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Lie bifurcation theory of the full extended Korteweg-de Vries equation

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Abstract

In this paper, the geometric method of symmetry is used to study the full extended Korteweg-de Vries equation: $u_t + 6uu_x + u_{xxx} + a_1u^2u_x + a_2uu_{xxx} + a_3u_xu_{xx} + a_4u_{xxxxx} = 0$. Based on the different states of the a_i s parameters, the problem is divided into different branches. While carefully examining each of these cases, the geometric properties of the set of solutions are studied and specific solutions are obtained for different cases.

Keywords. Bifurcation, Lie Transformation, Invariant Solution, Group Classifications.
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1. INTRODUCTION

In this paper, the geometric method of symmetry is used to study the full extended Korteweg-de Vries equation:

$$\text{eKdV}: \; u_t + 6uu_x + u_{xxx} + a_1u^2u_x + a_2uu_{xxx} + a_3u_xu_{xx} + a_4u_{xxxxx} = 0,$$

where a_1, a_2, a_3, and a_4 are arbitrary constants. The Korteweg-de Vries equation arises in the description of weakly nonlinear, long wave-length water-waves when terms of second order in wave amplitude in the water-wave equations are included [4, 5, 9]. Based on the different states of the a_i s parameters, the problem is divided into different branches. While carefully examining each of these cases, the geometric properties of the set of solutions are studied and specific solutions are obtained for different cases. All eKdV equations have a symmetric group with at least two dimensions. Some of them have the third order symmetric group and only the KdV equation has the fourth order symmetric group. The larger this group, the more geometric properties and exact solutions of the equation can be found.

2. LIE BIFURCATION ANALYSIS

This method is very successful and in many cases has led to useful facts. The method was first proposed by Sophus Lie (1842–1899) and is still being developed and used. We use the modern form of this theory, the way in which Peter Olver elaborated on [6] and described in [1, 2].

Suppose $v = T\,\partial_t + X\,\partial_x + U\,\partial_u \in \mathcal{X}(J^0(\mathbb{R}^2, \mathbb{R}))$ is a symmetry (infinitesimal generator for a group of symmetries) for the equation eKdV, where X, Y, and U are smooth functions of t, x and u. First we have to prolonged v to 5th order: $v^{(5)} \in \mathcal{X}(J^5(\mathbb{R}^3, \mathbb{R}))$. The result will be:

$$v^{(5)} = v + U^0\partial_{u_t} + U^1\partial_{u_x} + U^2\partial_{u_{xx}} + U^3\partial_{u_{xxx}} + U^5\partial_{u_{xxxxx}},$$

where

$$U^0 = D_tQ - Tu_{tt} - Xu_{tx} = U_t + U_uu_t - (T_uu_t + T_t)u_t - (X_uu_t + X_t)u_x,$$
$$U^1 = D_xQ - Tu_{xt} - Xu_{xx} = U_x + U_uu_x - (T_uu_x + T_x)u_t - (X_uu_x + X_x)u_x,$$

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$$\cdots$$

$$U^5 = D_x^5 Q - T u_{xxxxxt} - X u_{xxxxxx} = 5u_x T_{uu} u_t u_{xxxx} - 10 T_{uuuu} u_t u_x^3 u_{xx} + \cdots - 20 T_{xu,u,u} u_x^3 u_{tx},$$

and $Q = U - X u_x - T u_t$. The effect of $v^{(5)}$ on the eKdV equation is as follows:

$$\begin{aligned} &\Big(3a_1 u^2 u_x^2 T_{xxu} + 3a_3 u_x^2 u_{xx} T_{xxu} + \cdots + 30 a_2 a_4 u u_x^2 u_{xx} u_{xxx} T_{xuuu}\Big) u_t + \cdots \\ &\quad + \Big(3a_1 u^2 u_x^2 T_{xxu} + 3a_3 u_x^2 u_{xx} T_{xxu} + \cdots + 30 a_2 a_4 u u_x^2 u_{xxx} T_{xuuu}\Big) u_{xxxxxxx}. \end{aligned} \tag{2.1}$$

By placing $u_t = -6uu_x + u_{xxx} - \cdots - a_4 u_{xxxxx}$ in (2.1), we get a polynomial of higher order in terms of $u_x, u_{tt}, u_{tx}, u_{xx}, \cdots$ variables. Because X, Y, U are functions of t, x, u, this polynomial is only zero if the coefficients of this polynomial are all zero. The result is a set of partial equations composed of 84 first-order and homogeneous linear equations, which is called the determining system of the equation eKdV:

$$\begin{aligned} &U_t + (a_1 u_x^2 + 6u) U_x + (a_2 u + 1) U_{xxx} + a_4 U_{xxxxx} = 0, \\ &a_4 X_{uu} = 0, \qquad \cdots \qquad 3(a_2 u + 1) T_{xx} + 5 a_4 T_{xxxx} = 0. \end{aligned} \tag{2.2}$$

In the exact solution of this system of PDEs, depending on the conditions on a_1, a_2, a_3, and a_4, we perform the following 17 cases as follows:

Case 1: $a_4 \neq 0$, $a_3 \neq 0$, **and** $a_1 \neq 3a_2$. By simplifying the determining system (2.2) for this particular case, we conclude that

$$T_t = T_x = T_u = 0, \quad X_t = X_x = X_u = 0, \quad U = 0. \tag{2.3}$$

The general solution is

$$T = c_1, \quad X = c_2, \quad U = 0, \qquad c_1, c_2 \in \mathbb{R}. \tag{2.4}$$

Thus, the equation eKdV accepts the two-dimensional symmetry group G_1 with infinitesimal generators $v_1 = \partial_t$ and $v_2 = \partial_x$. The action of the group G_1 on $(t, x, u)-$space is as follows:

$$(s_1, s_2) \cdot (t, x, u) = \big(t + s_1, x + s_2, u\big).$$

Case 2: $a_4 \neq 0$, $a_3 \neq 0$, $a_1 = 3a_2$. By simplifying the determining system (2.2) for this particular case, we conclude that

$$\begin{aligned} &T_t = \frac{-5a_2 U}{2(a_2 u + 1)}, \quad T_x = T_u = 0, \qquad X_t = \frac{6U}{a_2 u + 1}, \\ &X_x = \frac{-a_2 U}{2(a_2 u + 1)}, \quad X_u = U_t = U_x = 0, \quad U_u = \frac{a_2 U}{a_2 u + 1}. \end{aligned}$$

The general solution is $T = -5c_1 a_2 t + c_2$, $X = 2c_1(12t - a_2 x) + c_3$, and $U = c_1(a_2 u + 1)$, where $c_1, c_2, c_3 \in \mathbb{R}$. Thus, the equation eKdV accepts the $3-$dimensional symmetry group G_2 with infinitesimal generators

$$v_1 = \partial_t, \qquad v_2 = \partial_x, \qquad v_3 = -5a_2 t\, \partial_t + 2(12t - a_2 x)\, \partial_x + (a_2 u + 1)\, \partial_u.$$

The action of the group G_2 on $(t, x, u)-$space is as follows:

$$(s_1, s_2, s_3) \cdot (t, x, u) = \Big(te^{-5a_2 s_3} + s_1, \frac{(xa_2 + 8t)e^{-2a_2 s_3} - 8te^{-5a_2 s_3}}{a_2} + s_2, s_3(a_2 u + 1) + u\Big).$$

Case 3: $a_4 \neq 0$, $a_3 = 0$, $a_1 \neq 0$, $a_2 \neq 0$, $a_1 \neq 3a_2$. This case is exactly the same as Case 1.

Case 4: $a_4 \neq 0$, $a_3 = 0$, $a_1 \neq 0$, $a_2 \neq 0$, $a_1 = a_2$. This case is exactly the same as Case 1.

Case 5: $a_4 \neq 0$, $a_3 = 0$, $a_1 \neq 0$, $a_2 = 0$. This case is exactly the same as Case 1.

Case 6: $a_4 \neq 0$, $a_3 = 0$, $a_1 = 0$, $a_2 \neq 0$. This case is exactly the same as Case 1.

Case 7: $a_4 \neq 0$**,** $a_3 = 0$**,** $a_1 = 0$**,** $a_2 = 0$**.** By simplifying the determining system (2.2) for this particular case, we conclude that

$$T_t = 0, \qquad X_t = 6U, \qquad U_t = T_x = X_x = U_x = T_u = X_u = U_u = 0.$$

The general solution is $T = c_1$, $X = 6c_2t + c_3$, and $U = c_2$, where $c_1, c_2, c_3 \in \mathbb{R}$. Thus, the equation eKdV accepts the 3-dimensional symmetry group G_3 with infinitesimal generators $v_1 = \partial_t$, $v_2 = \partial_x$, $v_3 = 6t\,\partial_x + \partial_u$. The action of the group G_3 on $(t, x, u)-$space is as follows:

$$(s_1,s_2,s_3) \cdot (t,x,u) = \big(t + s_1, x + 6s_3t + s_2, u + s_3\big).$$

Case 8: $a_4 = 0$**,** $a_3(3a_2 - a_3) \neq 0$**,** $a_2 \neq 0$**,** $a_1(3a_2 - a_1) \neq 0$**.** By simplifying the determining system (2.2) for this particular case, we conclude that

$$\begin{array}{lll} T_t = -\dfrac{(5a_1a_2u + 3a_1 + 6a_2)U}{2(a_1u + 3)(a_2u + 1)}, & T_x = T_u = 0, & X_t = \dfrac{6U}{ua_2 + 1}, \\ X_x = \dfrac{-a_1U}{2(a_1u + 3)}, & X_u = U_t = U_x = 0, & U_u = \dfrac{a_2U}{a_2u + 1}, \end{array}$$

The general solution is (2.3), and the equation eKdV accepts the two-dimensional symmetry group with infinitesimal generators (2.4).

Case 9: $a_4 = 0$**,** $a_3(3a_2 - a_3) \neq 0$**,** $a_2 \neq 0$**,** $a_1(3a_2 - a_1) = 0$**.** This case is exactly the same as Case 1.

Case 10: $a_4 = 0$**,** $a_3(3a_2 - a_3) \neq 0$**,** $a_2 = 0$**,** $a_1 \neq 0$**.** By simplifying the determining system (2.2) for this particular case, we conclude that

$$T_t = T_x = T_u = 0, \qquad X_t = 6U, \qquad X_x = X_u = U_t = U_x = U_u = 0.$$

The general solution is $T = c_1$, $X = 6c_2t + c_3$, and $U = c_2$, where $c_1, c_2, c_3 \in \mathbb{R}$. Thus, the equation eKdV accepts the $3-$dimensional symmetry group G_3 with infinitesimal generators $v_1 = \partial_t$, $v_2 = \partial_x$, $v_3 = 6t\,\partial_x + \partial_u$. The action of the group G_3 on $(t, x, u)-$space is as follows:

$$(s_1,s_2,s_3) \cdot (t,x,u) = \big(t + s_1, x + 6s_3t + s_2, u + s_3\big).$$

Case 11: $a_4 = 0$**,** $a_3(3a_2 - a_3) \neq 0$**,** $a_2 = 0$**,** $a_1 = 0$**.** This case is exactly the same as Case 1.

Case 12: $a_4 = 0$**,** $a_3(3a_2 - a_3) = 0$**,** $a_3 \neq 0$**,** $a_1(a_1 - a_3) = 0$**.** By simplifying the determining system (2.2) for this particular case, we conclude that

$$\begin{array}{lll} T_t = -\dfrac{(5a_1a_3u + 9a_1 + 6a_3)U}{2(a_1u + 3)(a_3u + 3)}, & T_x = T_u = 0, & X_t = \dfrac{18U}{a_3u + 3}, \\ X_x = -\dfrac{a_1U}{2(a_1u + 3)}, & X_u = U_t = U_x = 0, & U_u = \dfrac{a_3U}{a_3u + 3}. \end{array}$$

The general solution is (2.3), and the equation eKdV accepts the two-dimensional symmetry group with infinitesimal generators (2.4).

Case 13: $a_4 = 0$**,** $a_3(3a_2 - a_3) = 0$**,** $a_3 \neq 0$**,** $a_1(a_1 - a_3) \neq 0$**.** This case is exactly the same as Case 1.

Case 14: $a_4 = 0$**,** $a_3(3a_2 - a_3) = 0$**,** $a_3 = 0$**,** $a_2 \neq 0$**,** $a_1(3a_2 - a_1) = 0$**.** By simplifying the determining system (2.2) for this particular case, we conclude that

$$\begin{array}{lll} T_t = \dfrac{-(5a_1a_2u + 3a_1 + 6a_2)U}{2(a_1u + 3)(a_2u + 1)}, & T_x = T_u = 0, & X_t = \dfrac{6U}{a_2u + 1}, \\ X_x = \dfrac{-a_1U}{2(a_1u + 3)}, & X_u = U_t = U_x = 0, & U_u = \dfrac{Ua_2}{a_2u + 1}, \end{array}$$

The general solution is (2.3), and the equation eKdV accepts the two-dimensional symmetry group with infinitesimal generators (2.4).

Case 15: $a_4 = 0$**,** $a_3(3a_2 - a_3) = 0$**,** $a_3 = 0$**,** $a_2 \neq 0$**,** $a_1(3a_2 - a_1) \neq 0$**.** This case is exactly the same as Case 1.

Case 16: $a_4=0$, $a_3(3a_2-a_3)=0$, $a_3=0$, $a_2=0$, $a_1\neq 0$. By simplifying the determining system (2.2) for this particular case, we conclude that

$$\begin{aligned} &T_t=\frac{-3a_1U}{a_1u+3}, && T_x=T_u=0, && X_t=\frac{18U}{a_1u+3},\\ &X_x7=\frac{-a_1U}{a_1u+3}, && X_u=U_t=U_x=0, && U_u=\frac{a_1U}{a_1u+3}. \end{aligned}$$

The general solution is $T=-3c_1a_1t+c_2$, $X=c_1(18t-a_1x)+c_3$, and $U=c_1a_1u+3c_1$, where $c_1,c_2,c_3\in\mathbb{R}$. Thus, the equation eKdV accepts the $3-$dimensional symmetry group G_4 with infinitesimal generators

$$v_1=\partial_t,\qquad v_2=\partial_x,\qquad v_3=-3a_1t\,\partial_t+(18t-a_1x)\,\partial_x+a_1u\,\partial_u.$$

The action of the group G_4 on $(t,x,u)-$space is as follows:

$$(s_1,s_2,s_3)\cdot(t,x,u)=\Big(te^{-3a_1s_3}+s_1,\frac{(a_1x+9t)e^{-a_1s_3}-9te^{-3a_1s_3}}{a_1}+s_2,e^{a_1s_3}u\Big).$$

Case 17: $a_4=0$, $a_3(3a_2-a_3)=0$, $a_3=0$, $a_2=0$, $a_1=0$. By simplifying the determining system (2.2) for this particular case, we conclude that

$$\begin{aligned} &T_t=-\frac{3}{2}U_u, && T_x=T_u=0, && X_t=-6uU_u+6U,\\ &X_x=-\frac{1}{2}U_u, && X_u=U_t=U_x=0, && U_{uu}=0, \end{aligned}$$

The general solution is $T=3c_1t+c_3$, $X=6c_2t+c_1x+c_4$, and $U=-c_1u+c_2$, where $c_1,c_2,c_3,c_4\in\mathbb{R}$. Thus, the equation eKdV accepts the $4-$dimensional symmetry group G_5 with infinitesimal generators

$$v_1=\partial_t,\qquad v_2=\partial_x,\qquad v_3=6t\,\partial_x+\partial_u,\qquad v_4=3t\,\partial_t+x\,\partial_x-2u\,\partial_u.$$

The action of the group G_5 on $(t,x,u)-$space is as follows:

$$(s_1,s_2,s_3,s_4)\cdot(t,x,u)=\Big(e^{3s_4}t+s_1,6s_3e^{3s_4}t+e^{s_4}x+s_2,e^{-s_4}u+s_3\Big).$$

The symmetry-bifurcation diagram related to the eKdV equation is given in the Figure 1. In each box $\boxed{X}$, the exit arrow pointing to the right and up means $X=0$ and pointing down means $X\neq 0$.

Therefore, the following Theorem was proved. This is the main point of this article, and many results are obtained from it.

Theorem 2.1. *The eKdV equation always assumes the two-dimensional symmetry group with infintesimal generators:* $v_1=\partial_t$ *and* $v_2=\partial_x$. *Furthermore, this equation is accepted only in the following states of the three-dimensional symmetry group:*

Case 2) $a_1=3a_2$, $a_4\neq 0$, $a_3\neq 0$. *In the other words, infinitesimal generators of symmetry group of the equation*

$$\text{eKdV}_1:\ u_t+6uu_x+u_{xxx}+a_2u(3uu_x+u_{xxx})+a_3u_xu_{xx}+a_4u_{xxxxx}=0,$$

with $a_3a_4\neq 0$, *are* v_1, v_2, *and* $v_3=-5a_2t\,\partial_t+2(12t-a_2x)\,\partial_x+(a_2u+1)\,\partial_u$.

Case 7) $a_1=a_2=a_3=0$, $a_4\neq 0$. *In the other words, infinitesimal generators of symmetry group of the equation*

$$\text{eKdV}_2:\ u_t+6uu_x+u_{xxx}+a_4u_{xxxxx}=0,$$

with $a_4\neq 0$, *are* v_1, v_2, *and* $v_3=6t\,\partial_x+\partial_u$.

Case 10) $a_1\neq 0$, $a_2=a_4=0$, $a_3\neq 0$. *In the other words, infinitesimal generators of symmetry group of the equation*

$$\text{eKdV}_3:\ u_t+6uu_x+u_{xxx}+a_1u^2u_x+a_3u_xu_{xx}=0,$$

with $a_1a_3\neq 0$, *are* v_1, v_2, *and* $v_3=6t\,\partial_x+\partial_u$. *Same as Case 7.*

Case 16) $a_1\neq 0$, $a_2=a_3=a_4=0$. *In the other words, infinitesimal generators of symmetry group of the equation*

$$\text{eKdV}_4:\ u_t+6uu_x+u_{xxx}+a_1u^2u_x=0,$$

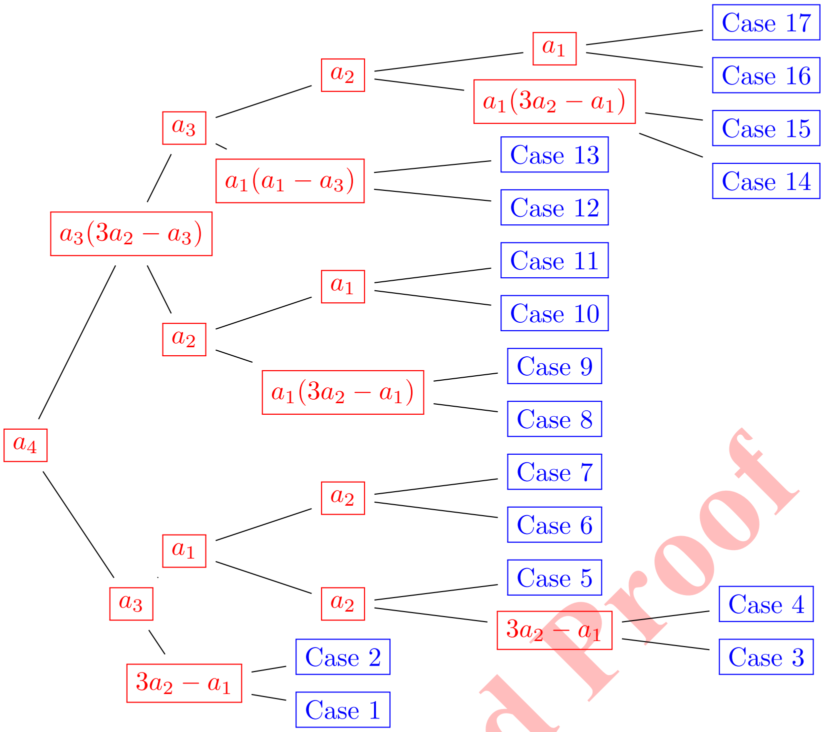

Figure 1. Different states in examining the determining system.

with $a_1 \neq 0$, *are* v_1, v_2, *and* $v_3 = -3a_1 t\,\partial_t + (18t - a_1 x)\,\partial_x + a_1 u\,\partial_u$.

And only in **Case 17** *is the symmetric group of this equation four-dimensional:* $a_1 = a_2 = a_3 = a_4 = 0$. *That is, for the classical KdV equation* $u_t + 6uu_x + u_{xxx} = 0$. *In this case, the infinitesimal generators of symmetry group of the equation are* v_1, v_2, $v_3 = 6t\,\partial_x + \partial_u$, *and* $v_4 = 3t\,\partial_t + x\,\partial_x - 2u\,\partial_u$. ☐

To improve readability and conciseness, we summarize these results in Table 1.

3. GEOMETRIC STRUCTURE OF SOLUTION SPACE

One of the results of this method is the extraction of some geometric properties of the set of solutions of the eKdV equation. For example, the symmetric group of the eKdV equation acts on the set of solutions. This means that if there is an answer to the equation and an element of the symmetric group, then the effect of the group element on the solutions still the solution. In this way, we get a geometric way to generate new solutions from existing solutions.

From the basic Theorem 2.1 the following Theorems are deduced.

Theorem 3.1. *The solutions of the eKdV equation with respect to the translations in the variables are invariant. That means, if* $u = f(t, x)$ *be a solution of the eKdV equation and* t_0, x_0 *are arbitrary numbers, then so is* $u = f(t+t_0, x+x_0)$.

Theorem 3.2. *The solutions of the eKdV*$_1$ *equation with respect to the* $3-$*dimensional action* G_2 *are invariant. That means, if* $u = f(t, x)$ *be a solution of the eKdV equation, then so is*

$$u = \lambda f\Big(\lambda^5 t + t_0, x + \frac{8\lambda^2}{a_2}(1 - \lambda^3)t + x_0\Big) + \frac{\lambda - 1}{\lambda a_2},$$

where $\lambda \neq 0$, t_0, *and* x_0 *are arbitrary constants.*

TABLE 1. Classification of Symmetry Algebras for the eKdV equation.

Group	**Parameter Conditions**	**Generators**
G_1 (2D)	**Generic Case:** All sets of parameters not listed below (Cases 1, 3–6, 8, 9, 11–15).	$v_1 = \partial_t,\ v_2 = \partial_x$
G_2 (3D)	**Case 2:** $a_4 \neq 0,\ a_3 \neq 0,\ a_1 = 3a_2$.	$v_1 = \partial_t,\ v_2 = \partial_x,$ $v_3 = -5a_2 t\partial_t + 2(12t - a_2 x)\partial_x + (a_2 u + 1)\partial_u$
G_3 (3D)	**Cases 7**: $a_1 = a_2 = a_3 = 0, a_4 \neq 0$. **Cases 10**: $a_2 = a_4 = 0, a_1 a_3 \neq 0$.	$v_1 = \partial_t,\ v_2 = \partial_x,$ $v_3 = 6t\partial_x + \partial_u$
G_4 (3D)	**Case 16:** $a_2 = a_3 = a_4 = 0, a_1 \neq 0$.	$v_1 = \partial_t,\ v_2 = \partial_x,$ $v_3 = -3a_1 t\partial_t + (18t - a_1 x)\partial_x + a_1 u\partial_u$
G_5 (4D)	**Case 17 (Classical KdV):** $a_1 = a_2 = a_3 = a_4 = 0$.	$v_1 = \partial_t,\ v_2 = \partial_x,$ $v_3 = 6t\partial_x + \partial_u,\ v_4 = 3t\partial_t + x\partial_x - 2u\partial_u$

Theorem 3.3. *The solutions of the eKdV$_2$ and eKdV$_3$ equations with respect to the 3−dimensional action G_3 are invariant. That means, if $u = f(t,x)$ be a solution of the eKdV$_2$ or eKdV$_3$ equation, then so is $u = f\big(t + t_0, x - 6st + x_0\big) + s$, where t_0, x_0, s are arbitrary constants.*

Theorem 3.4. *The solutions of the eKdV$_4$ equation with respect to the 3−dimensional action G_4 are invariant. That means, if $u = f(t,x)$ be a solution of the eKdV$_4$ equation, then so is*

$$u = \frac{1}{\lambda} f\Big(\lambda^3 t + t_0, x + \frac{9\lambda}{a_1}(1 - \lambda^2)t + x_0\Big),$$

where $\lambda \neq 0$, t_0, and x_0 are arbitrary constants. □

Theorem 3.5. *The solutions of the KdV equation with respect to the 4−dimensional action G_5 are invariant. That means, if $u = f(t,x)$ be a solution of the KdV equation, then so is $u = \lambda\, f\big(\lambda^3 t + t_0, 6s\lambda^3 t + \lambda^2 x + x_0\big) - s$, where $\lambda \neq 0$, t_0, x_0, and s are arbitrary constants.* □

4. SOME EXACT SOLUTIONS

In the case of partial differential equations, sometimes finding only one special solution can be very difficult and useful at the same time. One of the results of this method is the production of invariant solutions. That is, the solutions that are inaccurate to a particular subgroup of symmetries. For this purpose, it is enough to specify the subgroup (in fact, sub-Lie algebra) and apply invariant conditions to it. The most well-known type of these solutions are solitary (or wave-travelling) solutions, which are invariant to the group of translation of independent variables. Solarity solutions are so important in applications that they are still the answer if we move them in a certain direction. More precisely, if the equation has a solution in the form $u = f(x - ct)$, its translation in the $(1, c)$ direction is still the solution.

Since the infinitesimally generators of symmetry group the eKdV equation forms a Lie algebra, we conclude that

Corollary 4.1. *Any eKdV equation has solitary solutions. That is, if $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, then eKdV equation has a solution in the form of $u = f(x - ct)$.*

Proof. Since $I_1 = x - ct$ and $I_2 = u$ are inversions of the G_1 group action, by the Thorem 3.1, then $u = f(x - ct)$ is an invariant solution of the eKdV equation. □

While previous sections established the existence of invariant solutions, we now derive explicit closed-form solutions by solving the reduced ODEs for the traveling wave [7] reduction $u(t,x) = f(\xi)$ where $\xi = x - ct$.

Substituting $u = f(\xi)$ into the general eKdV equation yields the 5th-order ODE:

$$-cf' + 6ff' + f''' + a_1 f^2 f' + a_2 f f''' + a_3 f' f'' + a_4 f^{(5)} = 0.$$

Integrating once with respect to ξ and setting the integration constant to zero (assuming localized wave conditions $f, f', \dots \to 0$ as $\xi \to \pm\infty$), we obtain:

$$-cf + 3f^2 + f'' + \frac{a_1}{3} f^3 + a_4 f^{(4)} + \int (a_2 f f''' + a_3 f' f'') d\xi = 0. \tag{4.1}$$

The integral term can be resolved if we impose the constraint $a_2 = a_3$. Using the identity $(ff'')' = f'f'' + ff'''$, the equation integrates to:

$$-cf + 3f^2 + f'' + \frac{a_1}{3} f^3 + a_4 f^{(4)} + a_2\left(ff'' - \frac{1}{2}(f')^2\right) = 0.$$

This reduced ODE can be solved using auxiliary equation methods (e.g., assuming a solution of the form $\sum A_i \tanh^i \xi$).

Corollary 4.2. *Let's $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, then eKdV$_1$ equation has a solution in the form of*

(1) $u = f(x - ct)$,

(2) $u = \dfrac{1}{\sqrt[5]{t}} f\left(\dfrac{2a_2cx + 16ct - 1}{2t^{2/5}a_2c}\right) - \dfrac{1}{a_2}$,

(3) $u = \dfrac{1}{\sqrt[5]{5a_2ct - 1}} f\left(\dfrac{a_2^2cx + 8a_2ct - 4}{(5a_2ct - 1)^{2/5}ca_2^2}\right) + \dfrac{1}{a_2}$.

Proof. The fact (1) has already been proven. We prove the fact (2). Since the differential invariants of the $v = v_1 + cv_3$ vector field (infinitesimal generators of G_2) are

$$I_1 = \frac{a_2^2cx + 8a_2ct - 4}{(5a_2ct - 1)^{2/5}ca_2^2}, \qquad I_2 = \frac{\sqrt[5]{5a_2ct - 1}}{a_2}(a_2u + 1),$$

then $I_2 = f(I_1)$ is an invariant solution of the equation, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. Now it suffices to solve this equation in terms of u. The fact (3) is similar, with $v = v_2 + cv_3$. □

Corollary 4.3. *Let's $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, then eKdV$_2$ (eKdV$_3$, respectively) equation has a solution in the form of*

1) $u = f(x - ct)$, **2)** $u = ct + f\left(t^2 - \dfrac{x}{3c}\right)$, **3)** $u = f(t) + \dfrac{cx}{6ct + 1}$.

Proof. We use the same argument to prove the Corollary 4.2. Of course for vector fields: $v_1 + cv_2$, $v_1 + cv_3$, and $v_2 + cv_3$, where $v_1 = \partial_t$, $v_2 = \partial_x$ and $v_3 = 6t\,\partial_x + \partial_u$. □

Corollary 4.4. *Let's $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, then eKdV$_4$ equation has a solution in the form of*

1) $u = f(x - ct)$, **2)** $u = \dfrac{1}{\sqrt[3]{t}} f\left(\dfrac{ca_1x + 9ct - 1}{ca_1\sqrt[3]{t}}\right)$, **3)** $u = \dfrac{1}{\sqrt[3]{3ca_1t - 1}} f\left(\dfrac{ca_1^2x + 9ca_1t - 9}{ca_1^2\sqrt[3]{3ca_1t - 1}}\right)$.

Proof. We use the same argument to prove the Corollary 4.2. Of course for vector fields: $v_1 + cv_2$, $v_1 + cv_3$, and $v_2 + cv_3$, where $v_1 = \partial_t$, $v_2 = \partial_x$ and $v_3 = -3a_1t\,\partial_t + (18t - a_1x)\,\partial_x + a_1u\,\partial_u$. □

Corollary 4.5. *Let's $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, then KdV equation has a solution in the form of*

1) $u = f(x - ct)$, **2)** $u = ct + f(x - 3ct^2)$, **3)** $u = \sqrt[3]{(3ct + 1)^2}\, f\left(\dfrac{x}{\sqrt[3]{3ct + 1}}\right)$,

4) $u = f(t) + \dfrac{cx}{6ct + 1}$, **5)** $u = \sqrt[3]{t^2}\, f\left(\dfrac{cx + 1}{c\sqrt[3]{t}}\right)$, **6)** $u = \sqrt[3]{t^2}\, f\left(\dfrac{cx - 3t}{c\sqrt[3]{t}}\right) - \dfrac{1}{2c}$.

Proof. We use the same argument to prove the Corollary 4.2. Of course for vector fields: $v_1 + cv_2$, $v_1 + cv_3$, $v_1 + cv_4$, $v_2 + cv_3$, $v_2 + cv_4$, and $v_3 + cv_4$, where $v_1 = \partial_t$, $v_2 = \partial_x$, $v_3 = 6t\,\partial_x + \partial_u$ and $v_4 = 3t\,\partial_t + x\partial_x - 2u\,\partial_u$. □

Therefore, all five types of equations have solitary solutions.

5. Conclusion

In this work, we have performed a complete Lie symmetry classification of the full extended Korteweg–de Vries equation. We demonstrated that the equation always admits a two-dimensional symmetry group, but specific parameter constraints (Cases 2, 7, 10, 16, and 17) lead to higher-dimensional symmetry algebras, culminating in the four-dimensional group for the classical KdV equation.

We condensed the 17 parametric cases into a concise classification table, clarifying the bifurcation structure. Furthermore, we moved beyond the implicit reductions to derive the reduced ODEs for traveling wave solutions. Specifically, we showed that the integrability of the reduced ODE depends heavily on the relationship between a_2 and a_3, and we recovered the classical sech^2 soliton solution [3, 8] for the KdV limit. These results provide a geometric foundation for classifying the integrability and exact solutions of extended KdV models appearing in fluid dynamics.

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