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Dynamics of wave propagation for (1+2)-dimensional Chiral nonlinear Schrödinger in theoritical physics

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Abstract

We investigate and extracted many solutions for the (1+2)-dimensional Chiral nonlinear Schrödinger equation by new modified unified auxiliary equation method in this paper. The considered method is a enhanced version of the unified auxiliary equation method. The mentioned model is used to discuss the wave in quantum field theory. Consequently, some different and more general solutions are obtained such as singular, periodic, hyperbolic, dark-bright, trigonometric, exponential, and Jacobi elliptic functions. These solutions are more general in nature and offer different approaches to address the problem at hand. To elucidate the dynamics of these solutions, their figures are presented. These figures provide a visual representation of the behavior of the solutions, enabling a better understanding of their characteristics.

Keywords. The (1+2)-dimensional Chiral non-linear Schrödinger equation, Modified unified auxiliary equation method, Exact solutions.
2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Partial differential equations have proven to be a valuable tool for expressing various phenomena across different disciplines. By using PDEs, scientists and mathematicians can describe complex systems in physics, engineering, economics, and other fields. These equations provide a way to model relationships between multiple variables and their rates of change, allowing researchers to better understand and predict how systems will behave under different conditions. The Schrödinger equation is a widely recognized mathematical tool that aids researchers in comprehending complex physical phenomena in various fields, including quantum physics, fluid dynamics, and optical physics. By solving the Schrödinger equation, scientists can obtain insights into the behavior of systems, including the wave-like nature of particles in quantum mechanics, the motion of fluids, and the propagation of light. The equation offers a framework to analyze and predict the behavior of these phenomena, making it an indispensable tool in the study of many scientific disciplines. Consequently, the Schrödinger equation plays a crucial role in advancing our understanding of the natural world [43]. Nonlinear evolution equations with nonlinear coefficients, such as the nonlinear Schrödinger equation, are widely utilized to describe nonlinear behavior in various scientific fields, including optical fibers, high-energy physics, electricity, mechanics, hydrodynamics, quantum field theory, and shallow water wave propagation. These equations are essential for modeling nonlinear effects such as wave dispersion, self-focusing, and soliton propagation. By using nonlinear evolution equations, researchers can accurately predict the behavior of complex systems under nonlinear conditions. The significance of these equations lies in their ability to capture nonlinear phenomena that are not captured by linear models. As a result, nonlinear evolution equations play a crucial

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role in advancing our understanding of the natural world and developing new technologies in many scientific fields [12, 21, 22, 36].Over time, numerous competent methods have been developed and utilized to analyze and examine various models. These methods include analytical techniques, numerical simulations, and experimental investigations. Analytical methods often involve solving equations using mathematical tools such as calculus and linear algebra to derive analytical solutions. On the other hand, numerical simulations employ computational methods to solve complex models that do not have analytical solutions. These simulations use algorithms and numerical techniques to simulate the behavior of the model. Finally, experimental investigations involve conducting physical experiments to validate or invalidate the model's predictions. By using these various methods, researchers can better understand the limitations of the models and their ability to accurately represent real-world phenomena. Ultimately, these methods are crucial for advancing scientific knowledge and facilitating the development of new technologies. These techniques inculide the Sub-equation method [2, 13, 44], the Homotopy perturbation method [8, 16, 20, 24, 28], the collocation method [29, 35], the simplified Hirota's method [4, 6, 23, 27], the Jacobi elliptic function expansion method [7, 18, 26, 39–41], and so. The motivation of this current study is based on the work of new modified unified auxiliary equation method (MUAEM) [42]. The considered method is a enhanced version of the unified auxiliary equation method.

Consider the (1+2)-dimensional Chiral non-linear Schrödinger equation by [31]

$$i\frac{\partial u}{\partial t}+\beta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+i\left(\alpha_1\left(u\frac{\partial u^*}{\partial x}-u^*\frac{\partial u}{\partial x}\right)+\alpha_2\left(u\frac{\partial u^*}{\partial y}-u^*\frac{\partial u}{\partial y}\right)\right)u=0,\quad i=\sqrt{-1}, \tag{1.1}$$

where $u=u(x,y,t)$ represents the complex field envelope that encapsulates the soliton profile, the first term is the evolution term, β represents the coefficient of dispersion term which controls how the wave propagates in the space, it has application in optics, quantum mechanics, and plasma physics. While α_1 and α_2 stand for nonlinear phase shift arising from chiral interaction in the $x-$ and $y-$direction, respectively. They have applications in Gyrotropic optics, and plasma physics. Recently, some researchers have investigated the (1+2)-dimensional Chiral non-linear Schrödinger equation as Wang *et al.* [43] have the Semi inverse method, Arshed *et al.* [9] used $\left(\frac{G'}{G},\frac{1}{G}\right)$- expansion method, $tan\left(\frac{\phi(\xi)}{2}\right)$- expansion approach to obtain periodic, hyperbolic and rational solutions, Neirameh in [30] have the functional variable method, Eslami used trial solution technique in [15], Hosseini and Mirzazadeh have the modified Jacobi elliptic expansion method in [19], Raza and Javid used different methods method in [25, 34], Awan *et al.* used Functional variable method and First integral method to soliton solutions [10], Raza and arshed used The first integral method and Sine-Gordon expansion method in [33] and other studies have been done via different methods [1, 3, 5, 11, 14, 17, 32, 37, 38].

This study is structured as follows: Section one provides an introduction to the research topic. In Section Two, we outline the fundamental concepts of the new MUAEM. Next, in Section Three, we employ this method to generate new exact solutions for the proposed Eq. (1.1). Finally, in Section Four, we present our conclusions based on the findings of this study. By following this structure, we aim to provide a clear and concise presentation of our research methods and results, making it easier for readers to understand and appreciate the contributions of our study.

2. Outline of the newly MUAEM

Consider a nonlinear partial differential equation

$$F\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial xt},\frac{\partial^2 u}{\partial xy},...\right)=0, \tag{2.1}$$

where F is a polynomial function including $u(x,y,t)$ and its partial derivatives. Taking the following wave transformation

$$\begin{aligned} &u(x,y,t)=U(\varsigma)e^{i\Phi(x,y,t)},\\ &u^*(x,y,t)=U(\varsigma)e^{i\Phi(x,y,t)},\\ &\Phi(x,y,t)=\lambda_1 x+\lambda_2 y+wt+\theta,\varsigma=b_1x+b_2y-vt. \end{aligned} \tag{2.2}$$

Eq. (2.1) becomes a nonlinear ordinary differential equation (NODE) as below:

$$G(U^{'}, U^{''}, U^{'''}, ...) = 0. \tag{2.3}$$

In terms of finite rational formal expansion, we introduce a new and more general solution in the following forms:

$$U(\varsigma) = \frac{H_0 + \sum_{i=1}^{M} f^{i-1}(\varsigma)(H_i f(\varsigma) + L_i g(\varsigma))}{K_0 + \sum_{i=1}^{N} f^{i-1}(\varsigma)(K_i f(\varsigma) + S_i g(\varsigma))}. \tag{2.4}$$

The following cases will be considered for $f(\varsigma)$ and $g(\varsigma)$ variables

$$f^{'}(\varsigma) = f(\varsigma)g(\varsigma), \tag{2.5}$$

$$g^{'}(\varsigma) = q_1 + g^2(\varsigma) + r_1 f^{-2}(\varsigma), \tag{2.6}$$

$$g^2(\varsigma) = -(q_1 + c_1 f^2(\varsigma) + \frac{r_1}{2} f^{-2}(\varsigma)), \tag{2.7}$$

where q_1, r_1 and c_1 are constants, $f^{'}(\varsigma) = \frac{d}{d\varsigma}f(\varsigma)$, $g^{'}(\varsigma) = \frac{d}{d\varsigma}g(\varsigma)$ are the functions of ς, H_0, H_i, L_i $(i = 1, ..., M)$, K_0, K_i, S_i $(i = 1, ..., N)$ are constants to be determined later. Then, in order to construct exact solutions, the following steps are taken:

Step 1: We calculate the numbers N and M in Eq. (2.4) by using the homogeneous balance method between the highest order derivatives and the nonlinear terms in Eq. (2.3).

Step 2: Substituting Eq. (2.3) along with Eq. (2.5), Eq. (2.6), and Eq. (2.7). Then, setting the coefficients of the $f^i(\varsigma)g^j(\varsigma), (i = 0, \pm 1, \pm 2, ..., j = 0, 1)$ equal to zero.

Step 3: We solve the obtained system to find the unknown parameters.

Step 4: Using the results from step 3, the exact solutions are determined by taking into account the cases of $f(\varsigma)$ and $g(\varsigma)$. Considering Eqs. (2.5) and (2.6), by taking

$$f(\varsigma) = \frac{1}{W(\varsigma)}, g(\varsigma) = \frac{f'(\varsigma)}{f(\varsigma)}. \tag{2.8}$$

After necessary operations are done, we have

$$(W'(\varsigma))^2 + q_1(W(\varsigma))^2 + \frac{r_1}{2}(W(\varsigma))^4 + c_1 = 0. \tag{2.9}$$

Under Eq. (2.9), the following Jacobi elliptic function solutions are obtained.

3. Application for newly MUAEM

In this section, we present the application of the newly MUAEM to Eq. (1.1). Firstly, we will apply the transformation which is given in Eq. (2.2) to Eq. (1.1), then we have

$$i\left(-v + 2\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right)U' - \left(\left(w + \beta\left({\lambda_1}^2 + {\lambda_2}^2\right)\right)U\right) + 2\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)U^3 + \left({b_1}^2 + {b_2}^2\right)\beta U'' = 0. \tag{3.1}$$

If we set the coefficients of the imaginary part of Eq. (3.1) to zero, we have $v = 2\beta\left(b_1\lambda_1 + b_2\lambda_2\right)$. And the real part is as follows:

$$-\left(w + \beta\left({\lambda_1}^2 + {\lambda_2}^2\right)\right)U + 2\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)U^3 + \left({b_1}^2 + {b_2}^2\right)\beta U'' = 0. \tag{3.2}$$

If the balance principle is used, then the balance term be $N = 1$. Since the relation between M and N is $M = N + 1$, so $M = 2$. Considering Eq. (2.4) with the values of M and N, we may express the solution of Eq. (3.2) as below:

$$P(\varsigma) = \frac{H_0 + H_1 f(\varsigma) + L_1 g(\varsigma) + H_2 f^2(\varsigma) + L_2 f(\varsigma)g(\varsigma)}{K_0 + K_1 f(\varsigma) + S_1 g(\varsigma)}. \tag{3.3}$$

Using Eq. (3.3) in Eq. (3.2), then the system of algebraic equations will obtained. The following situation is obtained by using computer software to solve the equation system:

Case 1:

$$\begin{aligned} &H_0 = 0, \quad K_0 = \frac{H_1 S_1}{L_2}, \quad L_1 = 0, \quad K_1 = \frac{H_2 S_1}{L_2}, \\ &w = -\beta\left({b_1}^2 q_1 + {b_2}^2 q_1 + {\lambda_1}^2 + {\lambda_2}^2\right), \\ &\alpha_2 = \frac{{b_1}^2 c_1 {S_1}^2 \beta + {b_2}^2 c_1 {S_1}^2 \beta - {L_2}^2 \alpha_1 \lambda_1}{{L_2}^2 \lambda_2}. \end{aligned} \tag{3.4}$$

With the help of case 1, we present the solutions to Eq. (1.1).
Family -1. If $q_1 = 1 + {m_1}^2, r_1 = -2{m_1}^2, c_1 = -1$ and $0 < m_1 < 1$, then

$$f_1(\varsigma) = ns(\varsigma, m_1) = \frac{1}{sn(\varsigma, m_1)}, \quad g_1(\varsigma) = \frac{-cn(\varsigma, m_1)\, dn(\varsigma, m_1)}{sn(\varsigma, m_1)}. \tag{3.5}$$

From (2.2), (3.3) and (3.5), we have

$$\begin{aligned} u_1(x,y,t) = &\frac{e^{i\left(\lambda_1 x + \lambda_2 y - \left(\beta({b_1}^2\left(1+{m_1}^2\right) + {b_2}^2\left(1+{m_1}^2\right) + {\lambda_1}^2 + {\lambda_2}^2\right) t + \theta\right)}}{\frac{H_1 S_1}{L_2} + \frac{H_2 S_1 ns(-tv + b_1 x + b_2 y)}{L_2} - \frac{S_1 cn(-tv + b_1 x + b_2 y) dn(-tv + b_1 x + b_2 y)}{sn(-tv + b_1 x + b_2 y)}} \\ &\times \Bigg(H_1 ns(-tv + b_1 x + b_2 y) + H_2 ns(-tv + b_1 x + b_2 y)^2 \\ &- \frac{L_2 cn(-tv + b_1 x + b_2 y)\, dn(-tv + b_1 x + b_2 y)\, ns(-tv + b_1 x + b_2 y)}{sn(-tv + b_1 x + b_2 y)}\Bigg). \end{aligned} \tag{3.6}$$

When $m_1 \to 1$, we have $ns(\varsigma, m_1) \to \coth(\varsigma)$, $\frac{-cn(\varsigma, m_1)dn(\varsigma, m_1)}{sn(\varsigma, m_1)} \to \frac{-\text{sech}(\varsigma)\text{sech}(\varsigma)}{tanh(\varsigma)}$, leads to

$$\begin{aligned} u_{1,1}(x,y,t) = &\frac{e^{i\left(\lambda_1 x + \lambda_2 y - t\beta\left(2{b_1}^2 + 2{b_2}^2 + {\lambda_1}^2 + {\lambda_2}^2\right) + \theta\right)}}{\left(\begin{array}{c} \frac{H_1 S_1}{L_2} + \frac{H_2 S_1 \coth(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2))}{L_2} \\ -S_1 \text{csch}(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2))\,\text{sech}(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2)) \end{array}\right)} \\ &\times \Big(H_1 \coth(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2)) + H_2 \coth(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2))^2 \\ &- L_2 \text{csch}(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2))^2\Big). \end{aligned} \tag{3.7}$$

When $m_1 \to 0$, $ns(\varsigma, m_1) \to csc(\varsigma)$, $\frac{-cn(\varsigma, m_1)dn(\varsigma, m_1)}{sn(\varsigma, m_1)} \to \frac{-cos(\varsigma)}{sin(\varsigma)}$, causes to

$$\begin{aligned} u_{1,0}(x,y,t) = &\frac{e^{i\left(\lambda_1 x + \lambda_2 y - t\beta\left({b_1}^2 + {b_2}^2 + {\lambda_1}^2 + {\lambda_2}^2\right) + \theta\right)}}{\left(\frac{H_1 S_1}{L_2} - S_1 \cot(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2)) + \frac{H_2 S_1 \csc(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2))}{L_2}\right)} \\ &\times \Big(H_1 \csc(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2)) + H_2 \csc(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2))^2 \\ &- L_2 \cot(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2)) \csc(b_1 x + b_2 y - 2t\beta(b_1\lambda_1 + b_2\lambda_2))\Big). \end{aligned} \tag{3.8}$$

Family-2. When $q_1 = 1 - 2{m_1}^2, r_1 = 2{m_1}^2, c_1 = {m_1}^2 - 1$ and $0 < m_1 < 1$, then

$$f_2(\varsigma) = nc(\varsigma, m_1) = \frac{1}{cn(\varsigma, m_1)}, \; g_2(\varsigma) = \frac{-sn(\varsigma, m_1)\, dn(\varsigma, m_1)}{cn(\varsigma, m_1)}. \tag{3.9}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.9), we have

$$
\begin{aligned}
u_2(x,y,t) = &\left(\frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(b_1{}^2\left(1-2m_1{}^2\right)+b_2{}^2\left(1-2m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\frac{H_1S_1}{L_2}+\frac{H_2S_1nc(-tv+b_1x+b_2y)}{L_2}+\frac{S_1dn(-tv+b_1x+b_2y)sn(-tv+b_1x+b_2y)}{cn(-tv+b_1x+b_2y)}} \right) \\
&\times \Big(H_1 nc\left(-tv+b_1x+b_2y\right)+H_2 nc(-tv+b_1x+b_2y)^2 \\
&+\frac{L_2 dn\left(-tv+b_1x+b_2y\right) nc\left(-tv+b_1x+b_2y\right) sn\left(-tv+b_1x+b_2y\right)}{cn\left(-tv+b_1x+b_2y\right)} \Big).
\end{aligned} \tag{3.10}
$$

When $m_1 \to 1$, $nc\left(\varsigma, m_1\right) \to cosh\left(\varsigma\right)$, $\frac{-sn(\varsigma,m_1)dn(\varsigma,m_1)}{cn(\varsigma,m_1)} \to -tanh\left(\varsigma\right)$, causes to

$$
\begin{aligned}
u_{2,1}(x,y,t) = &\frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-b_1{}^2-b_2{}^2+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\frac{H_1S_1}{L_2}+\frac{H_2S_1\cosh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}-S_1\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)} \\
&\times \Big(H_1\cosh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+H_2\cosh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2 \\
&-L_2\sinh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\Big).
\end{aligned} \tag{3.11}
$$

Graphical representation of Eq. (3.11) is shown in Figure 1.

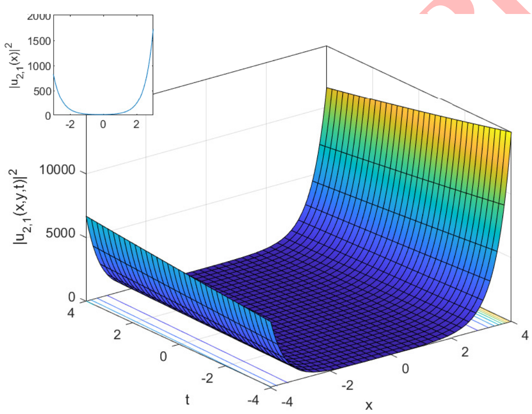

FIGURE 1. The graphic of Eq. (3.11) when $\theta = 0.5$, $\lambda_1 = 0.09$, $\lambda_2 = 1.5$, $\beta = 0.02$, $b_1 = 1$, $b_2 = 0.2$, $H_2 = 0.2$, $\alpha_1 = 1$, $H_1 = 2$, $S_1 = 0.3$ and $L_2 = 1.03$.

When $m_1 \to 0$, $nc\left(\varsigma, m_1\right) \to sec\left(\varsigma\right)$, $\frac{-sn(\varsigma,m_1)dn(\varsigma,m_1)}{cn(\varsigma,m_1)} \to \frac{-sin(\varsigma)}{cos(\varsigma)}$, causes to

$$
\begin{aligned}
u_{2,0}(x,y,t) = &\frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(b_1{}^2+b_2{}^2+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\frac{H_1S_1}{L_2}+\frac{H_2S_1\sec(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}-S_1\tan\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)} \\
&\times \Big(H_1\sec\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+H_2\sec\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2 \\
&-L_2\sec\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\tan\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\Big).
\end{aligned} \tag{3.12}
$$

Graphical representation of Eq. (3.12) is shown in Figure 2.

Family-3. When $q_1 = -2+m_1{}^2, r_1 = 2, c_1 = 1-m_1{}^2$ and $0 < m_1 < 1$, then

$$
f_3\left(\varsigma\right) = nd\left(\varsigma, m_1\right) = \frac{1}{dn\left(\varsigma, m_1\right)},\ g_3\left(\varsigma\right) = \frac{m_1{}^2 sn\left(\varsigma, m_1\right) cn\left(\varsigma, m_1\right)}{dn\left(\varsigma, m_1\right)}. \tag{3.13}
$$

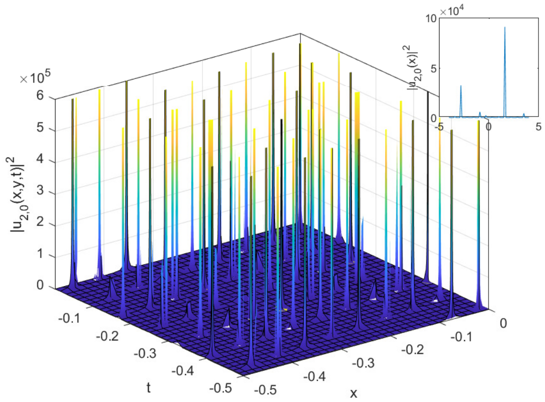

FIGURE 2. The graphic of Eq. (3.12) when $\theta = 0.5$, $\lambda_1 = 2$, $\lambda_2 = 1.5$, $\beta = 2$, $b_1 = 5$, $b_2 = 0.2$, $H_2 = 0.1$, $\alpha_1 = 3$, $H_1 = 2.5$, $S_1 = 0.3$ and $L_2 = 1.2$.

From Eq. (2.2), Eq. (3.3) and Eq. (3.12), we have

$$\begin{aligned} u_3(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(b_1{}^2\left(-2+m_1{}^2\right)+b_2{}^2\left(-2+m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\frac{H_1 S_1}{L_2}+\frac{H_2 S_1 nd(-tv+b_1x+b_2y)}{L_2}+\frac{m_1{}^2 S_1 cn(-tv+b_1x+b_2y)sn(-tv+b_1x+b_2y)}{dn(-tv+b_1x+b_2y)}} \\ & \times\Big(H_1 nd\left(-tv+b_1x+b_2y\right)+H_2 nd(-tv+b_1x+b_2y)^2 \\ & +\frac{L_2 m_1{}^2 cn\left(-tv+b_1x+b_2y\right) nd\left(-tv+b_1x+b_2y\right) sn\left(-tv+b_1x+b_2y\right)}{dn\left(-tv+b_1x+b_2y\right)}\Big). \end{aligned} \tag{3.14}$$

Family-4. When $q_1 = 1+m_1{}^2, r_1 = -2, c_1 = -m_1{}^2$, and $0 < m < 1$, then

$$f_4(\varsigma) = sn\left(\varsigma, m_1\right), \ g_4(\varsigma) = \frac{cn\left(\varsigma, m_1\right) dn\left(\varsigma, m_1\right)}{sn\left(\varsigma, m_1\right)}. \tag{3.15}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.14), we have

$$\begin{aligned} u_4(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(b_1{}^2\left(1+m_1{}^2\right)+b_2{}^2\left(1+m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\frac{H_1 S_1}{L_2}+\frac{S_1 cn(-tv+b_1x+b_2y)dn(-tv+b_1x+b_2y)}{sn(-tv+b_1x+b_2y)}+\frac{H_2 S_1 sn(-tv+b_1x+b_2y)}{L_2}} \\ & \times\Big(L_2 cn\left(-tv+b_1x+b_2y\right) dn\left(-tv+b_1x+b_2y\right) \\ & +H_1 sn\left(-tv+b_1x+b_2y\right)+H_2 sn(-tv+b_1x+b_2y)^2\Big). \end{aligned} \tag{3.16}$$

When $m_1 \to 1$, $sn\left(\varsigma, m_1\right) \to tanh\left(\varsigma\right)$, $\frac{cn(\varsigma,m_1)dn(\varsigma,m_1)}{sn(\varsigma,m_1)} \to \frac{\operatorname{sech}(\varsigma)\operatorname{sech}(\varsigma)}{tanh(\varsigma)}$, causes to

$$\begin{aligned} u_{4,1}(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(2b_1{}^2+2b_2{}^2+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+S_1\operatorname{csch}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{H_2 S_1 \tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}\right)} \\ & \times\Big(L_2\operatorname{sech}(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right))^2+H_1\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right) \\ & +H_2\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2\Big). \end{aligned} \tag{3.17}$$

Graphical representation of Eq. (3.17) is shown in Figure 3.

When $m_1 \to 0$, $sn\left(\varsigma, m_1\right) \to sin\left(\varsigma\right)$, $\frac{cn(\varsigma,m_1)dn(\varsigma,m_1)}{sn(\varsigma,m_1)} \to \frac{cos(\varsigma)}{sin(\varsigma)} = cot\left(\varsigma\right)$, causes to

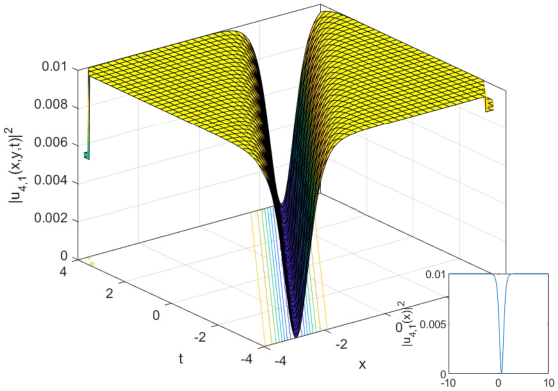

FIGURE 3. The graphic of Eq. (3.17) for $\theta = 0.5$, $\lambda_1 = 0.3$, $\lambda_2 = 1.5$, $\beta = 0.75$, $b_1 = 1.75$, $b_2 = 0.2$, $H_2 = 0.2$, $\alpha_1 = 2$, $H_1 = 0.5$, $S_1 = 0.3$ and $L_2 = 0.03$.

$$\begin{aligned}u_{4,0}(x,y,t) = &\frac{\mathrm{e}^{\mathrm{i}\left(\lambda_1 x+\lambda_2 y-t\beta\left({b_1}^2+{b_2}^2+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+S_1\cot\left(b_1 x+b_2 y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{H_2 S_1\sin\left(b_1 x+b_2 y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}{L_2}\right)} \\ &\times\left(L_2\cos\left(b_1 x+b_2 y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+H_1\sin\left(b_1 x+b_2 y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right. \\ &\left.+H_2\sin\left(b_1 x+b_2 y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2\right).\end{aligned} \tag{3.18}$$

Graphical representation of Eq. (3.18) is shown in Figure 4.

Family-5. When $q_1 = 1-2{m_1}^2, r_1 = -2+2{m_1}^2, c_1 = {m_1}^2$, and $0 < m_1 < 1$, then

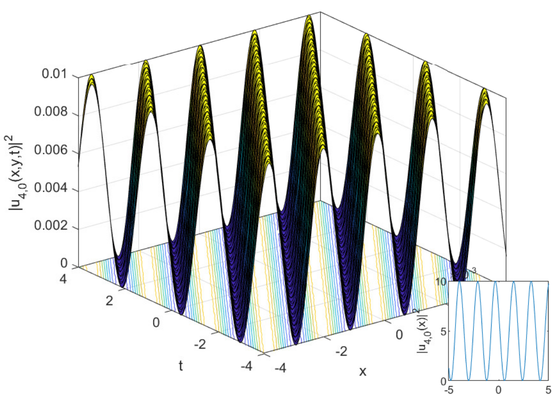

FIGURE 4. The graphic of Eq. (3.18) when $\theta = 0.5$, $\lambda_1 = 0.3$, $\lambda_2 = 1.5$, $\beta = 0.75$, $b_1 = 1.75$, $b_2 = 0.2$, $H_2 = 0.2$, $\alpha_1 = 2$, $H_1 = 0.5$, $S_1 = 0.3$ and $L_2 = 0.03$.

$$f_5(\varsigma) = cn(\varsigma, m_1),\ g_5(\varsigma) = -\frac{sn(\varsigma, m_1)\, dn(\varsigma, m_1)}{cn(\varsigma, m_1)}. \tag{3.19}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.19), we have

$$\begin{aligned} u_5(x,y,t) = & \frac{\mathrm{e}^{\mathrm{i}\left(\lambda_1 x+\lambda_2 y-t\beta\left(b_1{}^2\left(1-2m_1{}^2\right)+b_2{}^2\left(1-2m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+\frac{H_2 S_1 cn(-tv+b_1x+b_2y)}{L_2}-\frac{S_1 dn(-tv+b_1x+b_2y) sn(-tv+b_1x+b_2y)}{cn(-tv+b_1x+b_2y)}\right)} \\ & \times \Big(H_1 cn\left(-tv+b_1x+b_2y\right)+H_2 cn(-tv+b_1x+b_2y)^2 \\ & -L_2 dn\left(-tv+b_1x+b_2y\right) sn\left(-tv+b_1x+b_2y\right)\Big). \end{aligned} \tag{3.20}$$

When $m_1 \to 1$, $cn\,(\varsigma, m_1) \to \operatorname{sech}(\varsigma)$, $\frac{sn(\varsigma,m_1)dn(\varsigma,m_1)}{cn(\varsigma,m_1)} \to tanh\,(\varsigma)$, causes to

$$\begin{aligned} u_{5,1}(x,y,t) = & \frac{\mathrm{e}^{\mathrm{i}\left(\lambda_1 x+\lambda_2 y-t\beta\left(-b_1{}^2-b_2{}^2+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+\frac{H_2 S_1 \operatorname{sech}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}-S_1 \tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)} \\ & \times \Big(H_1 \operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+H_2 \operatorname{sech}(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right))^2 \\ & -L_2 \operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\Big). \end{aligned} \tag{3.21}$$

Graphical representation of Eq. (3.21) is shown in Figure 5.
When $m_1 \to 0$, $cn\,(\varsigma, m_1) \to cos\,(\varsigma)$, $\frac{sn(\varsigma,m_1)dn(\varsigma,m_1)}{cn(\varsigma,m_1)} \to tan\,(\varsigma)$, causes to

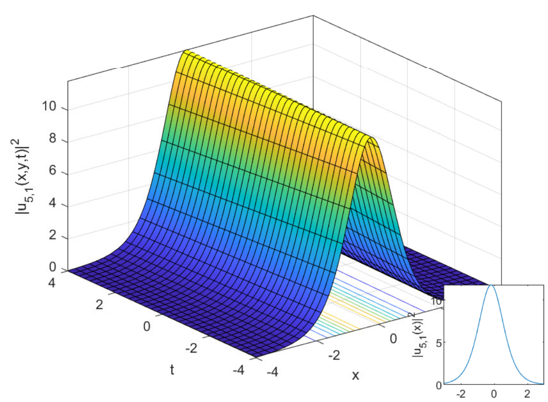

FIGURE 5. The graphic of Eq. (3.21) when $\theta = 0.5$, $\lambda_1 = 0.09$, $\lambda_2 = 1.5$, $\beta = 0.02$, $b_1 = 1$, $b_2 = 0.2$, $H_2 = 1$, $\alpha_1 = 1$, $H_1 = 2$, $S_1 = 0.3$ and $L_2 = 1.03$.

$$\begin{aligned} u_{5,0}(x,y,t) = & \frac{\mathrm{e}^{\mathrm{i}\left(\lambda_1 x+\lambda_2 y-t\beta\left(+b_1{}^2+b_2{}^2+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+\frac{H_2 S_1 \cos(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}-S_1 \tan\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)} \\ & \times \Big(H_1 \cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+H_2 \cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2 \\ & -L_2 \sin\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\Big). \end{aligned} \tag{3.22}$$

Graphical representation of Eq. (3.22) is shown in Figure 6.
Family-6. When $q_1 = -2 + m_1{}^2, r_1 = 2 - 2m_1{}^2, c_1 = 1$, and $0 < m_1 < 1$, then

$$f_6\left(\varsigma\right) = dn\left(\varsigma, m_1\right), \; g_6\left(\varsigma\right) = -m_1{}^2 \frac{sn\left(\varsigma, m_1\right) cn\left(\varsigma, m_1\right)}{dn\left(\varsigma, m_1\right)}. \tag{3.23}$$

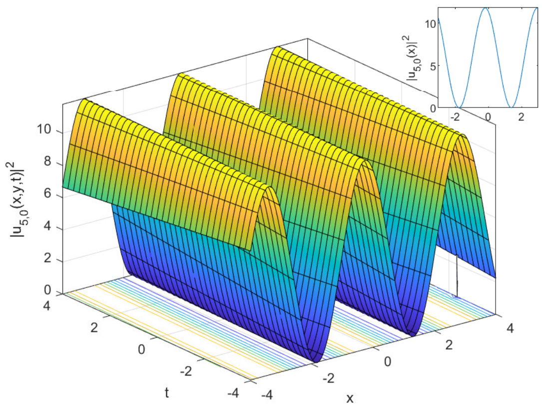

FIGURE 6. The graphic of Eq. (3.22) when $\theta = 0.5$, $\lambda_1 = 0.09$, $\lambda_2 = 1.5$, $\beta = 0.02$, $b_1 = 1$, $b_2 = 0.2$, $H_2 = 0.001$, $\alpha_1 = 1$, $H_1 = 2$, $S_1 = 0.3$ and $L_2 = 1.03$.

From Eq. (2.2), Eq. (3.3) and Eq. (3.23), we have

$$\begin{aligned} u_6(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(b_1{}^2\left(-2+m_1{}^2\right)+b_2{}^2\left(-2+m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+\frac{H_2 S_1 dn(-tv+b_1x+b_2y)}{L_2}-\frac{m_1{}^2 S_1 cn(-tv+b_1x+b_2y)sn(-tv+b_1x+b_2y)}{dn(-tv+b_1x+b_2y)}\right)} \\ & \times \Big(H_1 dn\left(-tv+b_1x+b_2y\right)+H_2 dn(-tv+b_1x+b_2y)^2 \\ & -L_2 m_1{}^2 cn\left(-tv+b_1x+b_2y\right) sn\left(-tv+b_1x+b_2y\right)\Big). \end{aligned} \tag{3.24}$$

When $m_1 \to 1$, $dn(\varsigma, m_1) \to \operatorname{sech}(\varsigma)$, $\frac{sn(\varsigma,m_1)cn(\varsigma,m_1)}{dn(\varsigma,m_1)} \to \frac{tanh(\varsigma)\operatorname{sech}(\varsigma)}{\operatorname{sech}(\varsigma)} = tanh(\varsigma)$, causes to

$$\begin{aligned} u_{6,1}(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-b_1{}^2-b_2{}^2+\lambda_1{}^2+\lambda_2{}^2\right)\right)}}{\left(\frac{H_1 S_1}{L_2}+\frac{H_2 S_1\ \operatorname{sech}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}-S_1\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)} \\ & \times \Big(H_1\ \operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+H_2\ \operatorname{sech}(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right))^2 \\ & -L_2\ \operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\Big). \end{aligned} \tag{3.25}$$

Family-7. When $q_1 = -2+m_1{}^2, r_1 = -2+2m_1{}^2, c_1 = -1$, and $0 < m_1 < 1$, then

$$f_7(\varsigma) = cs(\varsigma, m_1) = \frac{cn(\varsigma, m_1)}{sn(\varsigma, m_1)},\ g_7(\varsigma) = -\frac{dn(\varsigma, m_1)}{sn(\varsigma, m_1)\, cn(\varsigma, m_1)}. \tag{3.26}$$

From Eq. (2.2)), Eq. (3.3) and Eq. (3.26), we have

$$\begin{aligned} u_7(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(b_1{}^2\left(-2+m_1{}^2\right)+b_2{}^2\left(-2+m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)\right)}}{\left(\frac{H_1 S_1}{L_2}+\frac{H_2 S_1 cs(-tv+b_1x+b_2y)}{L_2}-\frac{S_1 dn(-tv+b_1x+b_2y)}{cn(-tv+b_1x+b_2y)sn(-tv+b_1x+b_2y)}\right)} \\ & \times \Big(H_1 cs\left(-tv+b_1x+b_2y\right)+H_2 cs(-tv+b_1x+b_2y)^2 \\ & -\frac{L_2 cs\left(-tv+b_1x+b_2y\right) dn\left(-tv+b_1x+b_2y\right)}{cn\left(-tv+b_1x+b_2y\right) sn\left(-tv+b_1x+b_2y\right)}\Big). \end{aligned} \tag{3.27}$$

When $m_1 \to 1$, $cs\left(\varsigma, m_1\right) \to \operatorname{csch}\left(\varsigma\right)$, $\frac{dn(\varsigma,m_1)}{sn(\varsigma,m_1)cn(\varsigma,m_1)} \to \frac{1}{tanh(\varsigma)}$, causes to

$$
\begin{aligned}
u_{7,1}(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left(-{b_1}^2-{b_2}^2+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2} - S_1 \coth\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right) + \frac{H_2 S_1 \operatorname{csch}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}\right)} \times \\
& \times \Big(H_1 \operatorname{csch}\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right) + H_2 \operatorname{csch}(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right))^2 \\
& - L_2 \coth\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right) \operatorname{csch}\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right)\Big).
\end{aligned} \tag{3.28}
$$

When $m_1 \to 0$, $cs\left(\varsigma, m_1\right) \to cot\left(\varsigma\right)$, $\frac{dn(\varsigma,m_1)}{sn(\varsigma,m_1)cn(\varsigma,m_1)} \to \frac{1}{sin(\varsigma)cos(\varsigma)}$, causes to

$$
\begin{aligned}
u_{7,0}(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left(-2{b_1}^2-2{b_2}^2+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\begin{array}{c}\frac{H_1 S_1}{L_2} + \frac{H_2 S_1 \cot(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2} \\ - S_1 \csc\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right) \sec\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right)\end{array}\right)} \\
& \times \Big(H_1 \cot\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right) + H_2 \cot\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right)^2 \\
& - L_2 \csc\left(b_1 x + b_2 y - 2t\beta\left(b_1\lambda_1 + b_2\lambda_2\right)\right)^2\Big).
\end{aligned} \tag{3.29}
$$

Family-8. When $q_1 = 1 - 2{m_1}^2, r_1 = 2{m_1}^2 - 2{m_1}^4, c_1 = -1$, and $0 < m_1 < 1$, then

$$
f_8\left(\varsigma\right) = ds\left(\varsigma, m_1\right) = \frac{dn\left(\varsigma, m_1\right)}{sn\left(\varsigma, m_1\right)}, \qquad g_8\left(\varsigma\right) = \frac{cn\left(\varsigma, m_1\right)}{sn\left(\varsigma, m_1\right) dn\left(\varsigma, m_1\right)}. \tag{3.30}
$$

From (2.2), (3.3), and (3.30), we have

$$
\begin{aligned}
u_8(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left({b_1}^2\left(1-2{m_1}^2\right)+{b_2}^2\left(1-2{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)\right)}}{\left(\frac{H_1 S_1}{L_2} + \frac{H_2 S_1 ds(-tv+b_1x+b_2y)}{L_2} - \frac{S_1 cn(-tv+b_1x+b_2y)}{dn(-tv+b_1x+b_2y)sn(-tv+b_1x+b_2y)}\right)} \\
& \times \Big(H_1 ds\left(-tv + b_1 x + b_2 y\right) + H_2 ds(-tv + b_1 x + b_2 y)^2 \\
& - \frac{L_2 cn\left(-tv + b_1 x + b_2 y\right) ds\left(-tv + b_1 x + b_2 y\right)}{dn\left(-tv + b_1 x + b_2 y\right) sn\left(-tv + b_1 x + b_2 y\right)}\Big).
\end{aligned} \tag{3.31}
$$

Family-9. When $q_1 = -2 + {m_1}^2, r_1 = -2, c_1 = -1 + {m_1}^2$, and $0 < m_1 < 1$, then

$$
f_9\left(\varsigma\right) = sc\left(\varsigma, m_1\right) = \frac{sn\left(\varsigma, m_1\right)}{cn\left(\varsigma, m_1\right)}, \; g_9\left(\varsigma\right) = \frac{dn\left(\varsigma, m_1\right)}{sn\left(\varsigma, m_1\right) cn\left(\varsigma, m_1\right)}. \tag{3.32}
$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.32), we have

$$
\begin{aligned}
u_9(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left({b_1}^2\left(-2+{m_1}^2\right)+{b_2}^2\left(-2+{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)\right)}}{\left(\frac{H_1 S_1}{L_2} + \frac{H_2 S_1 sc(-tv+b_1x+b_2y)}{L_2} + \frac{S_1 dn(-tv+b_1x+b_2y)}{cn(-tv+b_1x+b_2y)sn(-tv+b_1x+b_2y)}\right)} \\
& \times \Big(H_1 sc\left(-tv + b_1 x + b_2 y\right) + H_2 sc(-tv + b_1 x + b_2 y)^2 \\
& + \frac{L_2 dn\left(-tv + b_1 x + b_2 y\right) sc\left(-tv + b_1 x + b_2 y\right)}{cn\left(-tv + b_1 x + b_2 y\right) sn - tv + b_1 x + b_2 y}\Big).
\end{aligned} \tag{3.33}
$$

When $m \to 1$, $sc\left(\varsigma, m_1\right) \to sinh\left(\varsigma\right)$, $\frac{dn(\varsigma,m_1)}{sn(\varsigma,m_1)cn(\varsigma,m_1)} \to coth\left(\varsigma\right)$, causes to

$$\begin{aligned}\psi_{9,1}(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-{b_1}^2-{b_2}^2+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+S_1\coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{H_2S_1\sinh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}\right)}\\ & \times\left(L_2\cosh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+H_1\sinh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right.\\ & \left.+H_2\sinh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2\right).\end{aligned} \tag{3.34}$$

When $m \to 0$, $sc\left(\varsigma, m_1\right) \to tan\left(\varsigma\right)$, $\frac{dn(\varsigma,m_1)}{sn(\varsigma,m_1)cn(\varsigma,m_1)} \to \frac{1}{sin(\varsigma)cos(\varsigma)}$, leads to

$$\begin{aligned}\psi_{9,0}(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-2{b_1}^2-2{b_2}^2+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\begin{array}{c}\frac{H_1S_1}{L_2}+S_1\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\sec\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\\ +\frac{H_2S_1\tan(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}\end{array}\right)}\\ & \times\left(L_2\sec\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2+H_1\tan\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right.\\ & \left.+H_2\tan\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2\right).\end{aligned} \tag{3.35}$$

Family-10. When $q_1 = 1+{m_1}^2, r_1 = -2{m_1}^2, c_1 = -1$, and $0 < m_1 < 1$, then

$$f_{10}\left(\varsigma\right) = dc\left(\varsigma, m_1\right) = \frac{dn\left(\varsigma, m_1\right)}{cn\left(\varsigma, m_1\right)},\ g_{10}\left(\varsigma\right) = \left(1-{m_1}^2\right)\frac{sn\left(\varsigma, m_1\right)}{cn\left(\varsigma, m_1\right)dn\left(\varsigma, m_1\right)}. \tag{3.36}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.36), we have

$$\begin{aligned}u_{10}(x,y,t) = & \frac{e^{i\left(\lambda_1x+\lambda_2y-t\beta\left({b_1}^2\left(1+{m_1}^2\right)+{b_2}^2\left(1+{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+\frac{H_2S_1dc(-tv+b_1x+b_2y)}{L_2}+\frac{(1-{m_1}^2)S_1sn(-tv+b_1x+b_2y)}{cn(-tv+b_1x+b_2y)dn(-tv+b_1x+b_2y)}\right)}\\ & \times\left(H_1dc\left(-tv+b_1x+b_2y\right)+H_2dc(-tv+b_1x+b_2y)^2\right.\\ & \left.+\frac{L_2\left(1-{m_1}^2\right)dc\left(-tv+b_1x+b_2y\right)sn\left(-tv+b_1x+b_2y\right)}{cn\left(-tv+b_1x+b_2y\right)dn\left(-tv+b_1x+b_2y\right)}\right).\end{aligned} \tag{3.37}$$

Family-11. When $q_1 = 1-2{m_1}^2, r_1 = -2, c_1 = {m_1}^2-{m_1}^4$, and $0 < m_1 < 1$, then

$$f_{11}\left(\varsigma\right) = sd\left(\varsigma, m_1\right) = \frac{sn\left(\varsigma, m_1\right)}{dn\left(\varsigma, m_1\right)},\ g_{11}\left(\varsigma\right) = \frac{cn\left(\varsigma, m_1\right)}{sn\left(\varsigma, m_1\right)dn\left(\varsigma, m_1\right)}. \tag{3.38}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.38), we have

$$\begin{aligned}u_{11}(x,y,t) = & \frac{e^{i\left(\lambda_1x+\lambda_2y-t\beta\left({b_1}^2\left(1-2{m_1}^2\right)+{b_2}^2\left(1-2{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+\frac{H_2S_1sd(-tv+b_1x+b_2y)}{L_2}+\frac{S_1cn(-tv+b_1x+b_2y)}{dn(-tv+b_1x+b_2y)sn(-tv+b_1x+b_2y)}\right)}\\ & \times\left(H_1sd\left(-tv+b_1x+b_2y\right)+H_2sd(-tv+b_1x+b_2y)^2\right.\\ & \left.+\frac{L_2cn\left(-tv+b_1x+b_2y\right)sd\left(-tv+b_1x+b_2y\right)}{dn\left(-tv+b_1x+b_2y\right)sn\left(-tv+b_1x+b_2y\right)}\right).\end{aligned} \tag{3.39}$$

Family-12. When $q_1 = \frac{-1+{m_1}^2}{2}, r_1 = \frac{-1}{2}, c_1 = \frac{-1}{4}$, and $0 < m_1 < 1$, then

$$f_{13}\left(\varsigma\right) = \frac{cn\left(\varsigma, m_1\right)\pm 1}{sn\left(\varsigma, m_1\right)},\ g_{13}\left(\varsigma\right) = \mp ds\left(\varsigma, m_1\right). \tag{3.40}$$

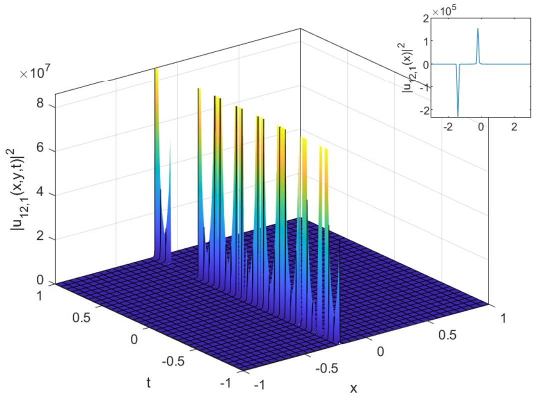

FIGURE 7. The graphic of Eq. (3.42) when $\theta = 0.5$, $\lambda_1 = 0.1$, $\lambda_2 = 1.5$, $\beta = 0.02$, $b_1 = 1$, $b_2 = 0.2$, $H_2 = 1$, $\alpha_1 = 1$, $H_1 = 2.5$, $S_1 = 0.3$ and $L_2 = 1$.

From Eq. (2.2), Eq. (3.3) and Eq. (3.40), we have

$$\begin{aligned} u_{12}(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left(\frac{1}{2}{b_1}^2\left(-1+2{m_1}^2\right)+\frac{1}{2}{b_2}^2\left(-1+2{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2} + S_1\left(\mp ds\left(-tv+b_1x+b_2y\right)\right) + \frac{H_2 S_1 (cn(-tv+b_1x+b_2y)\pm 1)}{L_2 sn(-tv+b_1x+b_2y)}\right)} \\ & \times \left(\frac{H_2 (cn\left(-tv+b_1x+b_2y\right)\pm 1)^2}{sn(-tv+b_1x+b_2y)^2} + \frac{H_1\left(cn\left(-tv+b_1x+b_2y\right)\pm 1\right)}{sn\left(-tv+b_1x+b_2y\right)}\right. \\ & \left. + \frac{L_2\left(\mp ds\left(-tv+b_1x+b_2y\right)\right)\left(cn\left(-tv+b_1x+b_2y\right)\pm 1\right)}{sn\left(-tv+b_1x+b_2y\right)}\right). \end{aligned} \tag{3.41}$$

When $m_1 \to 1$, $\frac{cn(\varsigma,m_1)\pm 1}{sn(\varsigma,m_1)} \to \coth(\varsigma) + \operatorname{csch}(\varsigma)$, $ds(\varsigma,m_1) \to \frac{\operatorname{sech}(\varsigma)}{\tanh(\varsigma)}$, causes to

$$\begin{aligned} u_{12,1}(x,y,t) = & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left(\frac{{b_1}^2}{2}+\frac{{b_2}^2}{2}+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\begin{array}{c}\frac{H_1 S_1}{L_2} + S_1 \operatorname{csch}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right) \\ + \frac{H_2 S_1 \coth(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))(1+\operatorname{sech}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))}{L_2}\end{array}\right)} \\ & \times \left(H_1 \coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\left(1+\operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)\right. \\ & + L_2 \coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\operatorname{csch}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right) \\ & \left(1+\operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right) + H_2 \coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2 \\ & \left.\left(1+\operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)^2\right). \end{aligned} \tag{3.42}$$

Graphical representation of Eq. (3.42) is shown in Figure 7.

When $m_1 \to 0$, $\frac{cn(\varsigma,m_1)\pm 1}{sn(\varsigma,m_1)} \to \cot(\varsigma) + \csc(\varsigma)$, $ds(\varsigma,m_1) \to \frac{1}{\sin(\varsigma)}$, causes to

$$u_{12,0}(x,y,t) = \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-\frac{b_1{}^2}{2}-\frac{b_2{}^2}{2}+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\begin{array}{c}\frac{H_1 S_1}{L_2}+S_1\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\\ +\frac{H_2S_1(1+\cos((b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))))\csc((b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))}{L_2}\end{array}\right)} \\ \times\left(H_1\left(1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right. \\ +L_2\left(1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2 \\ \left.+H_2(1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right))^2\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2\right). \tag{3.43}$$

Family-13. When $q_1 = -\frac{m_1{}^2+1}{2}, r_1 = \frac{1-m_1{}^2}{2}, c_1 = \frac{1-m_1{}^2}{4}$, and $0 < m_1 < 1$, then

$$f_{13}(\varsigma) = \frac{dn(\varsigma, m_1)}{1 \pm m_1 sn(\varsigma, m_1)},\ g_{13}(\varsigma) = \pm m_1 cd(\varsigma, m_1). \tag{3.44}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.46), we have

$$u_{13}(x,y,t) = \frac{e^{i\left(\lambda_1x+\lambda_2y-t\beta\left(\frac{1}{2}b_1{}^2\left(-1-m_1{}^2\right)+\frac{1}{2}b_2{}^2\left(-1-m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+(\mp m_1)S_1cd(-tv+b_1x+b_2y)+\frac{H_2S_1dn(-tv+b_1x+b_2y)}{L_2(m_1sn(-tv+b_1x+b_2y)\pm 1)}\right)} \\ \times\left(\frac{H_2dn(-tv+b_1x+b_2y)^2}{(m_1sn(-tv+b_1x+b_2y)\pm 1)^2}+\frac{H_1dn(-tv+b_1x+b_2y)}{m_1sn(-tv+b_1x+b_2y)\pm 1}\right. \\ \left.+\frac{(\mp m)L_2cd(-tv+b_1x+b_2y)dn(-tv+b_1x+b_2y)}{m_1sn(-tv+b_1x+b_2y)\pm 1}\right). \tag{3.45}$$

Family-14. When $q_1 = -\frac{1+m_1{}^2}{2}, r_1 = \frac{1-2m_1{}^2+m_1{}^4}{2}, c_1 = \frac{1}{4}$, and $0 < m_1 < 1$, then

$$f_{14}(\varsigma) = m_1cn(\varsigma, m_1) + dn(\varsigma, m_1),\ g_{14}(\varsigma) = -m_1sn(\varsigma, m_1). \tag{3.46}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.46), we have

$$u_{14}(x,y,t) = \frac{e^{i\left(\lambda_1x+\lambda_2y-t\beta\left(\frac{1}{2}b_1{}^2\left(-1-m_1{}^2\right)+\frac{1}{2}b_2{}^2\left(-1-m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+\frac{H_2S_1(m_1cn(-tv+b_1x+b_2y)+dn(-tv+b_1x+b_2y))}{L_2}-m_1S_1sn(-tv+b_1x+b_2y)\right)} \\ \times\left(H_1\left(m_1cn(-tv+b_1x+b_2y)+dn(-tv+b_1x+b_2y)\right)\right. \\ +H_2(m_1cn(-tv+b_1x+b_2y)+dn(-tv+b_1x+b_2y))^2 \\ \left.-L_2m_1\left(m_1cn(-tv+b_1x+b_2y)+dn(-tv+b_1x+b_2y)\right)sn(-tv+b_1x+b_2y)\right). \tag{3.47}$$

When $m_1 \to 1$, $m_1cn(\varsigma, m_1) + dn(\varsigma, m_1) \to 2sech(\varsigma)$, $m_1sn(\varsigma, m_1) \to tanh(\varsigma)$, causes to

$$u_{14,1}(x,y,t) = \frac{e^{i\left(\lambda_1x+\lambda_2y-t\beta\left(-b_1{}^2-b_2{}^2+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+\frac{2H_2S_1sech(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}-S_1\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)} \\ \times\left(2H_1\text{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+4H_2\text{sech}(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right))^2\right. \\ \left.-2L_2sech\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right). \tag{3.48}$$

Family-15. When $q_1 = -\frac{1+m_1{}^2}{2}, r_1 = \frac{1-2m_1{}^2+m_1{}^4}{2}, c_1 = \frac{-1}{4}$, and $0 < m_1 < 1$, then

$$f_{15}(\varsigma) = \frac{cn(\varsigma, m_1) \pm dn(\varsigma, m_1)}{sn(\varsigma, m_1)},\ g_{15}(\varsigma) = \mp ns(\varsigma, m_1). \tag{3.49}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.49), we have

$$\begin{aligned} u_{15}(x,y,t) = &\frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(\frac{1}{2}{b_1}^2\left(-1-{m_1}^2\right)+\frac{1}{2}{b_2}^2\left(-1-{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+S_1\left(\mp ns\left(-tv+b_1x+b_2y\right)\right)+\frac{H_2 S_1 (cn(-tv+b_1x+b_2y)\pm dn(-tv+b_1x+b_2y))}{L_2 sn(-tv+b_1x+b_2y)}\right)} \\ &\times\left(\frac{H_2(cn\left(-tv+b_1x+b_2y\right)\pm dn\left(-tv+b_1x+b_2y\right))^2}{sn(-tv+b_1x+b_2y)^2}\right. \\ &+\frac{H_1\left(cn\left(-tv+b_1x+b_2y\right)\pm dn\left(-tv+b_1x+b_2y\right)\right)}{sn\left(-tv+b_1x+b_2y\right)} \\ &\left.+\frac{L_2\left(\mp ns\left(-tv+b_1x+b_2y\right)\right)\left(cn\left(-tv+b_1x+b_2y\right)\pm dn\left(-tv+b_1x+b_2y\right)\right)}{sn\left(-tv+b_1x+b_2y\right)}\right). \end{aligned} \tag{3.50}$$

When $m_1 \to 1$, $\frac{cn(\varsigma,m_1)\pm dn(\varsigma,m_1)}{sn(\varsigma,m_1)} \to 2csch\left(\varsigma\right)$, $ns\left(\varsigma,m_1\right) \to \frac{1}{tanh(\varsigma)}$, causes to

$$\begin{aligned} u_{15,1}(x,y,t) = &\frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-{b_1}^2-{b_2}^2+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}-S_1\coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{2H_2 S_1 csch(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}\right)} \\ &\times\left(2H_1 csch\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+4H_2 csch(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right))^2\right. \\ &\left.-2L_2\coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)csch\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right). \end{aligned} \tag{3.51}$$

When $m_1 \to 0$, $\frac{cn(\varsigma,m_1)\pm dn(\varsigma,m_1)}{sn(\varsigma,m_1)} \to cot\left(\varsigma\right)+csc\left(\varsigma\right)$, $ns\left(\varsigma,m_1\right) \to \frac{1}{sin(\varsigma)}$, causes to

$$\begin{aligned} u_{15,0}(x,y,t) = &\frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-\frac{{b_1}^2}{2}-\frac{{b_2}^2}{2}+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\begin{array}{c}\frac{H_1 S_1}{L_2}-S_1\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right) \\ +\frac{H_2 S_1(1+\cos(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))\csc(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}\end{array}\right)} \\ &\times\left(H_1\left(1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right. \\ &-L_2\left(1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2 \\ &\left.+H_2(1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right))^2\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2\right). \end{aligned} \tag{3.52}$$

Graphical representation of Eq. (3.52) is shown in Figure 8.

Family-16. When $q_1 = {m_1}^2 - 6m_1 + 1, r_1 = -2, c_1 = 4m_1 - 8{m_1}^2 + 4{m_1}^3$, and $0 < m_1 < 1$, then

$$f_{16}\left(\varsigma\right) = \frac{sn\left(\varsigma,m_1\right)}{-1+m_1 sn(\varsigma,m_1)^2},\ g_{16}\left(\varsigma\right) = -cs\left(\varsigma,m_1\right)dn\left(\varsigma,m_1\right)\frac{m_1 sn(\varsigma,m_1)^2+1}{m_1 sn(\varsigma,m_1)^2-1}. \tag{3.53}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.53), we have

$$\begin{aligned} u_{16}(x,y,t) = &\frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left({b_1}^2\left(1-6m_1+{m_1}^2\right)+{b_2}^2\left(1-6m_1+{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+\frac{S_1 cn(-tv+b_1x+b_2y)dn(-tv+b_1x+b_2y)}{sn(-tv+b_1x+b_2y)}-\frac{H_2 S_1 cs(-tv+b_1x+b_2y)dn(-tv+b_1x+b_2y)\left(1+m_1 sn(-tv+b_1x+b_2y)^2\right)}{L_2\left(-1+m_1 sn-tv+b_1x+b_2y^2\right)}\right)} \\ &\times\left(-\frac{H_1 cs\left(-tv+b_1x+b_2y\right)dn\left(-tv+b_1x+b_2y\right)\left(1+m_1 sn(-tv+b_1x+b_2y)^2\right)}{-1+m_1 sn(-tv+b_1x+b_2y)^2}\right. \\ &-\frac{L_2 cn\left(-tv+b_1x+b_2y\right)cs\left(-tv+b_1x+b_2y\right)dn(-tv+b_1x+b_2y)^2\left(1+m_1 sn(-tv+b_1x+b_2y)^2\right)}{sn\left(-tv+b_1x+b_2y\right)\left(-1+m_1 sn(-tv+b_1x+b_2y)^2\right)} \\ &\left.+\frac{H_2 cs(-tv+b_1x+b_2y)^2 dn(-tv+b_1x+b_2y)^2\left(1+m_1 sn(-tv+b_1x+b_2y)^2\right)^2}{\left(-1+m_1 sn(-tv+b_1x+b_2y)^2\right)^2}\right). \end{aligned} \tag{3.54}$$

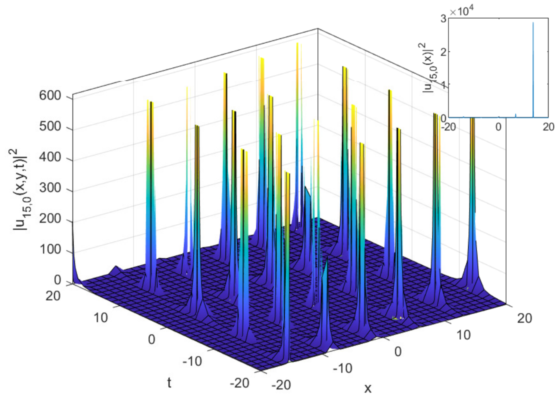

FIGURE 8. The graphic of Eq. (3.52) for $\theta = 0.5$, $\lambda_1 = 0.03$, $\lambda_2 = 0.5$, $\beta = 0.2$, $b_1 = 0.9$, $b_2 = 0.2$, $H_2 = 4$, $\alpha_1 = -2$, $H_1 = 2$, $S_1 = 4.3$ and $L_2 = 1.03$.

Family-17. When $q_1 = {m_1}^2 + 6m_1 + 1, r_1 = -2, c_1 = -4m_1 - 8{m_1}^2 - 4{m_1}^3$, and $0 < m_1 < 1$, then

$$f_{17}(\varsigma) = \frac{sn(\varsigma, m_1)}{1 + m_1 sn(\varsigma, m_1)^2},\ g_{17}(\varsigma) = -cs(\zeta, m_1)\, dn(\varsigma, m_1)\, \frac{m_1 sn(\varsigma, m_1)^2 - 1}{m_1 sn(\varsigma, m_1)^2 + 1}. \tag{3.55}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.55), we have

$$\begin{aligned} u_{17}(x,y,t) = {} & \frac{e^{i\left(\lambda_1 x + \lambda_2 y - t\beta\left(b_1^2\left(1+6m_1+m_1^2\right) + b_2^2\left(1+6m_1+m_1^2\right) + \lambda_1^2 + \lambda_2^2\right) + \theta\right)}}{\left(\frac{H_1 S_1}{L_2} + \frac{H_2 S_1 sn(-tv+b_1x+b_2y)}{L_2\left(1+m_1 sn(-tv+b_1x+b_2y)^2\right)} - \frac{S_1 cs(-tv+b_1x+b_2y) dn(-tv+b_1x+b_2y)\left(-1+m_1 sn(-tv+b_1x+b_2y)^2\right)}{1+m_1 sn(-tv+b_1x+b_2y)^2}\right)} \\ & \times \left(\frac{H_2 sn(-tv+b_1x+b_2y)^2}{\left(1+m_1 sn(-tv+b_1x+b_2y)^2\right)^2} + \frac{H_1 sn(-tv+b_1x+b_2y)}{1+m_1 sn(-tv+b_1x+b_2y)^2} - L_2 cs(-tv+b_1x+b_2y)\right. \\ & \left.\times \frac{dn(-tv+b_1x+b_2y)\, sn(-tv+b_1x+b_2y)\left(-1+m_1 sn(-tv+b_1x+b_2y)^2\right)}{\left(1+m_1 sn(-tv+b_1x+b_2y)^2\right)^2}\right). \end{aligned} \tag{3.56}$$

When $m_1 \to 1$, $\frac{sn(\varsigma, m_1)}{1+m_1 sn(\varsigma,m_1)^2} \to \frac{tanh(\varsigma)}{1+tanh(\varsigma)^2}$, $cs(\zeta, m_1)\, dn(\varsigma, m_1)\, \frac{m_1 sn(\varsigma,m_1)^2-1}{m_1 sn(\varsigma,m_1)^2+1} \to \frac{csch(\varsigma)\text{sech}(\varsigma)\left(-1+\tanh(\varsigma)^2\right)}{1+\tanh(\varsigma)^2}$, causes to

$$\begin{aligned} u_{17,1}(x,y,t) = {} & \frac{e^{i\left(\lambda_1 x + \lambda_2 y - t\beta\left(8b_1^2 + 8b_2^2 + \lambda_1^2 + \lambda_2^2\right) + \theta\right)}}{\left(\begin{array}{c} \frac{H_1 S_1}{L_2} + \frac{H_2 S_1 \tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2\left(1+\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2\right)} \\ -\frac{S_1 csch(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))\text{sech}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))\left(-1+\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2\right)}{1+\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2} \end{array}\right)} \\ & \times \left(\frac{H_2 \tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2}{\left(1+\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2\right)^2} + \frac{H_1 \tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{1+\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2}\right. \\ & \left. - \frac{L_2 \text{sech}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2\left(-1+\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2\right)}{\left(1+\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))^2\right)^2}\right). \end{aligned} \tag{3.57}$$

Family-18. When $q_1 = -\frac{1+{m_1}^2}{2}, r_1 = \frac{-1+{m_1}^2}{2}, c_1 = \frac{{m_1}^2-1}{4}$, and $0 < m_1 < 1$, then

$$f_{18}(\varsigma) = \frac{cn(\varsigma, m_1)}{\pm 1 + sn(\varsigma, m_1)^2},\ g_{20}(\varsigma) = \mp dc(\varsigma, m_1) \tag{3.58}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.58), we have

$$u_{18}(x,y,t) = \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(\frac{1}{2}{b_1}^2\left(-1-{m_1}^2\right)+\frac{1}{2}{b_2}^2\left(-1-{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+S_1\left(\mp dc\left(-tv+b_1x+b_2y\right)\right)+\frac{H_2S_1cn(-tv+b_1x+b_2y)}{L_2(sn(-tv+b_1x+b_2y)\pm 1)}\right)} \times\left(\frac{H_2cn(-tv+b_1x+b_2y)^2}{\left(sn\left(-tv+b_1x+b_2y\right)\pm 1\right)^2}+\frac{H_1cn\left(-tv+b_1x+b_2y\right)}{sn\left(-tv+b_1x+b_2y\right)\pm 1} +\frac{L_2cn\left(-tv+b_1x+b_2y\right)\left(\mp dc\left(-tv+b_1x+b_2y\right)\right)}{sn\left(-tv+b_1x+b_2y\right)\pm 1}\right). \tag{3.59}$$

When $m_1\to 1$, $\frac{cn(\varsigma,m_1)}{\pm 1+sn(\varsigma,m_1)^2}\to\frac{\operatorname{sech}(\varsigma)}{1+\tanh(\varsigma)}$, $dc\left(\varsigma,m_1\right)\to 1$, causes to

$$u_{18,1}(x,y,t) = \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-{b_1}^2-{b_2}^2+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{S_1+\frac{H_1S_1}{L_2}+\frac{H_2S_1\operatorname{sech}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2(1+\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))}} \left(\frac{H_2\operatorname{sech}(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right))^2}{\left(1+\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)^2}+\frac{H_1\operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}{1+\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)} +\frac{L_2\operatorname{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}{1+\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}\right) \tag{3.60}$$

When $m_1\to 0$, $\frac{cn(\varsigma,m_1)}{\pm 1+sn(\varsigma,m_1)^2}\to\frac{1}{sec(\varsigma)+\tan(\varsigma)}$, $dc\left(\varsigma,m_1\right)\to\frac{1}{cos(\varsigma)}$, leads to

$$u_{18,0}(x,y,t) = \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(-\frac{{b_1}^2}{2}-\frac{{b_2}^2}{2}+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}-S_1\sec\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{H_2S_1\cos(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2(1+\sin(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))}\right)} \times\left(\frac{H_2\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2}{\left(1+\sin\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)^2}-\frac{L_2}{1+\sin\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)} +\frac{H_1\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}{1+\sin\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}\right) \tag{3.61}$$

Family-19. When $q_1=\frac{2-{m_1}^2}{2}, r_1=-\frac{{m_1}^4}{2}, c_1=\frac{-1}{4}$, and $0<m_1<1$, then

$$f_{19}\left(\varsigma\right)=\frac{\pm 1+dn\left(\varsigma,m_1\right)}{sn(\varsigma,m_1)},\ g_{19}\left(\varsigma\right)=\mp cs\left(\varsigma,m_1\right). \tag{3.62}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.63), we have

$$u_{19}(x,y,t) = \frac{e^{i\left(\lambda_1 x+\lambda_2 y-t\beta\left(\frac{1}{2}{b_1}^2\left(2-{m_1}^2\right)+\frac{1}{2}{b_2}^2\left(2-{m_1}^2\right)+{\lambda_1}^2+{\lambda_2}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+S_1\left(\mp cs\left(-tv+b_1x+b_2y\right)\right)+\frac{H_2S_1(dn(-tv+b_1x+b_2y)\pm 1)}{L_2sn(-tv+b_1x+b_2y)}\right)} \times\left(\frac{H_2(dn\left(-tv+b_1x+b_2y\right)\pm 1)^2}{sn(-tv+b_1x+b_2y)^2}+\frac{H_1\left(dn\left(-tv+b_1x+b_2y\right)\pm 1\right)}{sn\left(-tv+b_1x+b_2y\right)} +\frac{L_2\left(\mp cs\left(-tv+b_1x+b_2y\right)\right)\left(dn\left(-tv+b_1x+b_2y\right)\pm 1\right)}{sn\left(-tv+b_1x+b_2y\right)}\right). \tag{3.63}$$

When $m_1 \to 1$, $\frac{\pm 1+dn(\varsigma,m_1)}{sn(\varsigma,m_1)} \to \coth(\varsigma)+csch(\varsigma)$, $cs(\varsigma,m_1) \to \frac{\text{sech}(\varsigma)}{tanh(\varsigma)}$, causes to

$$\begin{aligned}
u_{19,1}(x,y,t) = {} & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left(\frac{b_1^2}{2}+\frac{b_2^2}{2}+\lambda_1^2+\lambda_2^2\right)+\theta\right)}}{\left(\begin{array}{c}\frac{H_1S_1}{L_2}+S_1 csch\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\\ +\frac{H_2S_1\coth(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))(1+\text{sech}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))}{L_2}\end{array}\right)}\\
& \times\left(H_1\coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\left(1+\text{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)\right.\\
& +L_2\coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right) csch\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\\
& \left(1+\text{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)+H_2\coth\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2\\
& \left.\left(1+\text{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)^2\right).
\end{aligned} \tag{3.64}$$

When $m_1 \to 0$, $\frac{\pm 1+dn(\varsigma,m_1)}{sn(\varsigma,m_1)} \to 2csc(\varsigma)$, $cs(\varsigma,m_1) \to \frac{cos(\varsigma)}{sin(\varsigma)}$, causes to

$$\begin{aligned}
u_{19,0}(x,y,t) = {} & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left(b_1^2+b_2^2+\lambda_1^2+\lambda_2^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}-S_1\cot\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{2H_2S_1\csc(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2}\right)}\\
& \times\left(2H_1\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)-2L_2\cot\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right.\\
& \left.\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+4H_2\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2\right).
\end{aligned} \tag{3.65}$$

Family-20. When $q_1 = \frac{-1+2m_1^2}{2}, r_1 = -\frac{1}{2}, c_1 = \frac{-1}{4}$, and $0 < m_1 < 1$, then

$$f_{20}(\varsigma) = \frac{sn(\varsigma,m_1)}{1\pm cn(\varsigma,m_1)},\ g_{20}(\varsigma) = \pm ds(\varsigma,m_1) \tag{3.66}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.66), we have

$$\begin{aligned}
u_{20}(x,y,t) = {} & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left(\frac{1}{2}b_1^2\left(-1+2m_1^2\right)+\frac{1}{2}b_2^2\left(-1+2m_1^2\right)+\lambda_1^2+\lambda_2^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+S_1\left(\pm ds\left(-tv+b_1x+b_2y\right)\right)+\frac{H_2S_1 sn(-tv+b_1x+b_2y)}{L_2(1\pm cn(-tv+b_1x+b_2y))}\right)}\\
& \times\left(\frac{H_1 sn\left(-tv+b_1x+b_2y\right)}{1\pm cn\left(-tv+b_1x+b_2y\right)}+\frac{L_2\left(\pm ds\left(-tv+b_1x+b_2y\right)\right) sn\left(-tv+b_1x+b_2y\right)}{1\pm cn\left(-tv+b_1x+b_2y\right)}\right.\\
& \left.+\frac{H_2 sn\left(-tv+b_1x+b_2y\right)^2}{\left(1\pm cn\left(-tv+b_1x+b_2y\right)\right)^2}\right).
\end{aligned} \tag{3.67}$$

When $m_1 \to 1$, $\frac{sn(\varsigma,m_1)}{1\pm cn(\varsigma,m_1)} \to \frac{tanh(\varsigma)}{1\pm\text{sech}(\varsigma))}$, $ds(\varsigma,m_1) \to \frac{\text{sech}(\varsigma)}{tanh(\varsigma)}$, causes to

$$\begin{aligned}
u_{20,1}(x,y,t) = {} & \frac{e^{i\left(\lambda_1 x+\lambda_2 y - t\beta\left(\frac{b_1^2}{2}+\frac{b_2^2}{2}+\lambda_1^2+\lambda_2^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+S_1\text{csch}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{H_2S_1\tanh(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2(1+\text{sech}(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))}\right)}\\
& \times\left(\frac{L_2\text{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}{1+\text{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}+\frac{H_1\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}{1+\text{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}\right.\\
& \left.+\frac{H_2\tanh\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2}{\left(1+\text{sech}\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)^2}\right).
\end{aligned} \tag{3.68}$$

When $m_1 \to 0$, $\frac{sn(\varsigma,m_1)}{1\pm cn(\varsigma,m_1)} \to \frac{sin(\varsigma)}{1\pm cos(\varsigma,m_1)}$, $ds\,(\varsigma,m_1) \to \frac{1}{sin(\varsigma)}$, causes to

$$u_{20,0}(x,y,t) = \frac{\mathrm{e}^{\mathrm{i}\left(\lambda_1 x+\lambda_2 y-t\beta\left(-\frac{b_1{}^2}{2}-\frac{b_2{}^2}{2}+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}+S_1\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{H_2S_1\sin(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2(1+\cos(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))}\right)}$$
$$\left(\frac{L_2}{1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}+\frac{H_1\sin\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}{1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}\right.$$
$$\left.+\frac{H_2\sin\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2}{\left(1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)^2}\right). \tag{3.69}$$

Family-21. When $q_1 = \frac{-1-m_1{}^2}{2}, r_1 = -\frac{1}{2}, c_1 = -\frac{m_1{}^4-2m_1{}^2+1}{4}$, and $0 < m_1 < 1$, then

$$f_{21}\left(\varsigma\right) = \frac{sn\left(\varsigma,m_1\right)}{cn\left(\varsigma,m_1\right)\pm dn\left(\varsigma,m_1\right)},\ g_{21}\left(\varsigma\right) = \pm ns\left(\varsigma,m_1\right) \tag{3.70}$$

From Eq. (2.2), Eq. (3.3) and Eq. (3.70), we have

$$u_{21}(x,y,t) = \frac{\mathrm{e}^{\mathrm{i}\left(\lambda_1x+\lambda_2y-t\beta\left(\frac{1}{2}b_1{}^2\left(-1-m_1{}^2\right)+\frac{1}{2}b_2{}^2\left(-1-m_1{}^2\right)+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1S_1}{L_2}+S_1\left(\mp ns\left(-tv+b_1x+b_2y\right)\right)+\frac{H_2S_1sn(-tv+b_1x+b_2y)}{L_2(cn(-tv+b_1x+b_2y)\pm dn(-tv+b_1x+b_2y))}\right)}$$
$$\times\left(\frac{H_1sn\left(-tv+b_1x+b_2y\right)}{cn\left(-tv+b_1x+b_2y\right)\pm dn\left(-tv+b_1x+b_2y\right)}\right.$$
$$+\frac{L_2\left(\mp ns\left(-tv+b_1x+b_2y\right)\right)sn\left(-tv+b_1x+b_2y\right)}{cn\left(-tv+b_1x+b_2y\right)\pm dn\left(-tv+b_1x+b_2y\right)}$$
$$\left.+\frac{H_2sn(-tv+b_1x+b_2y)^2}{\left(cn\left(-tv+b_1x+b_2y\right)\pm dn\left(-tv+b_1x+b_2y\right)\right)^2}\right). \tag{3.71}$$

When $m_1 \to 0$, $\frac{sn(\varsigma,m_1)}{cn(\varsigma,m_1)\pm dn(\varsigma,m_1)} \to \frac{sin(\varsigma)}{cos(\varsigma)\pm 1}$, $ns\,(\varsigma,m_1) \to \frac{1}{sin(\varsigma)}$, causes to

$$u_{21,0}(x,y,t) = \frac{\mathrm{e}^{\mathrm{i}\left(\lambda_1 x+\lambda_2 y-t\beta\left(-\frac{b_1{}^2}{2}-\frac{b_2{}^2}{2}+\lambda_1{}^2+\lambda_2{}^2\right)+\theta\right)}}{\left(\frac{H_1 S_1}{L_2}-S_1\csc\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)+\frac{H_2S_1\sin(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2))}{L_2(1+\cos(b_1x+b_2y-2t\beta(b_1\lambda_1+b_2\lambda_2)))}\right)}$$
$$\left(-\frac{L_2}{1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}+\frac{H_1\sin\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}{1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)}\right.$$
$$\left.+\frac{H_2\sin\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)^2}{\left(1+\cos\left(b_1x+b_2y-2t\beta\left(b_1\lambda_1+b_2\lambda_2\right)\right)\right)^2}\right). \tag{3.72}$$

4. Conclusions

We have successfully applied the new modified unified auxiliary equation method to the (1+2)-dimensional Chiral non-linear Schrödinger equation which represents the wave in quantum field theory. Our approach has been successful in obtaining solutions for this equation. We give graphs of the solutions obtained in (3.11), (3.12), (3.17), (3.18), (3.21), (3.22), (3.42), and (3.52). The wave profile of hyperbolic function solution of Eq. (3.11) has been delineated in Figure 1 for $\theta = 0.5$, $\lambda_1 = 0.09$, $\lambda_2 = 1.5$, $\beta = 0.02$, $b_1 = 1$, $b_2 = 0.2$, $H_2 = 0.2$, $\alpha_1 = 1$, $H_1 = 2$, $S_1 = 0.3$ and $L_2 = 1.03$. The wave profile of periodic solution of Eq. (3.12) has been delineated in Figure 2 for $\theta = 0.5$, $\lambda_1 = 2$, $\lambda_2 = 1.5$, $\beta = 2$, $b_1 = 5$, $b_2 = 0.2$, $H_2 = 0.1$, $\alpha_1 = 3$, $H_1 = 2.5$, $S_1 = 0.3$ and $L_2 = 1.2$. The wave profile of dark-bright solution of Eq. (3.17) has been delineated in Figure 3 for $\theta = 0.5$, $\lambda_1 = 0.3$, $\lambda_2 = 1.5$, $\beta = 0.75$, $b_1 = 1.75$, $b_2 = 0.2$, $H_2 = 0.2$, $\alpha_1 = 2$, $H_1 = 0.5$, $S_1 = 0.3$ and $L_2 = 0.03$. The wave profile of trigonometric solution of Eq. (3.18) has been delineated in Figure 4 for $\theta = 0.5$, $\lambda_1 = 0.3$, $\lambda_2 = 1.5$, $\beta = 0.75$, $b_1 = 1.75$, $b_2 = 0.2$, $H_2 = 0.2$, $\alpha_1 = 2$, $H_1 = 0.5$,

$S_1 = 0.3$ and $L_2 = 0.03$. The wave profile of dark-bright solution of Eq. (3.21) has been delineated in Figure 5 for $\theta = 0.5$, $\lambda_1 = 0.09$, $\lambda_2 = 1.5$, $\beta = 0.02$, $b_1 = 1$, $b_2 = 0.2$, $H_2 = 1$, $\alpha_1 = 1$, $H_1 = 2$, $S_1 = 0.3$ and $L_2 = 1.03$. The wave profile of trigonometric solution of Eq. (3.22) has been delineated in Figure 6 for $\theta = 0.5$, $\lambda_1 = 0.09$, $\lambda_2 = 1.5$, $\beta = 0.02$, $b_1 = 1$, $b_2 = 0.2$, $H_2 = 0.001$, $\alpha_1 = 1$, $H_1 = 2$, $S_1 = 0.3$ and $L_2 = 1.03$. The wave profile of hyperbolic solution of Eq. (3.42) has been delineated in Figure 7 for $\theta = 0.5$, $\lambda_1 = 0.1$, $\lambda_2 = 1.5$, $\beta = 0.02$, $b_1 = 1$, $b_2 = 0.2$, $H_2 = 1$, $\alpha_1 = 1$, $H_1 = 2.5$, $S_1 = 0.3$ and $L_2 = 1$. The wave profile of hyperbolic solution of Eq. (3.52) has been delineated in Figure 8 for $\theta = 0.5$, $\lambda_1 = 0.03$, $\lambda_2 = 0.5$, $\beta = 0.2$, $b_1 = 0.9$, $b_2 = 0.2$, $H_2 = 4$, $\alpha_1 = -2$, $H_1 = 2$, $S_1 = 4.3$ and $L_2 = 1.03$. The obtained solutions are expressed as singular, periodic, dark-bright, exponential, trigonometric, and hyperbolic, Jacobi elliptic function solutions under certain parametric constraints. Lastly, the solutions demonstrate how to accurately control wave amplitude and width using to parameters such as hyperbolic, trigonometric, Jacobi elliptic, and others solutions. In particular, this model is helpful for comprehending chirality and wave dynamics. Its understanding of wave behavior makes it a useful instrument for enhancing the design of optical fibers. The findings of this study will be helpful to researchers in the domains of fluid mechanics, plasma physics, and nonlinear optics in the future. Understanding the basic physical principles of evolutionary processes is expected to benefit from the answers found.

5. Discuss and limitations of the proposed method

The new modified unified auxiliary equation method is one of the most power full method to solve all types of nonlinear partial differential equation such as KdV type and complex Schrodinger type equations. Furthermore, it can be applied to nonlinear PDEs including integrable and non-integrable equations and systems under various transformations. Multiple solution types (such as exponential, hyperbolic, Jacobi elliptic functions and trigonometric solutions) may be obtain through the new modified unified auxiliary equation method. This method provides unambiguous solutions, which aids theoretical understanding when compared to numerical approaches. There are several challenges. The complexity of the algebraic system increases with the polynomial's degree, making the solution increasingly challenging to solve. The best technique for getting multi-soliton solutions to nonlinear differential equations is Hirota's approach. Other techniques, such as the Painleve expansion method, Bäcklund transformation, Darboux transformation, or inverse scattering transformation, can also be used to study soliton solutions. In particular, the inverse scattering method is a particularly effective way to get exact solutions to nonlinear equations; yet, it takes some difficult work to apply to real-world issues. However, in this regard, Hirota's approach is far more adjustable.

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