$$

where

$$\eta=x-\nu t. \quad (3.37)$$

3.2. $\left(\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}\right)$ **expansion method.** Putting $N=2$ in equation 34 we get the following equation:

$$u(\eta)=g_2w(\eta)^2+g_1w(\eta)+g_0. \quad (3.38)$$

Inserting(3.38) and (2.35) into (3.2) then, get the following system

$$-\frac{\beta g_1^2\mu^2}{2b^4}+\frac{2\gamma g_2\mu^2\nu}{b^4}+\frac{g_1^2\mu^2}{2b^4}-\frac{2g_0g_2\mu^2}{b^4}+\frac{\gamma g_1\lambda\mu\nu}{b^3}-\frac{2\gamma g_1\mu^2\nu}{b^3}-\frac{g_0g_1\lambda\mu}{b^3}$$
$$+\frac{2g_0g_1\mu^2}{b^3}+2g_0k-g_0\nu-\frac{g_0^2\sigma}{2}=0,$$
$$-\frac{2\beta g_1g_2\mu^2}{b^4}-\frac{\beta g_1^2\lambda\mu}{b^3}+\frac{2\beta g_1^2\mu^2}{b^3}+\frac{6\gamma g_2\lambda\mu\nu}{b^3}-\frac{12\gamma g_2\mu^2\nu}{b^3}-\frac{6g_0g_2\lambda\mu}{b^3}+\frac{12g_0g_2\mu^2}{b^3}$$
$$+\frac{\gamma g_1\lambda^2\nu}{b^2}-\frac{6\gamma g_1\lambda\mu\nu}{b^2}+\frac{6\gamma g_1\mu^2\nu}{b^2}+\frac{2\gamma g_1\mu\nu}{b^2}-\frac{g_0g_1\lambda^2}{b^2}$$
$$+\frac{6g_0g_1\lambda\mu}{b^2}-\frac{6g_0g_1\mu^2}{b^2}-\frac{2g_0g_1\mu}{b^2}+2g_1k-g_1\nu-g_0g_1\sigma=0,$$
$$-\frac{2\beta g_2^2\mu^2}{b^4}-\frac{4\beta g_1g_2\lambda\mu}{b^3}+\frac{8\beta g_1g_2\mu^2}{b^3}-\frac{3g_1g_2\lambda\mu}{b^3}+\frac{6g_1g_2\mu^2}{b^3}-\frac{\beta g_1^2\lambda^2}{2b^2}$$
$$+\frac{3\beta g_1^2\lambda\mu}{b^2}-\frac{3\beta g_1^2\mu^2}{b^2}-\frac{\beta g_1^2\mu}{b^2}+\frac{4\gamma g_2\lambda^2\nu}{b^2}-\frac{24\gamma g_2\lambda\mu\nu}{b^2}+\frac{24\gamma g_2\mu^2\nu}{b^2}+\frac{8\gamma g_2\mu\nu}{b^2}-\frac{4g_0g_2\lambda^2}{b^2}$$
$$-\frac{g_1^2\lambda^2}{2b^2}+\frac{3g_1^2\lambda\mu}{b^2}+\frac{24g_0g_2\lambda\mu}{b^2}-\frac{3g_1^2\mu^2}{b^2}-\frac{24g_0g_2\mu^2}{b^2}-\frac{g_1^2\mu}{b^2}-\frac{8g_0g_2\mu}{b^2}-\frac{3\gamma g_1\lambda^2\nu}{b}+\frac{9\gamma g_1\lambda\mu\nu}{b}+\frac{3\gamma g_1\lambda\nu}{b}$$
$$-\frac{6\gamma g_1\mu^2\nu}{b}-\frac{6\gamma g_1\mu\nu}{b}+\frac{3g_0g_1\lambda^2}{b}-\frac{9g_0g_1\lambda\mu}{b}-\frac{3g_0g_1\lambda}{b}+\frac{6g_0g_1\mu^2}{b}+\frac{6g_0g_1\mu}{b}$$
$$+2g_2k-g_2\nu-g_0g_2\sigma-\frac{g_1^2\sigma}{2}=0,$$
$$-\frac{4\beta g_2^2\lambda\mu}{b^3}+\frac{8\beta g_2^2\mu^2}{b^3}-\frac{2g_2^2\lambda\mu}{b^3}+\frac{4g_2^2\mu^2}{b^3}-\frac{2\beta g_1g_2\lambda^2}{b^2}+\frac{12\beta g_1g_2\lambda\mu}{b^2}-\frac{12\beta g_1g_2\mu^2}{b^2}-\frac{4\beta g_1g_2\mu}{b^2}$$
$$-\frac{3g_1g_2\lambda^2}{b^2}+\frac{18g_1g_2\lambda\mu}{b^2}-\frac{18g_1g_2\mu^2}{b^2}-\frac{6g_1g_2\mu}{b^2}+\frac{\beta g_1^2\lambda^2}{b}-\frac{3\beta g_1^2\lambda\mu}{b}-\frac{\beta g_1^2\lambda}{b}+\frac{2\beta g_1^2\mu^2}{b}$$
$$+\frac{2\beta g_1^2\mu}{b}-\frac{10\gamma g_2\lambda^2\nu}{b}+\frac{30\gamma g_2\lambda\mu\nu}{b}+\frac{10\gamma g_2\lambda\nu}{b}-\frac{20\gamma g_2\mu^2\nu}{b}-\frac{20\gamma g_2\mu\nu}{b}+\frac{2g_1^2\lambda^2}{b}$$

$$
\begin{aligned}
&+\frac{10g_0g_2\lambda^2}{b}-\frac{6g_1^2\lambda\mu}{b}-\frac{30g_0g_2\lambda\mu}{b}-\frac{2g_1^2\lambda}{b}-\frac{10g_0g_2\lambda}{b}+\frac{4g_1^2\mu^2}{b}+\frac{20g_0g_2\mu^2}{b}+\frac{4g_1^2\mu}{b}\\
&+\frac{20g_0g_2\mu}{b}+2\gamma g_1\lambda^2\nu-4\gamma g_1\lambda\mu\nu-4\gamma g_1\lambda\nu+2\gamma g_1\mu^2\nu+4\gamma g_1\mu\nu+2\gamma g_1\nu-2g_0g_1\lambda^2+4g_0g_1\lambda\mu\\
&+4g_0g_1\lambda-2g_0g_1\mu^2-4g_0g_1\mu-g_1g_2\sigma-2g_0g_1=0,\\
&-\frac{2\beta g_2^2\lambda^2}{b^2}+\frac{12\beta g_2^2\lambda\mu}{b^2}-\frac{12\beta g_2^2\mu^2}{b^2}-\frac{4\beta g_2^2\mu}{b^2}-\frac{2g_2^2\lambda^2}{b^2}+\frac{12g_2^2\lambda\mu}{b^2}-\frac{12g_2^2\mu^2}{b^2}-\frac{4g_2^2\mu}{b^2}\\
&+\frac{4\beta g_1g_2\lambda^2}{b}-\frac{12\beta g_1g_2\lambda\mu}{b}-\frac{4\beta g_1g_2\lambda}{b}+\frac{8\beta g_1g_2\mu^2}{b}+\frac{8\beta g_1g_2\mu}{b}+\frac{9g_1g_2\lambda^2}{b}\\
&-\frac{27g_1g_2\lambda\mu}{b}-\frac{9g_1g_2\lambda}{b}+\frac{18g_1g_2\mu^2}{b}+\frac{18g_1g_2\mu}{b}-\frac{1}{2}\beta g_1^2\lambda^2+\beta g_1^2\lambda\mu+\beta g_1^2\lambda\\
&-\frac{1}{2}\beta g_1^2\mu^2-\beta g_1^2\mu-\frac{\beta g_1^2}{2}+6\gamma g_2\lambda^2\nu-12\gamma g_2\lambda\mu\nu-12\gamma g_2\lambda\nu+6\gamma g_2\mu^2\nu+12\gamma g_2\mu\nu+6\gamma g_2\nu\\
&-\frac{1}{2}3g_1^2\lambda^2-6g_0g_2\lambda^2+3g_1^2\lambda\mu+12g_0g_2\lambda\mu+3g_1^2\lambda+12g_0g_2\lambda-\frac{3}{2}g_1^2\mu^2-6g_0g_2\mu^2\\
&-3g_1^2\mu-12g_0g_2\mu-\frac{g_2^2\sigma}{2}-6g_0g_2-\frac{3g_1^2}{2}=0,\\
&\frac{4\beta g_2^2\lambda^2}{b}-\frac{12\beta g_2^2\lambda\mu}{b}-\frac{4\beta g_2^2\lambda}{b}+\frac{8\beta g_2^2\mu^2}{b}+\frac{8\beta g_2^2\mu}{b}+\frac{6g_2^2\lambda^2}{b}-\frac{18g_2^2\lambda\mu}{b}-\frac{6g_2^2\lambda}{b}+\frac{12g_2^2\mu^2}{b}\\
&+\frac{12g_2^2\mu}{b}-2\beta g_1g_2\lambda^2+4\beta g_1g_2\lambda\mu+4\beta g_1g_2\lambda-2\beta g_1g_2\mu^2-4\beta g_1g_2\mu-2\beta g_1g_2-6g_1g_2\lambda^2+12g_1g_2\lambda\mu\\
&+12g_1g_2\lambda-6g_1g_2\mu^2-12g_1g_2\mu-6g_1g_2=0,\\
&-2\beta g_2^2\lambda^2+4\beta g_2^2\lambda\mu+4\beta g_2^2\lambda-2\beta g_2^2\mu^2-4\beta g_2^2\mu-2\beta g_2^2-4g_2^2\lambda^2+8g_2^2\lambda\mu\\
&+8g_2^2\lambda-4g_2^2\mu^2-8g_2^2\mu-4g_2^2=0.
\end{aligned}
$$

Solve the previous system then, get the next solution:

Set 1:

$$
\begin{aligned}
&\beta=-2, g_2=\frac{bg_1(-\lambda+\mu+1)}{\lambda-2\mu}, \sigma=\frac{-\frac{12b\gamma\nu(\lambda-2\mu)(\lambda-\mu-1)}{g_1}+\lambda^2-4\mu}{b^2},\\
&g_0=\frac{g_1\mu}{b\lambda-2b\mu}, k=\frac{\nu\left(b^2-\gamma\lambda^2+4\gamma\mu\right)}{2b^2}.
\end{aligned}
\tag{3.39}
$$

Set 2:

$$
\begin{aligned}
&\gamma=\frac{g_1\sqrt{\lambda^2-4\mu}}{-2b\lambda\nu+2b\mu\nu+2b\nu}, \beta=-3, g_2=0, \sigma=-\frac{\lambda^2-4\mu}{b^2},\\
&g_0=-\frac{g_1\left(\sqrt{\lambda^2-4\mu}+\lambda-2\mu\right)}{2b(\lambda-\mu-1)}, k=\frac{1}{4}\left(\frac{g_1\left(\lambda^2-4\mu\right)^{3/2}}{b^3(\lambda-\mu-1)}+2\nu\right.
\end{aligned}
\tag{3.40}
$$

Set 3:

$$
\begin{aligned}
&\gamma=-\frac{g_1\sqrt{\lambda^2-4\mu}}{-2b\lambda\nu+2b\mu\nu+2b\nu}, \beta=-3, g_2=0, \sigma=-\frac{\lambda^2-4\mu}{b^2},\\
&g_0=\frac{g_1\left(\sqrt{\lambda^2-4\mu}-\lambda+2\mu\right)}{2b(\lambda-\mu-1)}, k=\frac{\nu}{2}-\frac{g_1\left(\lambda^2-4\mu\right)^{3/2}}{4b^3(\lambda-\mu-1)}.
\end{aligned}
\tag{3.41}
$$

Set 4:

$$\begin{aligned}&\beta=-2, g_2=\frac{bg_1(-\lambda+\mu+1)}{\lambda-2\mu}, \nu=0, \sigma=-\frac{2\left(\lambda^2-4\mu\right)}{b^2},\\&g_0=-\frac{g_1(\lambda-2\mu)}{4b(\lambda-\mu-1)}, k=\frac{g_1\left(\lambda^2-4\mu\right)^2}{8b^3(\lambda-2\mu)(\lambda-\mu-1)}.\end{aligned} \tag{3.42}$$

Set 5:

$$\beta=-2, g_2=\frac{bg_1(-\lambda+\mu+1)}{\lambda-2\mu}, \nu=0, \sigma=\frac{\lambda^2-4\mu}{b^2}, g_0=\frac{g_1\mu}{b\lambda-2b\mu}, k=0. \tag{3.43}$$

$$\begin{aligned}&\gamma=\frac{3\left(4bg_0(\lambda-\mu-1)+g_1(\lambda-2\mu)\right)\left(bg_0(\lambda-2\mu)-g_1\mu\right)}{2b\nu\left(6bg_0(\lambda-\mu-1)(\lambda-2\mu)+g_1\left(\lambda^2-6\lambda\mu+2\mu(3\mu+1)\right)\right)},\\&\beta=-2, g_2=\frac{bg_1(-\lambda+\mu+1)}{\lambda-2\mu}, \sigma=\frac{g_1\left(\lambda^2-4\mu\right)^2}{b^2\left(6bg_0(\lambda-\mu-1)(\lambda-2\mu)+g_1\left(\lambda^2-6\lambda\mu+2\mu(3\mu+1)\right)\right)},\end{aligned} \tag{3.44}$$

$$k=\frac{g_1\left(2b^3\nu(\lambda-2\mu)\left(\lambda^2-6\lambda\mu+6\mu^2+2\mu\right)-g_1\mu\left(\lambda^2-4\mu\right)^2\right)+bg_0(\lambda-2\mu)\left(12b^3\nu(\lambda-\mu-1)(\lambda-2\mu)+g_1\left(\lambda^2-4\mu\right)^2\right)}{4b^3(\lambda-2\mu)\left(6bg_0(\lambda-\mu-1)(\lambda-2\mu)+g_1\left(\lambda^2-6\lambda\mu+2\mu(3\mu+1)\right)\right)}.$$

From (2.36), (2.38), and (3.38) into (2.2), we get the following solution at:
when $\Delta=\lambda^2-4\mu>0$

$$\begin{aligned}q(x,t)=&\,g_2\left(\frac{\left(\lambda\left(v_2-v_1\right)-\sqrt{\Delta}\left(v_1+v_2\right)\right)\sinh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)+\left(\lambda\left(v_1+v_2\right)-\sqrt{\Delta}\left(v_2-v_1\right)\right)\cosh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)}{b\left((\lambda-2)\left(v_2-v_1\right)-\sqrt{\Delta}\left(v_1+v_2\right)\right)\sinh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)+b\left((\lambda-2)\left(v_1+v_2\right)-\sqrt{\Delta}\left(v_2-v_1\right)\right)\cosh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)}\right)^2\\&+\frac{g_1\left(\left(\lambda\left(v_2-v_1\right)-\sqrt{\Delta}\left(v_1+v_2\right)\right)\sinh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)+\left(\lambda\left(v_1+v_2\right)-\sqrt{\Delta}\left(v_2-v_1\right)\right)\cosh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)\right)}{b\left((\lambda-2)\left(v_2-v_1\right)-\sqrt{\Delta}\left(v_1+v_2\right)\right)\sinh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)+b\left((\lambda-2)\left(v_1+v_2\right)-\sqrt{\Delta}\left(v_2-v_1\right)\right)\cosh\left(\frac{\sqrt{\Delta}\eta}{2b}\right)}+g_0.\end{aligned} \tag{3.45}$$

When $\Delta=\lambda^2-4\mu<0$

$$\begin{aligned}q(x,t)=&\,g_2\left(\frac{\left(\sqrt{-\Delta}v_1+\lambda v_2\right)\sin\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)+\left(\lambda v_1-\sqrt{-\Delta}v_2\right)\cos\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)}{b\left(\sqrt{-\Delta}v_1+(\lambda-2)v_2\right)\sin\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)+b\left((\lambda-2)v_1-\sqrt{-\Delta}v_2\right)\cos\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)}\right)^2\\&+g_1\left(\frac{\left(\sqrt{-\Delta}v_1+\lambda v_2\right)\sin\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)+\left(\lambda v_1-\sqrt{-\Delta}v_2\right)\cos\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)}{b\left(\sqrt{-\Delta}v_1+(\lambda-2)v_2\right)\sin\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)+b\left((\lambda-2)v_1-\sqrt{-\Delta}v_2\right)\cos\left(\frac{\sqrt{-\Delta}\eta}{2b}\right)}\right)+g_0.\end{aligned} \tag{3.46}$$

$$\eta=x-\nu t. \tag{3.47}$$

3.3. **The Importance of methods.** ($H+\frac{G'(\eta)}{G(\eta)}$) expansion method and $\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$) expansion method are powerful analytical tools for obtaining exact solutions of nonlinear differential equations, such as the Gilson- pickering equation and other physically significant models in applied mathematics and physics. These methods are valuable because they transform complex nonlinear problems into solvable forms by introducing an auxiliary function $G(\eta)$ that satisfies a simpler differential equation. The flexibility of these approaches enables the construction of a wide range of exact solutions, including soliton-like, periodic, rational, and singular forms, which are essential for understanding nonlinear wave propagation, stability analysis, and the qualitative behavior of physical systems. ($H+\frac{G'(\eta)}{G(\eta)}$) expansion method is advantageous due to its simplicity and reduced computational effort, making it suitable for problems where a compact analytical expression is desired. In contrast, the ($\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$) expansion method offers greater generality and can capture more complex solution structures due to the presence of additional underlying parameters, though at the cost of more involved algebra. These methods are particularly relevant for equations describing nonlinear oscillations, fluid dynamics, optical pulse propagation, and Bose-Einstein condensates. In the case of Gilson pickering

Table 1. Comparison between $(H+\frac{G'(\eta)}{G(\eta)})$-expansion method and $(\frac{G'(\eta)}{a+b\,G'(\eta)+G(\eta)})$-expansion method.

Feature	$(H+\frac{G'(\eta)}{G(\eta)})$**-Expansion**	$(\frac{G'(\eta)}{a+b\,G'(\eta)+G(\eta)})$**-Expansion**
Basic structure	Constant H plus logarithmic derivative $\frac{G'}{G}$	Ratio of derivative G' to a linear combination of G' and G
Role of constants	Single constant H in addition to G-equation parameters	Two constants a and b in the denominator (more adjustable)
Complexity	Simpler form, fewer adjustable parameters	More complex form, more parameters to fit
Solution types	Typically yields hyperbolic, trigonometric, or exponential forms	Can yield those plus rational, singular, and kink-type solutions
Auxiliary equation	$G(\eta)$ usually satisfies a linear ODE	$G(\eta)$ can satisfy linear or Riccati-type ODEs
Advantages	Easy to compute and substitute	More general, can cover a wider set of exact solutions
Disadvantages	Less general, might miss some special solutions	Algebra can become more involved

equation , exact analytical solutions derived via these methods can provide deeper insight into the underlying physical mechanisms and assist in validating numerical simulations. However, the applicability of these methods is not universal. They may become cumbersome for high-dimensional systems or equations lacking a suitable auxiliary equation form. Moreover, if the targeted solutions are highly irregular or chaotic, these methods might not capture the full dynamics, and alternative techniques: such as numerical simulations, perturbation methods, or inverse scattering may be more appropriate.

3.4. **The comparison of methods.** Now, we comprise between two different method $(H+\frac{G'(\eta)}{G(\eta)})$-expansion method and $(\frac{G'(\eta)}{a+b\,G'(\eta)+G(\eta)})$-expansion method.

4. Modulation instability analysis[MI]

MI refers to a phenomenon that occurs in nonlinear optical systems, particularly in optical fibers. It is a process where small perturbations or fluctuations in an optical signal can grow rapidly, leading to the formation of multiple distinct pulses within the signal. This instability arises due to the interplay between the nonlinear effects in the medium and the dispersion of the signal. MI used to display the behavior of continuous waves over a long time of evolution.

We take the following transformation

$$q(x,t)=\big(v(x,t)+\sqrt{\tau}\big)\exp(ti\tau), \tag{4.1}$$

where τ is the optical power of normalized by inserting (4.1) into (1.1) then, we get the linear dispersion as following:

$$2kv_x+v_t-\gamma i\tau v_{\rm xx}+vi\tau-\gamma v_{\rm xxt}=0 \tag{4.2}$$

To get the linear expression, we used the tiny perturbations in the plane -wave

$$v(x,t)=A_1\exp(i(px-\rho\ {\rm t}))+A_2\exp(-i(px-\rho\ {\rm t})), \tag{4.3}$$

where p normalized wave number of perturbations, ρ is frequency of wave equation. Substituting (4.3) into (4.2) then, summing the coefficient of $\exp(i(px-\rho\ {\rm t})$ and $\exp(-i(px-\rho\ {\rm t})$, we get the determinant as following

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\begin{pmatrix} 2ikp+p^2\rho(-{\rm i}\gamma)+p^2\tau{\rm i}\gamma-i\rho+i\tau & 0 \\ 0 & -2ikp+p^2\rho(-{\rm i}\gamma)+p^2{\rm i}\gamma\tau+i\rho+i\tau \end{pmatrix}.$$

By solving the previous determinant, we get the following equation as:

$$4k^2p^2-4\gamma kp^3\rho-4kp\rho+\gamma^2p^4\rho^2-\gamma^2p^4\tau^2+2\gamma p^2\rho^2-2\gamma p^2\tau^2+\rho^2-\tau^2=0. \tag{4.4}$$

Determining the dispersion solution of (4.4) for τ results

$$\tau=\pm\frac{i\left(2kp-\gamma p^2\rho-\rho\right)}{\sqrt{-\gamma^2p^4-2\gamma p^2-1}}. \tag{4.5}$$

If the wave number τ is complex then the steady state solution will not be stable because the perturbation grow gradually, but if τ is real then steady state is stable. A steady state is not stable when

$$\pm\sqrt{-\gamma^2 p^4 - 2\gamma p^2 - 1} > 0, \tag{4.6}$$

then, the growth of modulation instability MI is achieved as:

$$\sigma = 2Im[\rho[p]]. \tag{4.7}$$

5. Graphical illustrating

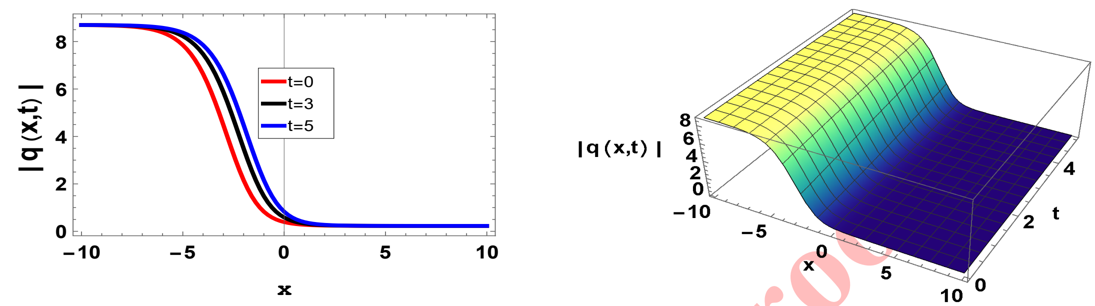

Figure 1. Optical singular soliton solution of case (1) (3.4) with $q(x,t)$ (3.9) using (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method with constants $A = 0.3, b_{-2} = 0.2, B = 0.3, c = 0.2, H = 0.6, k = 0.3$, and $\nu = 0.2$.

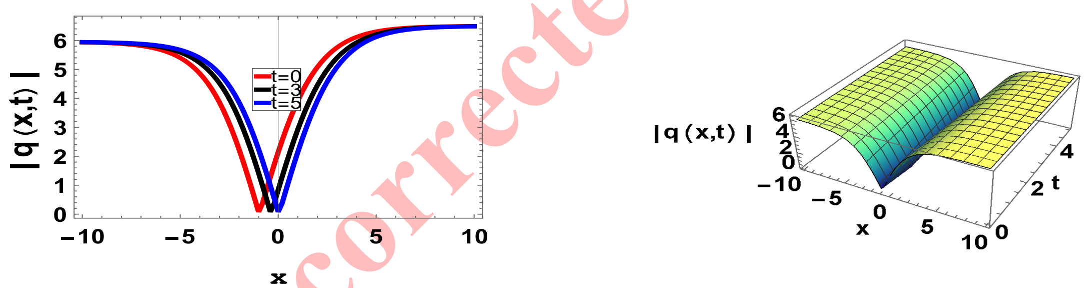

Figure 2. Optical singular soliton solution of case (1) (3.4) with $q(x,t)$ (3.11) using (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method with constants $A = 0.1, b_{-2} = 0.3, B = 0.2, c = 0.4, H = 0.2, k = -0.3$, and $\nu = 0.2$.

In this section, we introduce some figures in 2D and 3D about some solution of Gilson pickering equation by ($H+\frac{G'(\eta)}{G(\eta)}$) expansion method. In Figure 1, using (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method with constants $A = 0.3, (b_{-2} = 0.2,) B = 0.3, c = 0.2, H = 0.6, k = 0.3, \nu = 0.2.$. Figure 2, with constants$A = 0.1, b_{-2} = 0.3, B = 0.2, c = 0.4, H = 0.2, k = -0.3, \nu = 0.2$ we get the Optical soliton solution of case (1) (3.4) with $q(x,t)$ (3.11). In Figure 3 we get the Optical soliton solution of case (1) (3.4) with $q(x,t)$ (3.12) using (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method plots with constants $A = 0.2, b_{-2} = 0.2, B = 0.3, c = 0.2, H = 0.4, k = 0.1, \nu = 0.2.$. In Figure 4 using (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method Case (1) (3.4) with $q(x,t)$ (3.14) with constants $A = 0.2, b_{-2} = 0.2, B = 0.4, c = 0.4, H = 0.3, \nu = 0.4$, and so on in the method of (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method. Sing $\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$ expansion method in Figure 13 set (1) (3.39) with $q(x,t)$ (3.46) with constants $b = 1.06, g_1 = 0.6, \lambda = 0.001, \mu = 0.001, \nu = -0.5, v_1 = 0.5, v_2 = 0.03$. Figure 14 the optical soliton solution of set (2) (3.40) with $q(x,t)$ (3.46) $\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$ expansion method with constants

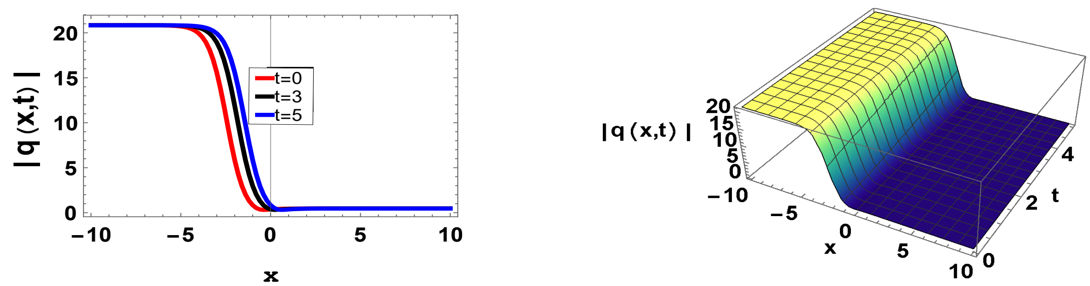

FIGURE 3. Optical singular soliton solution of case (1) (3.4) with $q(x,t)$ (3.12) using $(\text{H}+\frac{G'(\eta)}{G(\eta)})$ expanded method plots with constants $A = 0.2, b_{-2} = 0.2, B = 0.3, C = 0.2, H = 0.4, k = 0.1$, and $\nu = 0.2$..

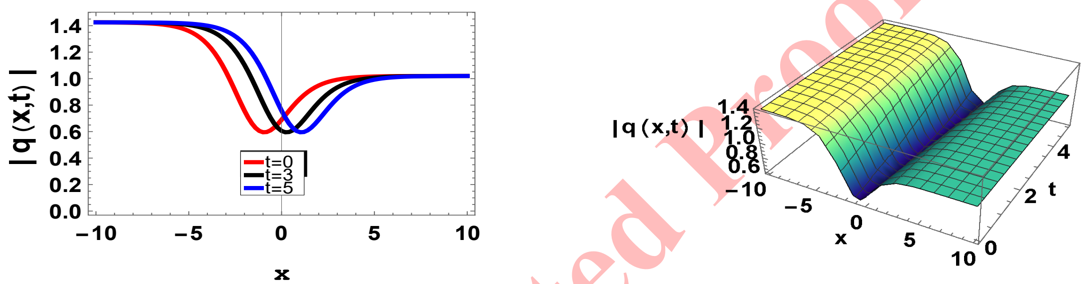

FIGURE 4. Optical singular soliton solution of Case (1) (3.4) with $q(x,t)$ (3.14) using $(\text{H}+\frac{G'(\eta)}{G(\eta)})$ expanded method with constants $A = 0.2, b_{-2} = 0.2;, B = 0.4, C = 0.4, H = 0.3$, and $\nu = 0.4$.

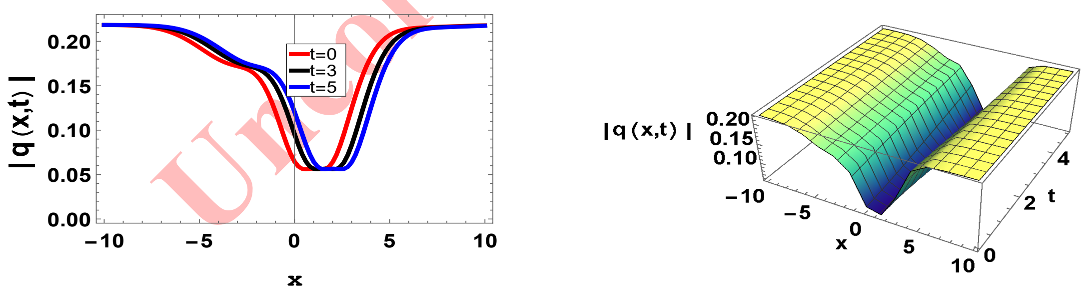

FIGURE 5. Optical singular soliton solution of Case (1) (3.4) with $q(x,t)$ (3.15) $(\text{H}+\frac{G'(\eta)}{G(\eta)})$ expanded method with constants $A = 0.4, b_{-2} = 0.2, B = 0.4, C = 0.7, F = 0.2, H = 0.05, k = 0.4, K = 0.4$, and $\nu = 0.2$.

$b = 1.2, g_1 = 0.09, \lambda = 0.04, \mu = 0.01, \nu = 0.03, v_1 = 0.1, v_2 = -0.01$, and so on in this method. We plot the graph of growth of modulation instability of Equation (4.4) with constants $k = 0.4, \tau = 0.2$. In Figure 15 modulation instability also Figure 17 with constants $\tau = -0.001, k = -0.005$ of Equation (4.5).

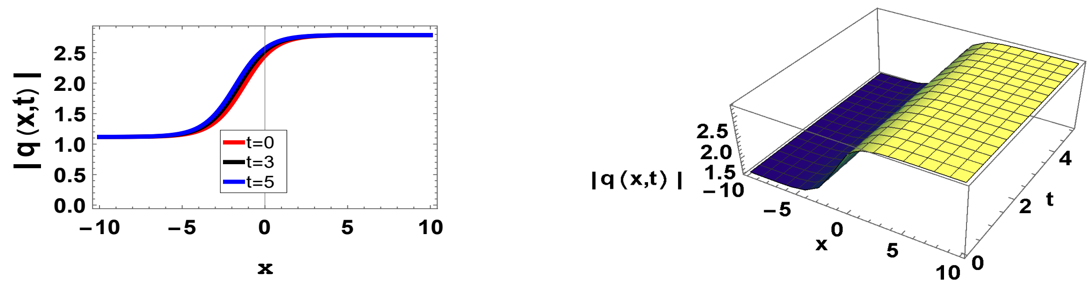

FIGURE 6. Optical bright soliton solution of Case (1) (3.4) with $q(x,t)$ (3.16) (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method with constants $A = 0.1, b_{-2} = 0.1, B = 0.5, C = 0.5, H = 0.4, k = 0.2$, and $\nu = -0.1$.

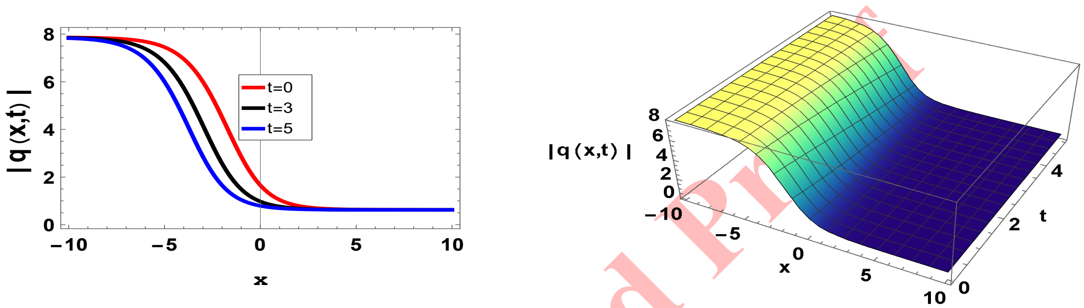

FIGURE 7. Optical bright soliton solution of Case (1) (3.4) with $q(x,t)$ (3.18) (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method with constants $b_{-2} = 0.5, A = 0.3, B = 0.2, C = 0.2, H = 0.6, k = 0.1$, and $\nu = -0.4$.

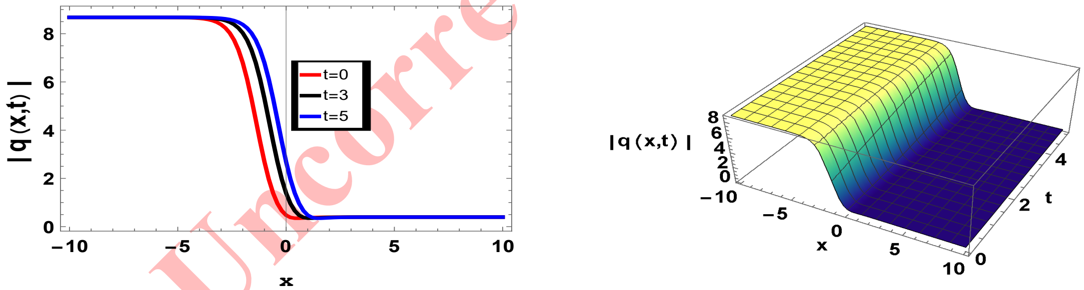

FIGURE 8. Optical bright soliton solution of Case (1) (3.4) with $q(x,t)$ (3.17) (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method with constants $A = 0.3, b_{-2} = 0.2, B = 0.3, C = 0.2, H = 0.3, k = 0.1$, and $\nu = 0.2$.

6. CONCLUSION

We solved partial differential equations by using effective and simple methods known as (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method and ($\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$) expansion method. These methods are powerful analytical tools for obtaining exact solutions of nonlinear differential equations. We introduced our solution by graphs in two and three dimension to show the behavior of our solution. Modulation instability obtained and showed how we can get the growth of modulation instability that very important and useful in the field of optical fiber.

These methods can be used to wide rang of nonlinear differential equations arising in physics, engineering, and applied

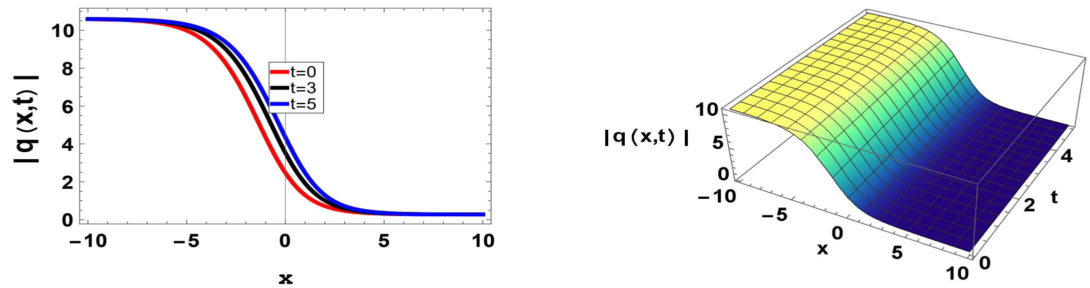

FIGURE 9. Optical bright soliton solution of Case (1) (3.4) with $q(x,t)$ (3.21) (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method with constants $A = 0.5, b_{-2} = 0.2, B = 0.1, C = 0.2, H = 0.3, k = 0.5$, and $\nu = 0.2$.

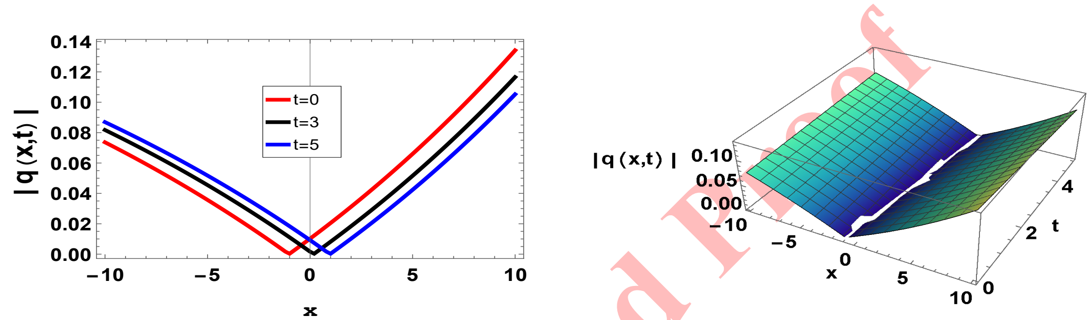

FIGURE 10. Optical rational soliton solution of Case (1) (3.4) with $q(x,t)$ (3.36) (H+$\frac{G'(\eta)}{G(\eta)}$) expanded method with constants $A = 0, b_{-2} = 0.1, b_{-1} = 0.1; B = 0, C = 1.097, H = 0.1, k = 0.2, \nu = 0.4, s = 0.1$, and $\sigma = 0.3$..

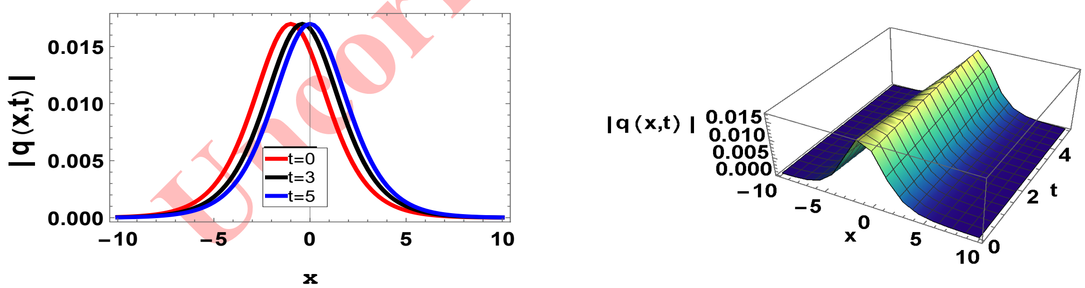

FIGURE 11. Optical rational soliton solution of set (1) (3.39) with $q(x,t)$ (3.45) ($\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)}$) expansion method with constants $b = 0.1, g_1 = 0.1, \lambda = 0.1, \mu = 0.001, \nu = 0.2, v_1 = 0.1$, and $v_2 = 0.2$.

mathematics. Future research could explore their application to higher-dimensional systems, fractional-order equations, and coupled nonlinear models. In addition, integrating these methods with numerical simulations may provide hybrid analytical-numerical schemes that enhance both accuracy and computational efficiency. Such advancements could contribute to practical developments in optical communication, fluid dynamics, Bose-Einstein condensates, and other areas where nonlinear wave phenomena play a main role.

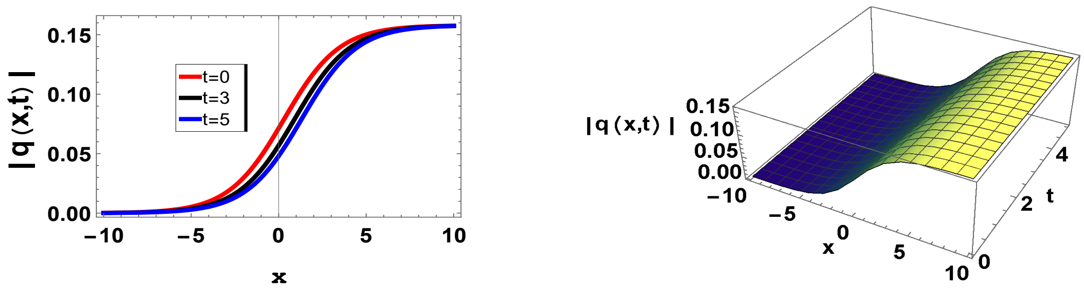

FIGURE 12. Optical soliton solution of set (2) (3.40) with $q(x,t)$ (3.45) $(\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)})$ expansion method with constants $b = 0.3, g_1 = 0.2, \lambda = 0.2, \mu = 0.001, \nu = 0.2, v_1 = 0.3$, and $v_2 = 0.2$..

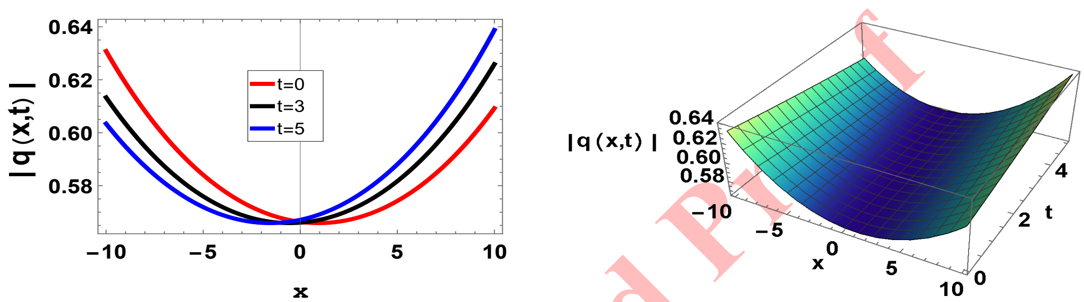

FIGURE 13. Optical singular soliton solution of set (1) (3.39) with $q(x,t)$ (3.46) $(\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)})$ expansion method with constants with constants $b = 1.06, g_1 = 0.6, \lambda = 0.001, \mu = 0.001, \nu = -0.5, v_1 = 0.5$, and $v_2 = 0.03$.

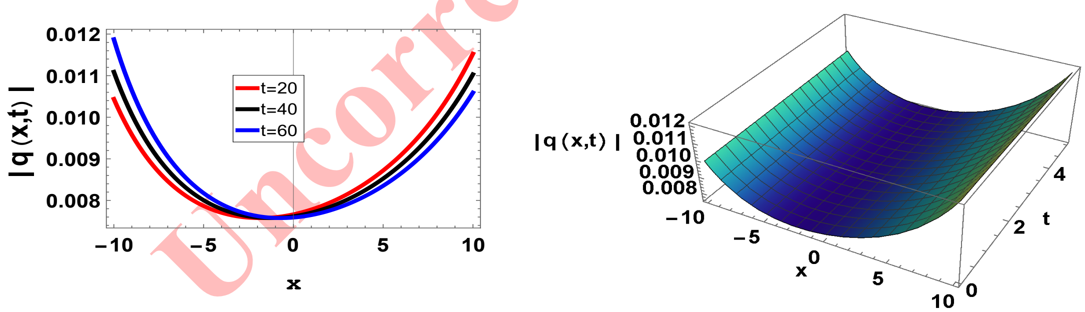

FIGURE 14. Optical singular soliton solution of set (2) (3.40) with $q(x,t)$ (3.46) $(\frac{G'(\eta)}{a+bG'(\eta)+G(\eta)})$ expansion method with constants $b = 1.2, g_1 = 0.09, \lambda = 0.04, \mu = 0.01, \nu = 0.03, v_1 = 0.1$, and $v_2 = -0.01$.

ETHICS APPROVAL AND CONSENT TO PARTICIPATE

The author confirm that all the results they obtained are new and there is no conflict of interest with anyone.

CONSENT FOR PUBLICATION

The paper has one author who agrees to publish.

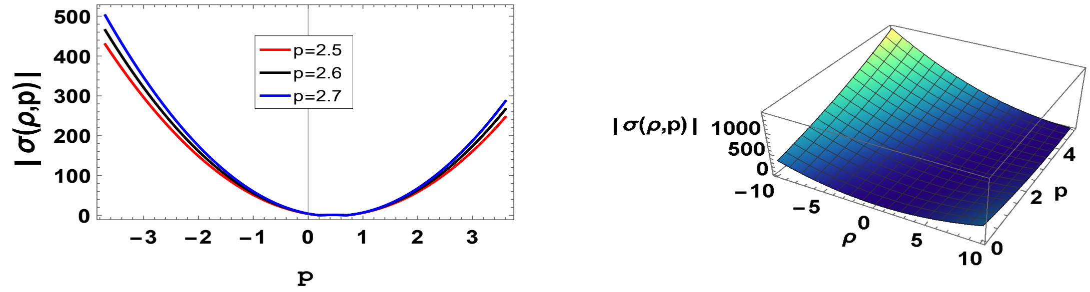

FIGURE 15. Growth modulation instability (4.4) $\tau = 0.5$ and $\gamma = 0.2$.

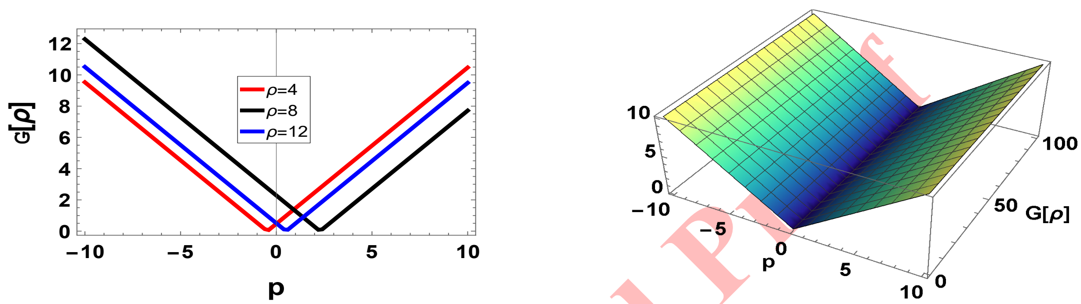

FIGURE 16. Modulation instability (4.5) $\gamma = 0.7$ and $k = 0.6$.

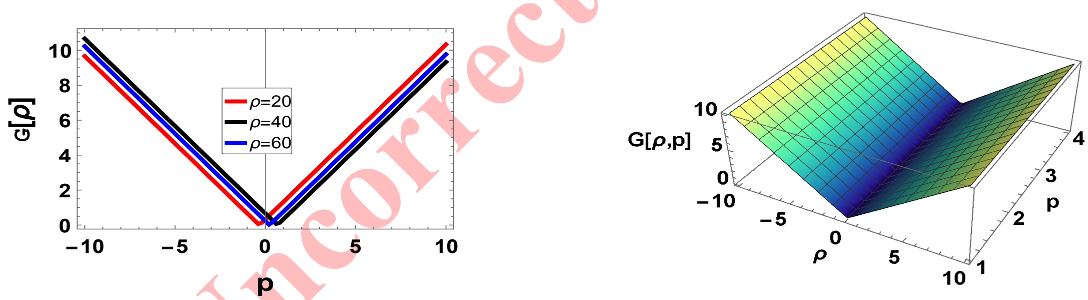

FIGURE 17. Modulation instability (4.5) $\gamma = -0.001$ and $k = -0.005$.

Availability of data and materials

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests

There is no conflict of interest between the authors or anyone else regarding this manuscript.

Authors' Contribution

If we look at the contribution of each author in this paper, we will find that each of them participated in the work from beginning to end in equal measure.

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