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An inverse problem of finding an absorption coefficient in a one-dimensional parabolic differential equation

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Abstract

This paper addresses an inverse problem related to the one-dimensional heat equation, incorporating the initial temperature and information from the heat flux and temperature on one of the boundaries of the domain and a supplementary temperature measurement at an instant of time. To tackle this problem, we utilize a discretization method, introducing approximations for both the temperature distribution and absorption coefficient functions. These approximations are established using Legendre basis functions and the operational matrix of differentiation corresponding to the selected bases. Subsequently, these estimations are incorporated into the residual function and then the least squares technique is applied to transform the main problem into the solution of a nonlinear system of algebraic equations. Notably, our proposed algorithm ensures accurate satisfaction of the given initial and boundary conditions of the problem. We provide proof of the method's convergence and showcase its effectiveness through illustrative test examples.

Keywords. Inverse heat equation, Least squares technique, Absorption coefficient.
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1. Introduction

In this work, we deal with the following mathematical model for retrieving the pair $\{H(z,t), c(z)\}$:

$$H_t(z,t) + b(z,t)H_{zz}(z,t) = c(z)H(z,t) + F(z,t), \quad (z,t) \in \Omega, \tag{1.1}$$

$$H(z,0) = \phi(z), \quad 0 \le z \le L, \tag{1.2}$$

$$H(L,t) = h_1(t), \quad H_z(L,t) = h_2(t), \qquad 0 \le t \le T, \tag{1.3}$$

$$H(z,T_1) = \psi(z), \qquad 0 \le z \le L, \tag{1.4}$$

where the space and time variables are denoted by z and t and the distribution of temperature along a thin rod or wire of length L is stated by $H(z,t)$. It is assumed that in the heat conduction process with the time span T, the initial temperature of the rod is given by $\phi(z)$ and the Dirichlet and Neumann boundary conditions are available only at the extremity $z = L$ denoted as $h_1(t)$ and $h_2(t)$, respectively. Indeed, the boundary conditions (1.3) imply that we only access to the temperature and heat flux of the medium at one end.

Additional temperature measurement given by condition (1.4) at $0 < t = T_1 \le T$, expresses that the inverse problem is not overdetermined as the nonzero function $\psi(z)$ is used to reconstruct the unknown absorption coefficient $c(z)$. The functions $b(z,t)$ and $F(z,t)$ are supposed to be given input data in the domain $\Omega = \{(z,t)|\ 0 \le z \le L,\ 0 \le t \le T\}$ and the conditions (1.2)-(1.4) are continuous in corners, that is

$$\phi(L) = h_1(0),\ \psi(L) = h_1(T_1),\ \phi^{'}(L) = h_2(0),\ \psi^{'}(L) = h_2(T_1). \tag{1.5}$$

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In other physical applications such as the piezometric head in groundwater flow or porous media, the unknown function $H(z,t)$ stands for the pressure. Additionally, the physical quantities such as conductivity/diffusivity of the medium $(0,L)$ is termed by negative function $b(z,t)$, and $F(z,t)$ represents the heat or pressure hydraulic source [3, 16].

There is a vast literature documenting the numerical and analytical solutions of inverse coefficient problems for recovering the unknown absorption/perfusion coefficients. Retrieving unknown coefficients is important in understanding and more accurately describing physical and engineering phenomena such as heat transfer in biological tissues, groundwater flow guidance, and oil recovery. Many authors have tried to deal with ill-posedness, as an associated difficulty with such problems, by establishing conditions for existence or uniqueness of a solution in the desired class of function spaces or providing a solution that continuously depends on input data [34]. In [36], the authors considered the problem of recovering an absorption coefficient of a parabolic equation supplemented with Neumann and Robin boundary conditions and solved it by the method of quasireversibility. An optimal control framework was applied for solving the inverse problem of approximating the absorption coefficient in the heat equation with Neumann boundary condition [38]. An approach based on inverse Sturm-Liouville problem was proposed by [30] to detect the absorption coefficient in an parabolic equation with Dirichlet boundary condition. An iterative technique based on proper solution space was proposed in [37] where the authors considered an initial guess for the unknown absorption coefficient and then necessary updates were made to approach the true value. The authors proposed numerical techniques based on the finite difference method (FDM) for determining the time-dependent coefficients including the thermal conductivity and absorption coefficients [13, 14, 16]. The authors considered the inverse problem of retrieving an additive space-time-dependent perfusion coefficient in the parabolic heat equation, discussed the uniqueness of the solution and suggested the FDM for solving it numerically [15]. The author investigated the sufficient conditions for uniquely determining absorption coefficient in a degenerate parabolic equation in two classes of functions such as L_2 and L_∞ [19, 20]. Authors studied some inverse coefficient problems in one and two-dimensional cases where special types of boundary conditions are supplemented with such problems, proved the uniqueness of the solution and presented pseudo-spectral technique to solve the problems [1, 2, 4, 5, 17, 18]. Additionally, approximating the solely time-dependent perfusion coefficients in the parabolic partial differential equations has been studied in several papers. To name a few, we refer to the application of the method of lines [8], the tau and pseudo-spectral collocation techniques [7, 31, 32, 35], the Bernstein spectral method [27, 28], the Adomian decomposition technique [9], the He's variational iteration method [10] and the radial basis functions method [11].

Assuming that consistency conditions (1.5) hold and the input data of the problem (1.1)-(1.4) including the initial and boundary conditions are sufficiently smooth to guarantee a unique solution, the main focus of this note is to present a numerical solution to determine the absorption coefficient which accurately satisfies all the initial and boundary conditions. This means that we need a small number of basis functions to reach acceptable solutions, which subsequently results in solving smaller systems of algebraic equations, and the amount of calculations is reduced. The method is easy to implement and the convergence analysis of the solution is provided. Additionally, perturbed boundary data is dealt by utilizing an appropriate regularization technique so that the robustness of the scheme is guaranteed.

2. Solution procedure

In this section, we provide an approximation based on the Legendre polynomials and we show that by increasing the number of approximation bases in a metric space equipped with a specific norm, the approximate solution approaches to its true value. First, by applying (1.4) in (1.1) we get

$$c(z) = \frac{H_t(z,T_1) + b(z,T_1)\psi''(z) - F(z,T_1)}{\psi(z)}, \tag{2.1}$$

provided $\psi(z) \neq 0$. Therefore, the governing Eq. (1.1) is modified as the following nonclassical parabolic equation

$$H_t(z,t) + b(z,t)H_{zz}(z,t) = \frac{H_t(z,T_1) + b(z,T_1)\psi''(z) - F(z,T_1)}{\psi(z)} \times H(z,t) + F(z,t), \quad (z,t) \in \Omega. \tag{2.2}$$

With the aid of satisfier function $S(z,t)$ given by

$$S(z,t) = h_1(t) - h_1(0) + (z-L)(h_2(t) - h_2(0)) + \phi(z),$$

and using it in transformation $v(z,t) = H(z,t) - S(z,t)$, the following system of equations is obtained

$$v_t(z,t) + b(z,t)v_{zz}(z,t) = A(z,v)(v(z,t) + S(z,t)) + F(z,t) - S_t(z,t) - b(z,t)\phi''(z), \quad (z,t) \in \Omega, \tag{2.3}$$

$$v(L,t) = v_z(L,t) = v(z,0) = 0, \quad 0 \le z \le L, \ 0 \le t \le T, \tag{2.4}$$

where

$$A(z,v) = \frac{b(z,T_1)\psi''(z) - F(z,T_1) + v_t(z,T_1) + S_t(z,T_1)}{\psi(z)},$$

and

$$S_t(z,t) = h_1'(t) + (z-L)h_2'(t).$$

We propose the solution of system (2.3)-(2.4) as

$$\hat{v}(z,t) = t(z-L)^2 P^\top(z) C Q(t), \tag{2.5}$$

such that the unknown matrix C includes the elements c_{ij} as follows:

$$\chi = \begin{pmatrix} c_{00} & \cdots & c_{0N'} \\ \vdots & & \vdots \\ c_{N0} & \cdots & c_{NN'} \end{pmatrix}, \tag{2.6}$$

and the vectors $P^\top(z) = [p_0(z), ..., p_N(z)]$ and $Q(t) = [q_0(t), ..., q_{N'}(t)]^\top$ include the shifted Legendre polynomials $p_i(z)$ and $q_j(t)$ [6, 12], defined over the intervals $[0,L]$ and $[0,T]$, respectively. If the Legendre polynomials of degree k are termed by $\chi_k(s)$, $s \in [-1,1]$, then, the shifted Legendre polynomials $p_k(z)$ and $q_k(t)$ are given by $p_k(z) = \chi_k(\frac{2z}{L} - 1)$ and $q_k(t) = \chi_k(\frac{2t}{T} - 1)$. Next, the approximations of $v_t(z,t)$ and $v_{zz}(z,t)$ are acquired utilizing the operational matrix of differentiation of the basis functions. That is we take the operational matrices D_L and D_T which satisfy the following equations

$$\frac{d^m}{dz^m} P^\top(z) = P^\top(z)(D_L^\top)^m, \quad \frac{d^k}{dt^k} Q(t) = D_T^k Q(t), \quad m \in \{1, ..., N\}, \ k \in \{1, ..., N'\}. \tag{2.7}$$

Using the following relations [33]

$$\begin{cases} p_k'(z) = \frac{2\sqrt{2k+1}}{L} \sum_{j=0}^{r^*} \sqrt{2k-4j-1} p_{k-2j-1}(z), \\ q_k'(t) = \frac{2\sqrt{2k+1}}{T} \sum_{j=0}^{r^*} \sqrt{2k-4j-1} q_{k-2j-1}(t), \end{cases} \tag{2.8}$$

the elements of D_L and D_T can be obtained where $r^* \in \{\mathbb{Z} | \ 2r^* + 1 \le k\}$. Specific examples of operational matrices of differentiation when $N = 4$, $N' = 5$ are presented as follows

$$D_L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{2\sqrt{3}}{L} & 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{15}}{L} & 0 & 0 & 0 \\ \frac{2\sqrt{7}}{L} & 0 & \frac{2\sqrt{35}}{L} & 0 & 0 \\ 0 & \frac{6\sqrt{3}}{L} & 0 & \frac{6\sqrt{7}}{L} & 0 \end{bmatrix}, \quad D_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2\sqrt{3}}{T} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{15}}{T} & 0 & 0 & 0 & 0 \\ \frac{2\sqrt{7}}{T} & 0 & \frac{2\sqrt{35}}{T} & 0 & 0 & 0 \\ 0 & \frac{6\sqrt{3}}{T} & 0 & \frac{6\sqrt{7}}{T} & 0 & 0 \\ \frac{2\sqrt{11}}{T} & 0 & \frac{2\sqrt{55}}{T} & 0 & \frac{6\sqrt{11}}{T} & 0 \end{bmatrix}.$$

Accordingly, we can write

$$\hat{v}_t(z,t) = (z-L)^2 P^\top(z) C (I + tD_T) Q(t), \tag{2.9}$$

$$\hat{v}_{zz}(z,t) = tP^\top(z)(2I + 4(z-L)D_L^\top + (z-L)^2(D_L^\top)^2) C Q(t), \tag{2.10}$$

$$A(z,\hat{v}) = \frac{b(z,T_1)\psi''(z) - F(z,T_1) + (z-L)^2 P^\top(z) C (I + T_1 D_T) Q(T_1) + S_t(z,T_1)}{\psi(z)}. \tag{2.11}$$

By defining the residual function

$$res(z,t,A(z,v)) = v(z,t) + \int_0^t \Big(b(z,u)v_{zz}(z,u) - A(z,v)(v(z,u)+S(z,u)) - f(z,u)\Big)du, \tag{2.12}$$

where $f(z,t) = F(z,t) - S_t(z,t) - b(z,t)\phi^{''}(z)$, the approximations $\hat{v}(z,t)$, $\hat{v}_{zz}(z,t)$ and $A(z,\hat{v})$ given by Eqs. (2.13) and (2.10)-(2.11) are substituted in the residual function (2.12) to get

$$res(z,t,A(z,\hat{v})) = \hat{v}(z,t) + \int_0^t \Big(b(z,u)\hat{v}_{zz}(z,u) - A(z,\hat{v})(\hat{v}(z,u)+S(z,u)) - f(z,u)\Big)du. \tag{2.13}$$

Then, a system of algebraic equations in terms of the coefficients $\{c_{ij}\}$, $i=0,...,N$, $j=0,...,N'$ is obtained, which we can either use the collocation equations

$$res(z_i,t_j,A(z_i,\hat{v})) = 0, \ z_i \in (0,L), \ t_j \in (0,T), \tag{2.14}$$

or take advantage of the least squares procedure by calculating the following functional

$$J_v[c_{ij}] = \int_0^L \int_0^T res^2(z,t,A(z,v))dtdz, \tag{2.15}$$

and then use the necessary conditions for minimization in Eq. (2.15) as follows

$$\frac{\partial J_{\hat{v}}[c_{ij}]}{\partial c_{ij}} = \frac{\partial}{\partial c_{ij}} \int_0^L \int_0^T res^2(z,t,A(z,\hat{v}))dtdz = 0, \quad i=0,...,N, \ j=0,...,N'. \tag{2.16}$$

Finally, a nonlinear system of equations in terms of elements c_{ij} is produced from (2.14) or (2.16) which is solved by Newton's iteration method.

In the sequel, we present the convergence of the solution in a special norm denoted by $\|.\|_\circ$, provided that it is obtained using the least squares scheme over the domain $\Omega_1 = [0,1]\times[0,1]$, i.e. the unknowns c_{ij} are found by Eqs. (2.15)-(2.16). In this regard, the following definitions are required:
Let

$$C^{2,1}(\Omega_1) := \{y(z,t) : \Omega_1 \to \mathbb{R} | \ y_t, \ y_{zz} \in C(\Omega_1)\},$$

be the Banach space equipped with the norm

$$\|y\|_\circ := \|y\|_\infty + \|\frac{\partial y}{\partial t}\|_\infty + \|\frac{\partial^2 y}{\partial z^2}\|_\infty, \tag{2.17}$$

and consider the space

$$\Delta(\Omega_1) := \{y(z,t) \in C^{2,1}(\Omega_1) | \ y(1,t) = y_z(1,t) = 0\}. \tag{2.18}$$

Lemma 2.1. *The polynomials of the space $\Delta(\Omega_1)$ are dense in the space $\Big(\Delta(\Omega_1), \|\cdot\|_\circ\Big)$.*

Proof. Considering $y_{zz}(z,t)$, $y_z(z,t) \in C(\Omega_1)$, from the Weierstrass approximation theorem, the existence of sequences of polynomials, namely $\{s^0_{m,n}(z,t)\}_{m,n\in\mathbb{N}}$ and $\{s^{00}_{m,n}(z,t)\}_{m,n\in\mathbb{N}}$ are guaranteed such that for given $\epsilon > 0$ we have

$$\|s^0_{m,n}(z,t) - y_z(z,t)\|_\infty < \epsilon, \quad \|s^{00}_{m,n}(z,t) - y_{zz}(z,t)\|_\infty < \epsilon.$$

Definition of $\Delta(\Omega_1)$ implies that any $y(z,t)$ of this space can be represented by

$$y(z,t) = \int_1^z y_p(p,t)dp, \quad y(z,t) = \int_z^1 \int_w^1 y_{pp}(p,t)dpdw. \tag{2.19}$$

Meanwhile, by defining

$$s^1_{m,n}(z,t) := \int_1^z \frac{\partial}{\partial w} s^0_{m,n}(w,t)dw, \quad s^2_{m,n}(z,t) := \int_z^1 \int_w^1 \frac{\partial^2}{\partial w \partial p} s^{00}_{m,n}(p,t)dpdw, \tag{2.20}$$

$$s_{m,n}(z,t) := (z-1)^2 s^1_{m,n}(z,t) + (2z - z^2)s^2_{m,n}(z,t), \tag{2.21}$$

and paying attention to

$$y(z,t) = (z-1)^2 y(z,t) + (2z - z^2) y(z,t), \tag{2.22}$$

one can observe that $s_{m,n}(1,t) = \frac{\partial s_{m,n}(z,t)}{\partial z}|_{z=1} = 0$ and also

$$\|y(z,t) - s^1_{m,n}(z,t)\|_\infty < \epsilon, \quad \|y(z,t) - s^2_{m,n}(z,t)\|_\infty < \epsilon.$$

Furthermore,

$$\begin{aligned}\|y(z,t) - s_{m,n}(z,t)\|_\infty &= \|(z-1)^2\Big(y(z,t) - s^1_{m,n}(z,t)\Big) + (2z-z^2)\Big(y(z,t) - s^2_{m,n}(z,t)\Big)\|_\infty \\ &\leq \|y(z,t) - s^1_{m,n}(z,t)\|_\infty + 2\|y(z,t) - s^2_{m,n}(z,t)\|_\infty < 3\epsilon,\end{aligned} \tag{2.23}$$

hence

$$\|y(z,t) - s_{m,n}(z,t)\|_\infty \longrightarrow 0 \quad as \quad m,n \longrightarrow \infty. \tag{2.24}$$

Simple calculations conclude that

$$\max\{\|\frac{\partial y(z,t)}{\partial t} - \frac{\partial s_{m,n}(z,t)}{\partial t}\|_\infty, \|\frac{\partial^2 y(z,t)}{\partial z^2} - \frac{\partial^2 s_{m,n}(z,t)}{\partial z^2}\|_\infty = 14\epsilon. \tag{2.25}$$

Accordingly, as $m,n \longrightarrow \infty$, Eqs. (2.24) and (2.25) indicate that $\|y(z,t) - s_{m,n}(z,t)\|_\circ \to 0$. □

Now, consider the multi-dimensional cell $\Upsilon = \Omega_1 \times [-l-\tau, l+\tau]^3$ with

$$l = \sup\{\|y\|_\infty, \|y_t\|_\infty, \|y_{zz}\|_\infty\}, \quad \tau > 0,$$

where $y := y(z,t)$ is an arbitrary function of $C^{2,1}(\Omega_1)$ and define the element $Y = (z,t,y,y_t,y_{zz}) \in \Upsilon$. Recall that functions $b(z,t)$ and $F(z,t)$ as the elements of $C^{2,1}(\Omega_1)$ are continuous on Ω_1 and $\psi(z) \neq 0$. If we suppose that $y^* \in C^{2,1}(\Omega_1)$ such that $\|y - y^*\|_\circ < \delta$, then it is implied that $Y^* = (z,t,y^*,y^*_t,y^*_{zz}) \in \Upsilon$ because

$$\max\{\|y-y^*\|_\infty, \|y_t - y^*_t\|_\infty, \|y_{zz} - y^*_{zz}\|_\infty\} \leq \|y - y^*\|_\circ < \delta \Longrightarrow \|Y - Y^*\|_\infty < \delta. \tag{2.26}$$

On the other hand, Υ is a compact set and $res^2(z,t,A(z,v))$ is continuous with respect to its arguments, so $J_y[c_{ij}]$ presented by Eq. (2.15) is uniformly continuous [29] on Υ and this implies that for any $\epsilon > 0$, there exists sufficiently small value of $\delta > 0$ such that for $\|Y - Y^*\|_\infty < \delta$ we have

$$|res(z,t,A(z,y)) - res(z,t,A(z,y^*))| < \epsilon \Longrightarrow |J_y(c_{ij}) - J_{y^*}(c_{ij})| < \epsilon.$$

Theorem 2.2. *Assuming λ_{mn} as the minimum of $J_y[c_{ij}]$ on the metric subspace*

$$K^{m,n}(\Omega_1) := \Delta(\Omega_1) \bigcap Span\{p_i(z)q_j(t)| \quad i = 0,...,m, \ j = 0,...,n\},$$

then

$$\lim_{m,n \longrightarrow \infty} \lambda_{mn} = 0.$$

Proof. Let

$$\min_{y \in K^{m,n}(\Omega_1)} J_y[c_{ij}] = \lambda_{mn}. \tag{2.27}$$

The property of minimum implies that

$$\forall \ \epsilon > 0, \ \exists \ y^* \in \Delta(\Omega_1) \quad s.t \quad J_{y^*}[c_{ij}] < \epsilon. \tag{2.28}$$

Furthermore, $J_y[c_{ij}]$ is continuous on $(\Delta(\Omega_1), \|.\|_\circ)$, so for small enough $\delta > 0$, fulfilling $\|y - y^*\|_\circ < \delta$ we get

$$|J_y[c_{ij}] - J_{y^*}[c_{ij}]| < \epsilon. \tag{2.29}$$

As guaranteed by Lemma 2.1, for large enough values of m and n there exists the element $q_{m,n}(z,t)$ such that $\|y^*(z,t) - q_{m,n}(z,t)\|_\circ < \delta$. Now, considering

$$J_{y^{mn}}[c_{ij}] = \lambda_{mn}, \quad y^{mn}(z,t) \in K^{m,n}(\Omega_1) \subseteq \Delta(\Omega_1), \tag{2.30}$$

and paying attention to (2.28) and (2.29) we have

$$0 \le J_{y^{mn}}[c_{ij}] \le J_{q_{m,n}}[c_{ij}] + \epsilon < 2\epsilon.$$

Accordingly,

$$\lambda_{m,n} \longrightarrow 0 \quad \text{if} \quad m, n \longrightarrow \infty.$$

□

Following scheme presents the regularized solution for the determination of the second-order derivative of perturbed data corresponding to the condition (1.4).

2.1. **Numerical differentiation schedule.** Given a set of N_1 discrete data points $z_k \in [0, L]$, we consider $\psi_{\delta_1}(z)$ as the measured data of $\psi(z)$ such that $\|\psi(z) - \psi_{\delta_1}(z)\|_\infty \le \delta_1$ and utilize the mollification technique proposed by [22] to compute the function $\psi''_{\delta_1}(z)$. The method is based on the convolution smoothing with a Gaussian mollifier given by $G_\sigma(t) = \frac{\exp^{-\frac{t^2}{\sigma^2}}}{\sigma\sqrt{\pi}}$ with the regularization parameter $\sigma > 0$. Recalling the convolution formula

$$\Big(G_\sigma * \psi\Big)(z) = \int_{-\infty}^{\infty} G_\sigma(\tau)\psi(z-\tau)d\tau, \tag{2.31}$$

and paying attention to the following property

$$\int_{-\infty}^{\infty} G_\sigma(\tau)\psi''(z-\tau)d\tau = \int_{-\infty}^{\infty} G''_\sigma(\tau)\psi(z-\tau)d\tau, \tag{2.32}$$

the mollified derivative is found by

$$\Big(G_\sigma * \psi''_{\delta_1}\Big)(z) = \int_{-\infty}^{\infty} G''_\sigma(\tau)\psi_{\delta_1}(z-\tau)d\tau. \tag{2.33}$$

After recovering $\Big(G_\sigma * \psi''_{\delta_1}\Big)(z_k)$ from (2.33) we apply the curve fitting scheme and get the approximation $\hat{\psi}''_{\delta_1}(z) = \sum_{k=0}^{N_2} \gamma_k p_k(z)$ by solving the following system of equations for the elements γ_k

$$\frac{\partial}{\partial \gamma_k} \sum_{m=1}^{N_1} \left\{ \hat{\psi}''_{\delta_1}(z_m) - \Big(G_\sigma * \psi''_{\delta_1}\Big)(z_m) \right\}^2 = 0, \quad k = 0, 1, ..., N_2. \tag{2.34}$$

We select the regularization parameter σ in such a way for given $\eta > 0$ the following inequality is fulfilled

$$\|res(z, t, A(z, \hat{v}))\|_\infty \le \eta.$$

3. Numerical tests

In this section, three examples are solved where numerical simulations are implemented by the Mathematica software version 12.3. Some routine command such as "FindRoot" is used to solve the systems of algebraic Eqs. (2.14) or (2.16). The functions of absolute error of $c(z)$ and $H(z,t)$ termed by $Abs(c)$ and $Abs(H)$ are used to exhibit the accuracy and the convergence of the approximate solutions numerically.

Example 3.1. We consider the numerical solution of the following system of equations [15]

$$\begin{cases} H_t(z,t) - ztH_{zz}(z,t) = c(z)H(z,t) + F(z,t), & (z,t) \in [0,1] \times [0,1], \\ H(z,0) = 2 + \cos(\pi z), & 0 \le z \le 1, \\ H(1,t) = e^{\frac{t^2}{1+t}}, \quad H_z(1,t) = 0, & 0 \le t \le 1, \\ H(z, \frac{3}{4}) = (2 + \cos(\pi z))e^{\frac{9}{28}}, & 0 \le z \le 1, \end{cases} \tag{3.1}$$

with

$$F(z,t) = e^{\frac{t^2}{1+t}}(\pi^2 tz\cos(\pi z) - \pi^2\cos(\pi z) + (2 + \cos(\pi z))(\frac{2t}{1+t} - \frac{t^2}{(1+t)^2})),$$

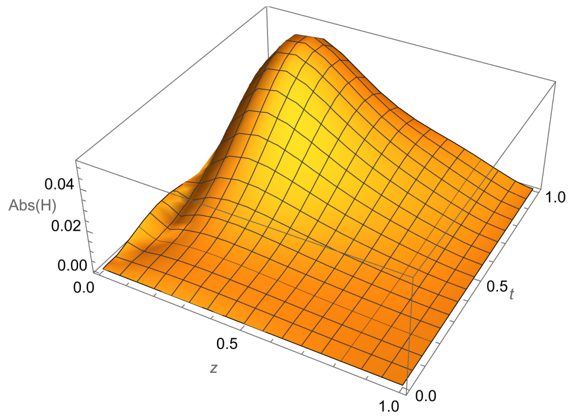

FIGURE 1. Graph of the absolute error for $H(z,t)$ when accurate boundary data are applied in Example 3.1.

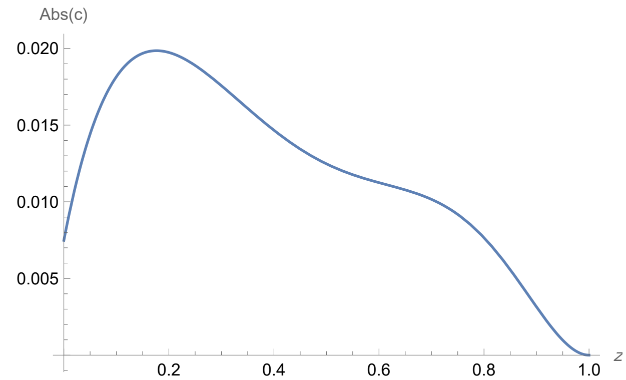

FIGURE 2. Graph of the absolute error for $c(z)$ when accurate boundary data are applied in Example 3.1.

TABLE 1. The results of computations for $\|Abs(H)\|_2$ and $\|Abs(c)\|_2$ when different number of parameters N and N' and exact boundary data are applied in Example 3.1.

(N,N')	$\|Abs(H)\|_2$	$\|Abs(c)\|_2$
$(2,2)$	0.0187631	0.0132264
$(4,4)$	0.00119509	0.000312243
$(6,6)$	0.00010948	0.0000748154
$(8,8)$	1.85×10^{-6}	7.5×10^{-7}

to approximate the pair solution $(c(z),H(z,t)) = (\frac{\pi^2\cos(\pi z)}{2+\cos(\pi z)}, e^{\frac{t^2}{1+t}}(2+\cos(\pi z)))$.

We apply the numerical scheme proposed in section 2 with $N = N' = 3$ in the presence of exact data $\psi(z)$ and utilize the least squares technique to find the unknown coefficients $c_{ij},\ i,j = 0,...,3$. The results are pictured in Figures 1 and 2. By repeating the experiment with greater values of N and N', as shown in Table 1, the accuracy of the numerical solutions is improved by increasing the number of basis functions.

Example 3.2. Consider the inverse problem presented by Eqs. (1.1)–(1.4), defined on the finite domain $\Omega_1 = [0,1]\times[0,1]$ with the following input data [15]:

$$\begin{cases} b(z,t)=1,\ h_1(t)=1+t,\ h_2(t)=0, \\ \phi(z)=1+z^2(z-1)^2, \\ T_1=\frac{1}{4},\ \psi(z)=1.25(1+z^2(z-1)^2), \\ F(z,t)=-\frac{(1+t)(-2+12z-14z^2+16z^3-38z^4+36z^5-12z^6)}{1+z^2-2z^3+z^4}-1-2t+12z(1+t)-z^2(11+12t)-2z^3+z^4. \end{cases} \tag{3.2}$$

The numerical scheme of section 2 is applied to approximate the exact solutions

$$H(z,t)=(1+t)(1+z^2(z-1)^2), \quad c(z)=\frac{-2+12z-12z^2}{1+z^2-2z^3+z^4},$$

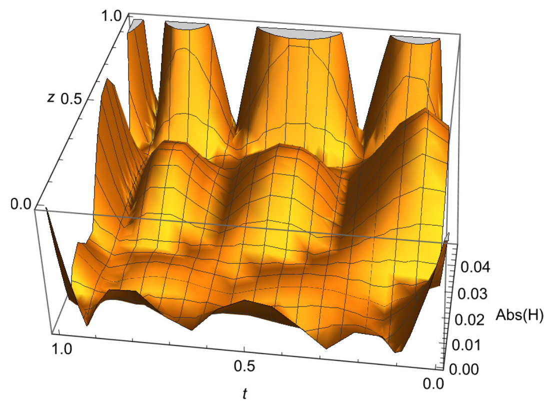

FIGURE 3. Graph of the absolute error for $H(z, t)$ when accurate boundary data are applied in Example 3.2.

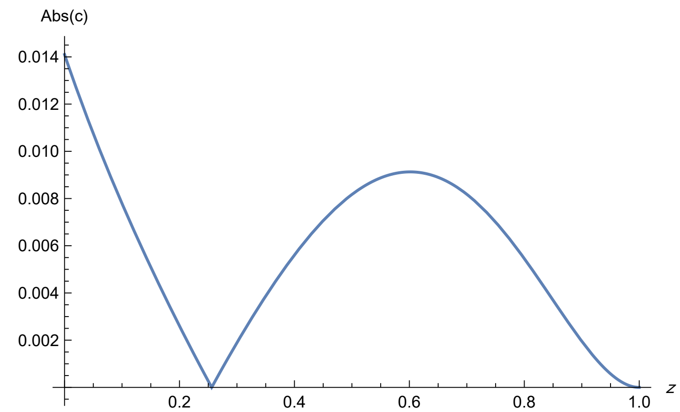

FIGURE 4. Graph of the absolute error for $c(z)$ when accurate boundary data are applied in Example 3.2.

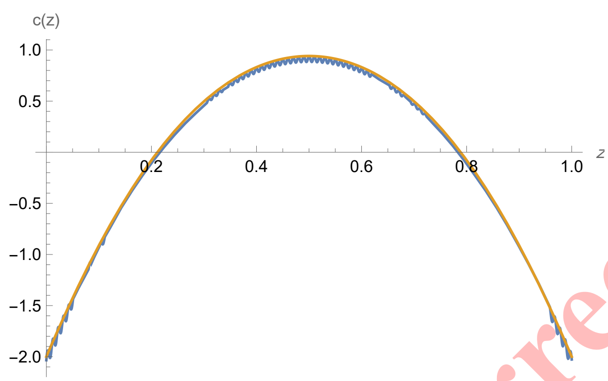

FIGURE 5. The orange curve represents the exact solution, and the approximate solution of $c(z)$ is pictured by the blue color when perturbed boundary data are applied in Example 3.2.

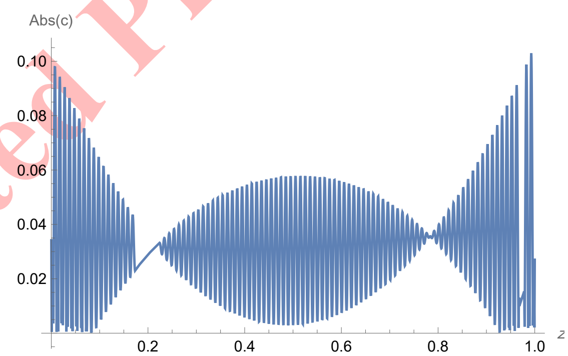

FIGURE 6. Graph of the absolute error for $c(z)$ when perturbed boundary data are applied in Example 3.2.

when it is assumed that $N = N' = 3$ and no perturbation is applied to the boundary condition (1.4). The outcomes are depicted in Figures 3 and 4. The applicability of proposed method in the presence of inaccurate boundary data is also studied. In this direction, we denote the perturbed data of (1.4) by applying the formula [21] $\psi_{\delta_1}(z_m) = \psi(z_m) + \delta_1 \sin(\frac{z_m}{\delta_1^2})$ with $\delta_1 = 0.04$ and employ the numerical differentiation technique described in subsection 2.1 with

$$N_1 = 50,\ N_2 = 3,\ \sigma = 0.08,\ \eta = 0.15,$$

to produce $\hat{\psi}''_{\delta_1}(z) = 2.54655 - 14.9925z + 14.9882z^2 + 0.00582484z^3$. Finally, by including $\hat{\psi}''_{\delta_1}(z)$ in the calculations and using the formulas in section 2 with $N = N' = 3$, we reached the results illustrated in Figures 5–7 showing good agreement with the exact solutions.

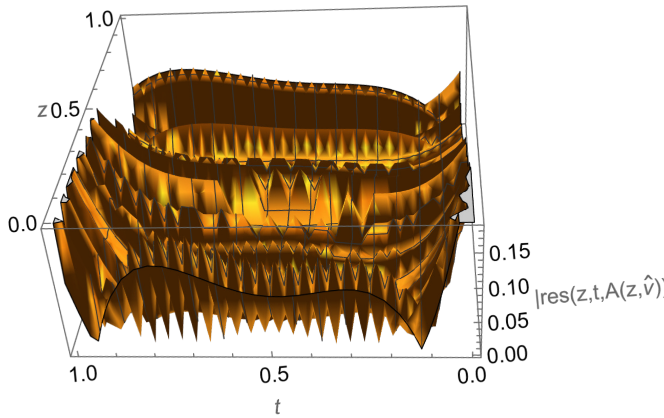

FIGURE 7. Graph of absolute value for $res(z,t,A(z,\hat{v}))$ when perturbed boundary data are applied in Example 3.2.

Example 3.3. Consider the following input data to approximate the absorption coefficient $c(z) = \cosh(z)$ in the inverse problem (1.1)-(1.4):

$$\begin{cases} b(z,t) = (1+z^2+t^2),\ h_1(t) = 1,\ h_2(t) = t, \\ \phi(z) = 1,\ T_1 = \frac{1}{2}, \\ F(z,t) = -2.0519947100620595 + 0.405954004559216 04t + 4.1602802414172615t^2 \\ \qquad +1.1908648714325598z + 2.077328406162695tz - 4.597302624636829t^2 z \\ \qquad -0.6819067050142174z^2 - 2.7636530705190414tz^2 + 0.47509028799102904t^2z^2. \end{cases} \tag{3.3}$$

There is no analytical solution $H(z,t)$ corresponding to this problem. Thus, to retrieve the measurement (1.4), the system of Eqs. (1.1)-(1.3) is solved with known function $c(z)$ using the Ritz approximation technique [23–26]. That is we assume the approximation of (1.1)-(1.3) as:

$$H^{direct}(z,t) = \sum_{i,j=0}^{6} t(z-1)^2 \varrho_{ij} p_i(z) q_j(t) + 1 + t(z-1),$$

and solve it for the elements ϱ_{ij}. The approximation of $\psi(z)$ is obtained as

$$\begin{aligned} \psi(z) \simeq H^{direct}(z,\frac{1}{2}) &= 0.3777427243649015 + 0.8906753334074937z - 0.37240786564966855z^2 \\ &+ 0.019723829193526665z^3 + 0.1215160472440315z^4 - 0.017460450974330494z^5 \\ &- 0.039788683273925644z^6 + 0.025053449653425836z^7 - 0.005054383965454664z^8. \end{aligned} \tag{3.4}$$

Then, we solve the inverse problem using the proposed method with $N = N' = 8$ and get the approximation whose absolute error is shown in Figure 8.

4. Conclusion

This paper proposes a numerical solution based on the combination of the Legendre basis functions and the operational matrix of differentiation of such bases to determine the absorption coefficient and temperature in a parabolic equation. By using an appropriate transformation, the initial problem is converted to the solution of a nonclassical partial differential equation in terms of the temperature function with homogeneous initial and boundary conditions. The numerical solution of the new problem is substituted into the residual function and then a nonlinear system of algebraic equations is produced which can be solved by Newton's iteration. Our proposed method accurately satisfies all the initial and boundary conditions of the problem which significantly reduces the amount of calculations since a

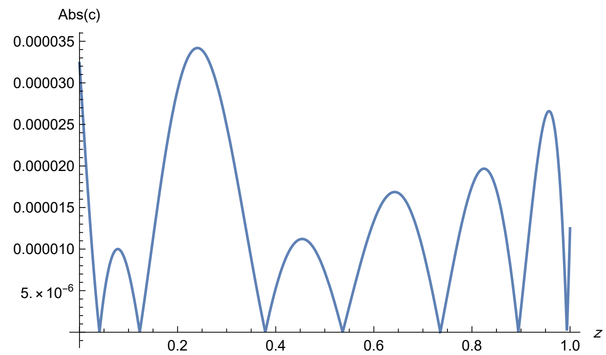

FIGURE 8. Graph of the absolute error for $c(z)$ in Example 3.3.

small number of basis functions are needed to reach the acceptable solutions. We provide a proof of the method's convergence and demonstrate its effectiveness through illustrative test examples. The increasing accuracy of approximate solutions as the number of bases increases, along with the robustness and consistency of numerical solutions when boundary conditions are accompanied by perturbations, are demonstrated in numerical examples. The method can be adapted to solve a wide range of partial differential equations as the governing equations with nonstandard boundary conditions.

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