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Efficient implicit numerical methods for nonlinear Fisher equation

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Abstract

This paper explores various numerical methods for solving the one-dimensional nonlinear Fisher equation using the finite difference and Newton methods. The study focuses on achieving higher accuracy in numerical solutions, the proposed approach being first-order accurate in time and second-order accurate in space. The numerical results for different values of α closely match the exact solutions. Several examples are presented, comparing L_2 and L_∞ errors with the exact solution and the existing methods from the literature and leading to high accuracy. These types of equation arise in various fields of sciences and engineering, the main application of this equation has been found in biomedical sciences. The solution of this equation helps determine the size of the brain tumor.

Keywords. Fisher's equation, Finite difference method, Newton metho, Crank-Nicolson.
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1. INTRODUCTION

In this paper, we focus on the one-dimensional generalized Fisher equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u^\beta (1 - u^\nu), \quad 0 \le x \le 1, \quad t > 0, \tag{1.1}$$

with initial condition

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1,$$

and the boundary conditions

$$\begin{aligned} u(0,t) &= f_3(t), \qquad && 0 \le t \le T, \\ u(1,t) &= f_4(t), \qquad && 0 \le t \le T. \end{aligned}$$

Here, the functions $u_0(x)$, f_3, and f_4 are given smooth functions and α is the reactive factor. The nonlinear reaction-diffusion equation was originally introduced by Fisher in 1937 [10]. The Fisher equation is commonly known as the KPP equation, an abbreviation for Kolmogorov-Petrovsky-Piscounov. However, the Fisher equation is the more well-known name for it. Equation (1.1) characterizes a nonlinear model of a physical system featuring linear diffusion and nonlinear evolution, as described in [2]. Numerous disciplines, including science and industry, have made significant use of Fisher's equation [5–7, 16, 18]. The interplay between diffusion and reaction is thus described by equation (1.1) [8].

The mathematical features of Fisher's equation have been thoroughly discussed in the literature [1, 3, 4, 9, 14, 15, 17]. The overviews of Fisher's equation provided by Brazhnik and Tyson [28], Kawahara and Tanaka [21] and Larson [22] are highly informative and well-regarded. Subsequently, numerous researchers have conducted numerical solutions for Fisher's equation. To investigate numerical approaches for Fisher's equation, Parekh and Puri [25] and Twizell et al. [27] introduced both implicit and explicit finite difference algorithms. The modified form of a nonlinear Fisher's

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reaction-diffusion equation solved by RBFs based on DQMs [12]. The Fisher equation in bounded domains, by Faedo–Galerkin's method and with a homogeneous Dirichlet conditions [11], The Fisher equation is solved by extended homogeneous balance method and it is used to solve many non-linear equation liker Fisher's equation and Burgers-Fisher equation [9]. The Fisher's equation is solved by the Lie symmetries of the generalized Fisher equation in [26]. We have outlined a comprehensive approach in five sections to derive numerical solutions for Equation (1.1). In section 2, we discuss various numerical schemes, including explicit, semi-implicit, implicit, semi-implicit Crank-Nicolson, implicit Crank-Nicolson type-1 and implicit Crank-Nicolson type-2 schemes. Section 3 covers consistency and error analysis, while section 4 presents the results of numerical experiments. Finally, Section 5 provides a conclusion.

2. NUMERICAL SCHEME

We employ a regular grid to divide the solution domain of Equation (1.1). The space between $[0, 1]$ is divided into N equal subintervals, and the time interval $[0, T]$ is divided into M equal subintervals. In the spatial dimension, we set the mesh width as $\Delta x = 1/N$, and the points x_i are defined as $x_i = i\Delta x$ for $i = 0, 1, ..., N$. For the temporal dimension, we set $t^n = n\Delta t$ for $n = 0, 1, ..., M$, where $\Delta t = T/M$ represents the mesh width in time.

2.1. **Fully Explicit Scheme (ES).**

In this approach, we compute derivatives at the $(n+1)^{th}$ time level, while in the explicit method, derivatives are calculated at the n^{th} time level. We apply the central difference formula in space and the forward difference formula in the time direction. Consider modified Fisher's Equation (1.1) for $\beta = \nu = 1$, we get

$$u_t = \mu u_{xx} + \alpha u(1-u), \tag{2.1}$$

for the time derivative we use the forward difference approximation in the form

$$(u_t)|_{(x_i,t^n)} = \frac{U_i^{n+1} - U_i^n}{\Delta t} + \mathscr{O}(\Delta t),$$

by applying the central difference formula with respect to x, the second derivative $\frac{\partial^2 u}{\partial x^2}$ can be approximated. Assuming α is a real number within the interval $[0, 1]$, we use the following α-family for the diffusion term.

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \mu \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2} + \alpha U_i^n(1 - U_i^n),$$

$$U_i^{n+1} = \mu \frac{\Delta t}{(\Delta x)^2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) + \alpha \Delta t\, U_i^n(1 - U_i^n) + U_i^n, \tag{2.2}$$

$$U_i^{n+1} = \gamma U_{i-1}^n + \left(1 - 2\gamma + \alpha\Delta t(1 - U_i^n)\right) U_i^n + \gamma U_{i+1}^n, \tag{2.3}$$

where,

$$\gamma = \mu \frac{\Delta t}{(\Delta x)^2}, \qquad n \geq 0, \quad i = 1, 2, ..., N.$$

Equation (2.3) may be expressed in the form

$$N_i U_{i-1}^n + E_i U_i^n + H_i U_{i+1}^n = A_i. \tag{2.4}$$

where, N_i, $A_i = U_i^{n+1}$ and H_i are constants. So $N_i = H_i = (\gamma)$ and $E_i = (1 - 2\gamma + \alpha\Delta t(1 - U_i^n))U_i^n$. Equation (2.4) is termed as the explicit finite difference approximation to Fisher's equation. We can express this equation in the form of a tridiagonal matrix with dimensions $(N \times (N + 2))$

$$\begin{bmatrix} E_1 & \gamma & 0 & \dots & 0 \\ \gamma & E_2 & \gamma & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & E_{N-1} & \gamma \\ 0 & \dots & \dots & \gamma & E_N \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ \vdots \\ U_N^n \end{bmatrix} + \begin{bmatrix} \gamma U_0^n \\ 0 \\ \vdots \\ \vdots \\ \gamma U_{N+1}^n \end{bmatrix} = \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_N^{n+1} \end{bmatrix}.$$

2.2. **Semi Implicit Scheme (SIS).** Here we calculate the derivative at the time level of $(n+1)^{th}$, but in the explicit FD method, the derivatives are determined at the time level of n^{th}. We employ the forward difference formula for time and the central difference formula for space. In this partially implicit scheme. While using the explicit method, derivatives are computed at the n^{th} time level, they are computed at the $(n+1)^{th}$ time level. Consider the Fisher Equation (2.1) will become,

$$\begin{aligned} \frac{U_i^{n+1}-U_i^n}{\Delta t} &= \mu \frac{U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{(\Delta x)^2} + \alpha U_i^{n+1}(1-U_i^n), \\ U_i^{n+1} &= \mu \frac{\Delta t}{(\Delta x)^2}(U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}) + \alpha \Delta t U_i^{n+1}(1-U_i^n) + U_i^n, \qquad (2.5) \end{aligned}$$

consider $\gamma = \mu \frac{\Delta t}{(\Delta x)^2}$ and $\eta = \alpha \Delta t$. So, we get

$$-\gamma U_{i-1}^{n+1} + (1+2\gamma-\eta(1-U_i^n))U_i^{n+1} - \gamma U_{i+1}^{n+1} = U_i^n, \qquad (2.6)$$

$$P_i U_{i-1}^{n+1} + B_i U_i^{n+1} + Q_i U_{i+1}^{n+1} = D_i. \qquad (2.7)$$

Here, P_i and Q_i are constants. So $P_i = Q_i = (-\gamma)$ and $B_i = (1+2\gamma-\eta(1-U_i^n))$, $i = 1, 2, ..., N$. Equation (2.7) written in the tridigonal matrix $(N \times (N+2))$ form

$$\begin{bmatrix} B_1 & -\gamma & 0 & 0 & \dots & 0 \\ -\gamma & B_2 & -\gamma & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -\gamma & B_{N-1} & -\gamma \\ 0 & \dots & \dots & 0 & -\gamma & B_N \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_{N-1}^{n+1} \\ U_N^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ \vdots \\ U_{N-1}^n \\ U_N^n \end{bmatrix} - \begin{bmatrix} -\gamma U_0^{n+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -\gamma U_{N+1}^{n+1} \end{bmatrix}.$$

2.3. **Fully Implicit Scheme (IS).** In this implicit scheme, the derivatives are evaluated at the $(n+1)^{th}$ time step, whereas in the explicit method, they are calculated at the n^{th} time step. We apply a central difference scheme for spatial derivatives and a forward difference scheme for time derivatives. If $\alpha = 0$ in Equation (2.1), the equation reduces to a parabolic heat equation, commonly used in problems involving heat and mass transfer, consider the Fisher Equation (2.1) will become,

$$\begin{aligned} \frac{U_i^{n+1}-U_i^n}{\Delta t} &\approx \mu \frac{U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{(\Delta x)^2} + \alpha U_i^{n+1}(1-U_i^{n+1}), \\ U_i^{n+1} &= \mu \frac{\Delta t}{(\Delta x)^2}(U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}) + \alpha \Delta t U_i^{n+1}(1-U_i^{n+1}) + U_i^n, \qquad (2.8) \end{aligned}$$

consider $\gamma = \mu \frac{\Delta t}{(\Delta x)^2}$ and $\eta = \alpha \Delta t$

$$-\gamma U_{i-1}^{n+1} + \big(1+2\gamma-\eta(1-U_i^{n+1})\big)U_i^{n+1} - \gamma U_{i+1}^{n+1} = U_i^n, \qquad (2.9)$$

$$-\gamma U_{i-1}^{n+1} + I_i U_i^{n+1} + -\gamma U_{i+1}^{n+1} = U_i^n. \qquad (2.10)$$

where, $I_i = (1+2\gamma-\eta(1-U_i^{n+1})), i = 1, 2, ..., N$. Equation (2.10) written in the tridigonal matrix (N × (N + 2)) form

$$\begin{bmatrix} I_1 & -\gamma & 0 & 0 & \dots & 0 \\ -\gamma & I_2 & -\gamma & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & I_{N-1} & -\gamma \\ 0 & \dots & \dots & 0 & -\gamma & I_N \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_{N-1}^{n+1} \\ U_N^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ \vdots \\ U_{N-1}^n \\ U_N^n \end{bmatrix} + \begin{bmatrix} \gamma U_0^{n+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \gamma U_{N+1}^{n+1} \end{bmatrix}.$$

2.4. **Semi-Implicit Crank-Nicolson Scheme (SICNS).** Higher-order derivatives are approximated using central difference methods,

$$\frac{\partial^2 u}{\partial x^2} = \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta x)^2} + \frac{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}{(\Delta x)^2}.$$

A widely used family of implicit schemes is based on the Crank-Nicolson method, which computes a weighted average of the spatial derivatives at both the n^{th} and $(n+1)^{th}$ time levels. Consequently, the discretization of Equation (2.1) is given as follows:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{2}\Big(\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta x)^2} + \frac{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}{(\Delta x)^2}\Big) + \alpha U_i^{n+1}(1 - U_i^n),$$

where, $\mu = 1$, $i = 2, 3, ..., l$ and $n = 1, 2, ..., m$. Then, we obtain

$$U_i^{n+1} - U_i^n = \frac{k}{2h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}) + \alpha k U_i^{n+1}(1 - U_i^n), \tag{2.11}$$

$$\frac{-k}{h^2}U_{i-1}^{n+1} + (2 + \frac{2k}{h^2})U_i^{n+1} - \frac{k}{h^2}U_{i+1}^{n+1} = \frac{k}{h^2}U_{i-1}^{n} + (2 - \frac{2k}{h^2})U_i^n + \frac{k}{h^2}U_{i+1}^n + 2k\alpha U_i^{n+1}(1 - U_i^n), \tag{2.12}$$

$$SU_{i-1}^{n+1} + TU_i^{n+1} + SU_{i+1}^{n+1} = WU_{i-1}^n + YU_i^n + WU_{i+1}^n + 2Z_i^n. \tag{2.13}$$

where, $\Delta t = k$, $\Delta x = h$, $i = 1, 2, ..., N$ and $n = 1, 2, ..., m$.

Equation (2.13) written in the tridigonal matrix $(N \times (N+2))$ form

$$\begin{bmatrix} T & S & \dots & \dots & \dots & \dots \\ S & T & S & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & S & T & S \\ \dots & \dots & \dots & \dots & S & T \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_{N-1}^{n+1} \\ U_N^{n+1} \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ \vdots \\ V_{N-1} \\ V_N \end{bmatrix} + \begin{bmatrix} WU_0^n \\ 0 \\ \vdots \\ \vdots \\ 0 \\ Wu_{N+1}^n \end{bmatrix} - \begin{bmatrix} SU_1^{n+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ SU_{N+1}^{n+1} \end{bmatrix} + 2 \begin{bmatrix} Z_1^n \\ Z_2^n \\ \vdots \\ \vdots \\ Z_{n-1}^n \\ Z_N^n \end{bmatrix},$$

where, $i = 1, 2, ..., N$ and $n = 2, 3, .., m+1$,

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ \vdots \\ V_{N-1} \\ V_N \end{bmatrix} = \begin{bmatrix} Y & W & & \dots & \dots & \dots \\ W & Y & W & \dots & \dots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & W & Y & W \\ 0 & \dots & \dots & & W & Y \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ \vdots \\ U_{N-1}^n \\ U_N^n \end{bmatrix},$$

where, $T = 2 + \frac{2k}{h^2}$, $S = \frac{-k}{h^2}$, $Y = 2 - \frac{2k}{h^2}$, $Z_i^n = k\alpha U_i^{n+1}(1 - U_i^n)$.

2.5. **Implicit Crank-Nicolson Discretization of Type -1 (ICNS-1).** Higher-order derivatives are approximated using central difference methods,

$$\frac{\partial^2 u}{\partial x^2}\Big|_{(x_i, t^n)} \approx \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta x)^2} + \frac{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}{(\Delta x)^2}.$$

A widely used family of implicit schemes is based on the Crank-Nicolson method, which computes a weighted average of the spatial derivatives at both the n^{th} and $(n+1)^{\text{th}}$ time levels. Consequently, the discretization of Equation (2.1) is given as follows:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{2}\Big(\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta x)^2} + \frac{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}{(\Delta x)^2}\Big) + \alpha U_i^{n+1}(1 - U_i^{n+1}), \tag{2.14}$$

where, $\mu = 1,\quad i = 1, 2, ..., N$ and $n = 1, 2, ..., m,\ \Delta t = k,\ \Delta x = h,\ \frac{k}{h^2} = \lambda,\ \alpha\Delta t = \eta$.

$$U_i^{n+1} - U_i^n = \frac{k}{2h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^n - 2U_i^n + U_{i-1}^n) + \alpha\Delta t U_i^{n+1}(1 - U_i^{n+1}), \tag{2.15}$$

$$-\lambda U_{i-1}^{n+1} + (2 + 2\lambda - 2\eta(1 - U_i^{n+1}))U_i^{n+1} - \lambda U_{i+1}^{n+1} = \lambda U_{i-1}^n + (2 - 2\lambda)U_i^n + \lambda U_{i+1}^n, \tag{2.16}$$

$$-\lambda U_{i-1}^{n+1} + L_i U_i^{n+1} - \lambda U_{i+1}^{n+1} = \lambda U_{i-1}^n + F U_i^n + \lambda U_{i+1}^n. \tag{2.17}$$

where, $L_i = 2 + 2\lambda - 2\eta(1 - U_i^{n+1}),\ F = 2 - 2\lambda,\ i = 1, 2, ..., N,\ n = 1, 2, ..., m$.

Equation (2.17) written in the tridigonal matrix $(N \times (N + 2))$ form

$$\begin{bmatrix} L_1 & -\lambda & \dots & \dots & \dots & \dots \\ -\lambda & L_2 & -\lambda & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -\lambda & L_{N-1} & -\lambda \\ 0 & \dots & \dots & & -\lambda & L_N \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_{l-1}^{n+1} \\ U_N^{n+1} \end{bmatrix} - \begin{bmatrix} \lambda U_0^{n+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \lambda U_{N+1}^{n+1} \end{bmatrix} = \begin{bmatrix} F & \lambda & & \dots & \dots & \dots \\ \lambda & F & \lambda & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & \lambda & F & \lambda \\ \dots & \dots & \dots & \dots & \lambda & F \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ \vdots \\ U_{l-1}^n \\ U_N^n \end{bmatrix} + \begin{bmatrix} \lambda U_0^{n+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \lambda U_{N+1}^{n+1} \end{bmatrix},$$

where, $i = 1, 2, ..., N$ and $n = 2, 3, .., m + 1$.

2.6. **Implicit Crank-Nicolson Discretization of Type-2 (ICNS-2).** In this method (2.5), we have discretized by Crank-Nicolson the term

$$\alpha u(1 - u)\big|_{(x_i, t^n)} \approx \alpha U_i^{(n+1)}(1 - U_i^{n+1}),$$

but in the (2.6) method, we are discretizing by Crank-Nicolson the term

$$\alpha u(1 - u)\big|_{(x_i, t^n)} \approx \alpha\Big(\frac{U_i^{n+1} + U_i^n}{2}\Big)\Big(1 - \frac{U_i^{n+1} + U_i^n}{2}\Big).$$

Consider the Fisher Equation (2.1) will become,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{2}\Big(\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta x)^2} + \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}\Big) + \alpha\Big(\frac{U_i^{n+1} + U_i^n}{2}\Big)\Big(1 - \frac{U_i^{n+1} + U_i^n}{2}\Big), \tag{2.18}$$

where, $\mu = 1,\quad i = 1, 2, ..., N$ and $n = 1, 2, ..., m,\ \Delta t = k,\ \Delta x = h,\ \frac{k}{h^2} = \lambda,\ \alpha\Delta t = \eta$

$$U_i^{n+1} - U_i^n = \frac{k}{2h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^n - 2U_i^n + U_{i-1}^n) + \eta\Big(\frac{U_i^{n+1} + U_i^n}{2}\Big)\Big(1 - \frac{U_i^{n+1} + U_i^n}{2}\Big), \tag{2.19}$$

hence, we reach at

$$\begin{aligned} &-2\lambda U_{i-1}^{n+1} + \big(4 + 4\lambda - 2\eta + \eta U_i^{(n+1)} + 2\eta U_i^n\big)U_i^{n+1} - 2\lambda U_{i+1}^{n+1} = 2\lambda U_{i-1}^n + \big(4 - 4\lambda + 2\eta\big)U_i^n + 2\lambda U_{i+1}^n, \\ &-2\lambda U_{i-1}^{n+1} + K U_i^{n+1} - 2\lambda U_{i+1}^{n+1} = 2\lambda U_{i-1}^n + J U_i^n + 2\lambda U_{i+1}^n. \end{aligned} \tag{2.20}$$

Equation (2.20) written in the tridigonal matrix $(N \times (N + 2))$ form

$$\begin{bmatrix} K_1 & -2\lambda & \dots & \dots & \dots & \dots \\ -2\lambda & K_2 & -2\lambda & \dots & \dots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & & -2\lambda & K_{N-1} & -2\lambda \\ \dots & \dots & \dots & \dots & -2\lambda & K_N \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_{N-1}^{n+1} \\ U_N^{n+1} \end{bmatrix} - \begin{bmatrix} 2\lambda U_0^{n+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 2\lambda U_{N+1}^{n+1} \end{bmatrix} = \begin{bmatrix} J_1 & 2\lambda & \dots & \dots & \dots & \dots \\ 2\lambda & J_2 & 2\lambda & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & 2\lambda & J_{N-1} & 2\lambda \\ \dots & \dots & \dots & \dots & 2\lambda & C_N \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ \vdots \\ U_{l-1}^n \\ U_N^n \end{bmatrix} + \begin{bmatrix} 2\lambda U_0^{n+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 2\lambda U_{N+1}^{n+1} \end{bmatrix},$$

where, $K_i = \big(4 + 4\lambda - 2\eta + \eta U_i^{(n+1)} + 2\eta U_i^n\big)$ and $J_i = \big(4 - 4\lambda + 2\eta\big)$.

3. Consistency and Error Analysis

LTE(Local Truncation Errors)
This occurs when a numerical method is employed to estimate the solution of a partial differential equation. Let's assume that the function u is smooth at the point (x_i, t_n). The Taylor series method is then utilized.

$$u_i^{n+1} = u_i^n + \frac{k}{1!}u_{t,i}^n + \frac{k^2}{2!}u_{tt,i}^n + \frac{k^3}{3!}u_{ttt,i}^n + ..., \tag{3.1}$$

$$\begin{aligned} u_{i-1}^{n+1} = {} & u_i^n - hu_{x,i}^n + ku_{t,i}^n + \frac{h^2}{2!}u_{xx,i}^n - khu_{tx,i}^n + \frac{k^2}{2!}u_{tt,i}^n \\ & - \frac{h^3}{3!}u_{xxx,i}^n + \frac{kh^2}{2!}u_{txx,i}^n - \frac{hk^2}{2!}u_{ttx,i}^n + \frac{h^4}{4!}u_{xxxx,i}^n + \frac{k^3}{3!}u_{ttt,i}^n \\ & - \frac{kh^3}{3!}u_{txxx,i}^n + \frac{k^2h^2}{2!}u_{ttxx,i}^n - \frac{hk^3}{3!}u_{xttt,i}^n + ..., \end{aligned} \tag{3.2}$$

$$\begin{aligned} u_{i+1}^{n+1} = {} & u_i^n + hu_{x,i}^n + ku_{t,i}^n + \frac{h^2}{2!}u_{xx,i}^n + khu_{tx,i}^n + \frac{k^2}{2!}u_{tt,i}^n \\ & + \frac{h^3}{3!}u_{xxx,i}^n + \frac{kh^2}{2!}u_{txx,i}^n + \frac{hk^2}{2!}u_{ttx,i}^n + \frac{h^4}{4!}u_{xxxx,i}^n + \frac{k^3}{3!}u_{ttt,i}^n \\ & + \frac{kh^3}{3!}u_{txxx,i}^n + \frac{k^2h^2}{2!}u_{ttxx,i}^n + \frac{hk^3}{3!}u_{xttt,i}^n + ..., \end{aligned} \tag{3.3}$$

$$u_{i-1}^n = u_i^n - hu_{x,i}^n + \frac{h^2}{2!}u_{xx,i}^n - \frac{h^3}{3!}u_{xxx,i}^n + \frac{h^4}{4!}u_{xxxx,i}^n + ..., \tag{3.4}$$

$$u_{i+1}^n = u_i^n + hu_{x,i}^n + \frac{h^2}{2!}u_{xx,i}^n + \frac{h^3}{3!}u_{xxx,i}^n + \frac{h^4}{4!}u_{xxxx,i}^n + ... \tag{3.5}$$

Explicit scheme(ES). Substituting Equations (3.1), (3.4), and (3.5) in Equation (2.2), we get

$$\frac{1}{\Delta t}[u_t(\Delta t) + O(\Delta t)^2] = \frac{1}{(\Delta x)^2}[u_{xx}(\Delta x)^2 + O(\Delta x)^4] + \alpha U_i^n\big(1 - U_i^n\big). \tag{3.6}$$

Semi-implicit scheme (SIS). Substituting Equations (3.1), (3.2), and (3.3) in Equation (2.5), we get

$$\frac{1}{\Delta t}[u_t(\Delta t) + O(\Delta t)^2] = \frac{1}{(\Delta x)^2}[u_{xx}(\Delta x)^2 + O(\Delta x)^4] + \alpha U_i^{n+1}\big(1 - U_i^n\big). \tag{3.7}$$

Implicit scheme(IS). Substituting Equations (3.1), (3.2), and (3.3) in Equation (2.8), we get

$$\frac{1}{\Delta t}[u_t(\Delta t) + O(\Delta t)^2] = \frac{1}{(\Delta x)^2}[u_{xx}(\Delta x)^2 + O(\Delta x)^4] + \alpha U_i^{n+1}\big(1 - U_i^{n+1}\big). \tag{3.8}$$

Semi-implicit Crank-Nicolson scheme (SICNS). Substituting Equations (3.1), (3.2), (3.3), (3.4), and (3.5) in Equation (2.11), we get

$$\frac{1}{\Delta t}[u_t(\Delta t) + O(\Delta t)^2] = \frac{1}{(\Delta x)^2}[u_{xx}(\Delta x)^2 + O(\Delta x)^4] + \alpha U_i^{n+1}\big(1 - U_i^n\big). \tag{3.9}$$

Implicit Crank-Nicolson Discretization of Type -1 (ICNS-1). Substituting Equations (3.1), (3.2), (3.3), (3.4), and (3.5) in Equation (2.15), we get

$$\frac{1}{\Delta t}[u_t(\Delta t) + O(\Delta t)^2] = \frac{1}{(\Delta x)^2}[u_{xx}(\Delta x)^2 + O(\Delta x)^4] + \alpha U_i^{n+1}\big(1 - U_i^{n+1}\big). \tag{3.10}$$

Implicit Crank-Nicolson Discretization of Type-2 (ICNS-2). Substituting Equations (3.1), (3.2), (3.3), (3.4), and (3.5) in Equation (2.19), we get

$$\frac{1}{\Delta t}[u_t(\Delta t)+O(\Delta t)^2]=\frac{1}{(\Delta x)^2}[u_{xx}(\Delta x)^2+O(\Delta x)^4]+\alpha\Big(\frac{U_i^{n+1}+U_i^n}{2}\Big)\Big(1-\frac{U_i^{n+1}+U_i^n}{2}\Big). \tag{3.11}$$

For the above equation, the local truncation error is expressed as,

$$LTE=\lim_{\Delta x,\Delta t\to 0}\frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}+\frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}+\frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}+\frac{(\Delta x)^5}{5!}\frac{\partial^5 u}{\partial x^5}+...+=0. \tag{3.12}$$

Consistency In all methods we discussed here, a numerical approach is considered consistent when the difference between a partial differential equation (PDE) and its corresponding finite difference equation (FDE) approaches zero with an increasing number of subdivisions.

$$\lim_{\Delta x,\Delta t\to 0}(PDE-FDE)=\lim_{\Delta x,\Delta t\to 0}(LTE)=0.$$

As Δt and Δx both approach zero, the local truncation error becomes zero based on Equation (3.12). Consequently, the proposed scheme is consistent. And solving the Equations (3.6), (3.7), (3.8), (3.9), (3.10), and (3.11) comparing with Equations (2.2), (2.5), (2.8), (2.11), (2.15), and (2.19). Our proposed scheme can be described as having a first-order accuracy in time and a second-order accuracy in space.

3.1. **Stability Analysis:** Numerical errors introduced during the discretization of equations should not be amplified. This property is known as stability. The von-Neumann stability analysis of finite difference methods for non-linear problems, such as the reaction-diffusion model, has been explored in [13, 19, 24]. In von-Neumann stability analysis, it is assumed that the solution to the Equation (2.1) is expressed in terms of Fourier modes.

$$\begin{aligned}
U_i^n &= \xi^{\alpha nk}e^{\mathbf{i}rih},\\
U_{i+1}^n &= \xi^{\alpha nk}e^{\mathbf{i}r(i+1)h},\\
U_{i-1}^n &= \xi^{\alpha nk}e^{\mathbf{i}r(i-1)h},\\
U_i^{n+1} &= \xi^{\alpha(n+1)k}e^{\mathbf{i}rih},\\
U_{i-1}^{n+1} &= \xi^{\alpha(n+1)k}e^{\mathbf{i}r(i-1)h},
\end{aligned}$$

where $\mathbf{i}=\sqrt{-1}$, r is the wave number, and $\xi=\xi(r)$ represents the amplification factor. The stability condition requires that $|\xi(r)|\leq 1$.

Fully Explicit Scheme (ES). We consider only von-Neumann stability analysis to explain this method on fully explicit scheme (ES). The Equation (2.2) is considered in the following way.

$$U_i^{n+1}=\mu\frac{k}{h^2}(U_{i+1}^n-2U_i^n+U_{i-1}^n)+\alpha k(U_i)^n(1-(U_i^\nu)^n)+U_i^n.$$

Linear form of the above equation is.

$$U_i^{n+1}=\mu\frac{k}{h^2}(U_{i+1}^n-2U_i^n+U_{i-1}^n)+\alpha k(U_i)^n(1-constant)+U_i^n, \tag{3.13}$$

Substituting the Fourier mode $U_i^n=\xi^{\alpha nk}e^{\mathbf{i}rih}$ into the linearized form of the difference Equation (3.13) yields:

$$\xi^{\alpha k}=1+\mu\frac{k}{h^2}\cdot 2(\cos(rh)-1)+\alpha k(1-\text{constant}).$$

For stability, we require $|\xi|\leq 1$. Therefore, $|\xi^{\alpha k}|\leq 1$, which imposes a condition on the parameters. The stability condition becomes:

$$\left|1+\mu\frac{k}{h^2}\cdot 2(\cos(rh)-1)+\alpha k(1-\text{constant})\right|\leq 1.$$

The stability condition depends on the values of μ, α, k, and h. Typically, for the linear diffusion part, stability is ensured by the condition:

$$\mu \frac{k}{h^2} \leq \frac{1}{2}.$$

Semi Implicit Scheme (SIS). Equation (2.5) converting into linear form by considering as follows.

$$U_i^{n+1} = \mu \frac{k}{h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) + \alpha k U_i^{(n+1)}(1 - constant) + U_i^n. \tag{3.14}$$

Substituting the Fourier mode $U_i^n = \xi^{\alpha n k} e^{\mathbf{i} r i h}$ into the linearized form of the difference Equation (3.14) yields:

$$\xi^{\alpha k} = \frac{1 + \mu \frac{k}{h^2} \cdot 2(\cos(rh) - 1)}{1 - \alpha k(1 - \text{constant})}.$$

For stability, we require $|\xi| \leq 1$. Therefore:

$$\left| \frac{1 + \mu \frac{k}{h^2} \cdot 2(\cos(rh) - 1)}{1 - \alpha k(1 - \text{constant})} \right| \leq 1.$$

Fully Implicit Scheme (IS). Equation (2.8) converting into linear form as follows.

$$U_i^{n+1} = \mu \frac{k}{h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) + \alpha k U_i^{n+1}(1 - constant) + U_i^n. \tag{3.15}$$

Substituting the Fourier mode $U_i^n = \xi^{\alpha n k} e^{\mathbf{i} r i h}$ into the linearized form of the difference Equation (3.15) yields:

$$\xi^{\alpha k} = \frac{1 + \alpha k \left(\xi^{n+1}(1 - \text{constant})\right)}{1 - \mu \frac{k}{h^2} \cdot 2(\cos(rh) - 1)}.$$

The stability condition is:

$$\left| \frac{1 + \alpha k \left(\xi^{n+1}(1 - \text{constant})\right)}{1 - \mu \frac{k}{h^2} \cdot 2(\cos(rh) - 1)} \right| \leq 1.$$

Semi-Implicit Crank-Nicolson Scheme (SICNS). Equation (2.11) converting into linear form as follows.

$$U_i^{n+1} = \frac{k}{2h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^n - 2U_i^n + U_{i-1}^n) + \alpha k U_i^{(n+1)}(1 - constant) + U_i^n. \tag{3.16}$$

Substituting the Fourier mode $U_i^n = \xi^{\alpha n k} e^{\mathbf{i} r i h}$ into the linearized form of the difference Equation (3.16) yields:

$$\xi^{\alpha k} = \frac{\frac{k}{h^2}(\cos(rh) - 1) \cdot (\xi^{\alpha k} + 1) + \alpha k \left(\xi^{n+1}(1 - \text{constant})\right) + 1}{1 - \frac{k}{h^2}(\cos(rh) - 1)}.$$

The stability condition is:

$$\left| \frac{\frac{k}{h^2}(\cos(rh) - 1) \cdot (\xi^{\alpha k} + 1) + \alpha k \left(\xi^{n+1}(1 - \text{constant})\right) + 1}{1 - \frac{k}{h^2}(\cos(rh) - 1)} \right| \leq 1.$$

Implicit Crank-Nicolson Discretization of Type -1 (ICNS-1). Equation (2.15) converting into linear form as follows.

$$U_i^{n+1} = \frac{k}{2h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}) + \alpha k U_i^{n+1}(1 - constant) + U_i^n. \tag{3.17}$$

Substituting the Fourier mode $U_i^n = \xi^{\alpha n k} e^{\mathbf{i} r i h}$ into the linearized form of the difference Equation (3.17) yields,

$$\xi^{\alpha k} = \frac{\frac{k}{h^2}(\cos(rh) - 1)\left(\xi^{\alpha k} + 1\right) + \alpha k\left(\xi^{\alpha k}\right)(1 - \text{constant}) + 1}{1 - \frac{k}{h^2}(\cos(rh) - 1)}.$$

The stability condition is:

$$\left|\frac{\frac{k}{h^2}(\cos(rh) - 1)\left(\xi^{\alpha k} + 1\right) + \alpha k\left(\xi^{\alpha k}\right)(1 - \text{constant}) + 1}{1 - \frac{k}{h^2}(\cos(rh) - 1)}\right| \leq 1.$$

Implicit Crank-Nicolson Discretization of Type-2 (ICNS-2). Equation (2.19) converting into linear form as follows.

$$U_i^{n+1} - U_i^n = \frac{k}{2h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}) + \eta(\frac{U_i^{n+1} + U_i^n}{2})(1 - constant). \tag{3.18}$$

Substituting the Fourier mode $U_i^n = \xi^{\alpha n k} e^{\mathbf{i} r i h}$ into the linearized form of the difference Equation (3.18) yields,

$$\xi^{\alpha k} = \frac{\frac{k}{h^2}(\cos(rh) - 1)\left(\xi^{\alpha k} + 1\right) + \eta\left(\frac{(\xi^{\alpha k}) + (\xi^{\alpha n k})}{2}\right)(1 - \text{constant})}{1}.$$

The stability condition is:

$$\left|\frac{\frac{k}{h^2}(\cos(rh) - 1)\left(\xi^{\alpha k} + 1\right) + \eta\left(\frac{(\xi^{\alpha k}) + (\xi^{\alpha n k})}{2}\right)(1 - \text{constant})}{1}\right| \leq 1.$$

If $1 - \text{constant} \leq 0$, (the reaction term is nonpositive), then it is unconditionally stable.

$$|\xi| \leq 1, \quad \text{for all } k > 0, \text{ and } h > 0.$$

Otherwise, conditionally stable.

4. Numerical Experiments

In this context, we simulate test examples following the presentation of the suggested scheme to validate the theoretical findings facilitated by the new method. Analyzing the Fisher equation proves challenging in this analysis. The proposed scheme's accuracy is ensured by the sufficiently smooth boundary conditions. We implement and generate numerical solutions for boundary value problems using MATLAB(R2013a) software. Presented are the numerical results of the different methods applied to various instances of the Fisher Equation (1.1) using MATLAB. With the help of the exact solution, we measured the accuracy of the numerical method. Assess the accuracy and efficiency of the proposed method by evaluating the l_2 and l_∞ at final time step t_N.

$$l_2 = \left[\frac{1}{M}\sum_{m=0}^{M}(U_m - u_m)^2\right]^{1/2}, \quad l_\infty = \max_{0 \leq m \leq M}|U_m - u_m|.$$

Here, u_m is numerical solution and U_m as the similarity solution corresponding to the node at position x_m.

The rate of numerical convergence for these methods was determined by using the following formula:

$$ROC \approx \frac{log(E(N_2)/E(N_1))}{log(N_1/N_2)}, \tag{4.1}$$

where the L_∞ error value obtained by employing N_j number of grid points is denoted as $E(N_j)$. The L_2 and L_∞ error norms are computed for Table 18 at the final time T = 0.1, with $\alpha = 1$. The table also displays the rate of convergence (ROC) for the suggested methods across various grid sizes.

4.1. **Results and Discussion.** This paper employs the proposed schemes, which include explicit, semi-implicit, and implicit methods, alongside semi-implicit Crank-Nicolson, implicit Crank-Nicolson type-1, and implicit Crank-Nicolson type-2 schemes, to numerically solve the Fisher equation. A comparison is made between the numerical results and the exact solution. In Table 1, 3, and 7, the numerical results obtained from the implicit Crank-Nicolson type-2 schemes demonstrate better alignment with the exact solution for Example 4.1, considering a time step of $\Delta t = 0.000005$ and a final time of $T = 0.1$ for both $\alpha = 6$ and $\alpha = 1$. Conversely, in Table 5, the numerical results from the implicit Crank-Nicolson type-1 scheme prove more accurate in replicating the exact solution for example-2 under a time step of $\Delta t = 0.000005$ and a final time of $T = 0.1$ for $\nu = \frac{1}{\sqrt{2}}$. Tables 9, 10, and 11 display the solutions obtained from the mentioned schemes and compare them with a few existing methods [23]. The proposed ICNS-1 and ICNS-2 schemes achieve second-order spatial accuracy with reduced computational effort, requiring only a single matrix solve per step, while ICNS-2 further improves accuracy by minimizing nonlinear truncation errors.

4.2. **Illusrative examples.** In this section, we provide three examples to showcase the validity and effectiveness of our proposed method. For each case, we obtain both initial and boundary conditions directly from analytical solutions. The calculations for these examples are conducted using MATLAB.

Example 4.1. Consider Fisher's Equation (1.1) for $\beta = \nu = 1$.

$$u_t = u_{xx} + \alpha u(1-u), \tag{4.2}$$

subject to the initial condition

$$u(x,0) = \frac{1}{(1+e^{\sqrt{\frac{\alpha}{6}}x})^2}, \tag{4.3}$$

and the boundary conditions

$$u(0,t) = \frac{1}{(1+e^{-5t})^2},$$
$$u(1,t) = \frac{1}{(1+e^{1-5t})^2},$$

where the exact solution is given [23] by

$$u(x,t) = \frac{1}{(1+e^{\sqrt{\frac{\alpha}{6}}x-\frac{5}{6}\alpha t})^2}, \tag{4.4}$$

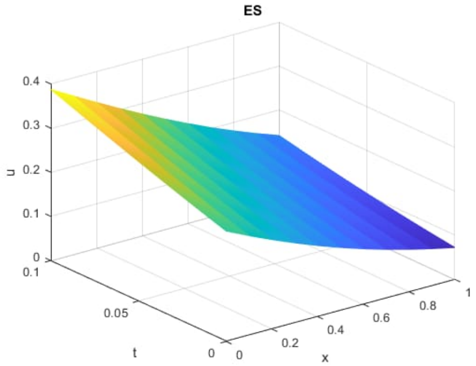

FIGURE 1. Solution at $\alpha = 6$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

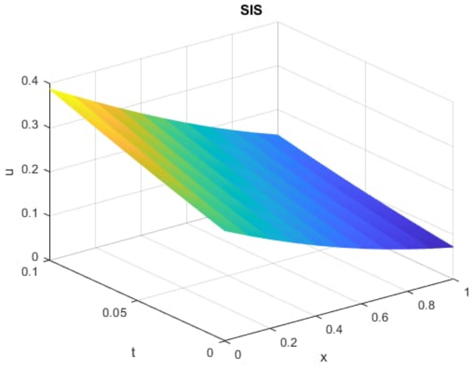

FIGURE 2. Solution at $\alpha = 6$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

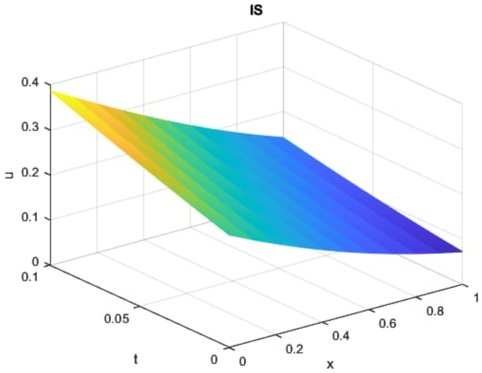

FIGURE 3. Solution at $\alpha = 6$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

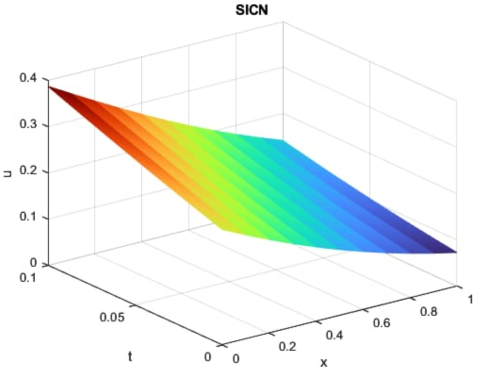

FIGURE 4. Solution at $\alpha = 6$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

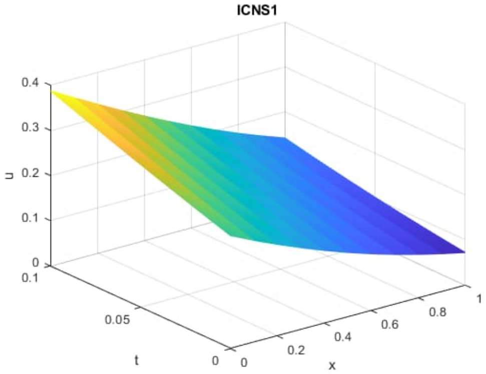

FIGURE 5. Solution at $\alpha = 6$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

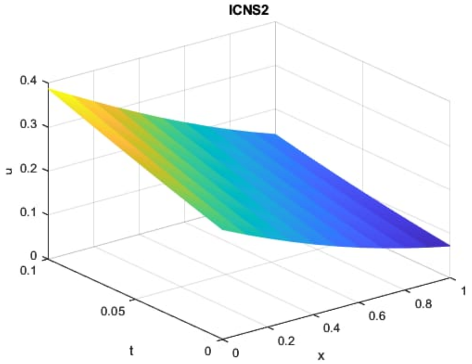

FIGURE 6. Solution at $\alpha = 6$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

Table 1. Numerical and exact solution of Example 4.1 at $\Delta t = 0.000005$ and $T = 0.1$ for $\alpha = 6$.

	Computed solution						**Exact solution**
x	ES	SIS	IS	SICNS	ICNS-1	ICNS-2	
0	0.387455619	0.387455619	0.387455619	0.387455619	0.387455619	0.387455619	0.387455619
0.1	0.358420950	0.358421753	0.358421429	0.358420902	0.358421477	0.358424323	0.358426914
0.2	0.329973045	0.329974448	0.329973933	0.329972971	0.329974007	0.329976117	0.329984205
0.3	0.302302174	0.302303995	0.302303386	0.302302091	0.30230347	0.302304964	0.302317425
0.4	0.275585137	0.275587205	0.275586575	0.275585056	0.275586655	0.275587693	0.275603147
0.5	0.249980707	0.249982862	0.249982262	0.249980637	0.249982331	0.249983095	0.250000000
0.6	0.225625778	0.225627862	0.225627329	0.225625724	0.225627383	0.225628071	0.225644772
0.7	0.202632409	0.202634257	0.202633819	0.202632371	0.202633857	0.202634669	0.20264943
0.8	0.181085897	0.181087332	0.181087014	0.181085875	0.181087036	0.18108816	0.181099172
0.9	0.161043954	0.161044779	0.161044606	0.161043944	0.161044615	0.161046218	0.161051594
1	0.142536957	0.142536957	0.142536957	0.142536957	0.142536957	0.142536957	0.142536957

Table 2. Absolute errors of Example 4.1 at $\Delta t = 0.000005$ and $T = 0.1$ for $\alpha = 6$.

x	ES	SIS	IS	SICNS	ICNS-1	ICNS-2
0.1	5.96450E-06	5.16190E-06	5.48500E-06	6.01200E-06	5.43750E-06	2.59160E-06
0.2	1.11605E-05	9.75670E-06	1.02720E-05	1.12343E-05	1.01981E-05	8.08820E-06
0.3	1.52504E-05	1.34296E-05	1.40386E-05	1.53338E-05	1.39551E-05	1.24605E-05
0.4	1.80101E-05	1.59421E-05	1.65727E-05	1.80909E-05	1.64920E-05	1.54547E-05
0.5	1.92931E-05	1.71381E-05	1.77385E-05	1.93628E-05	1.76687E-05	1.69049E-05
0.6	1.89942E-05	1.69105E-05	1.74435E-05	1.90486E-05	1.73891E-05	1.67010E-05
0.7	1.70213E-05	1.51731E-05	1.56107E-05	1.70591E-05	1.55729E-05	1.47614E-05
0.8	1.32745E-05	1.18397E-05	1.21577E-05	1.32970E-05	1.21352E-05	1.10117E-05
0.9	7.64000E-06	6.81480E-06	6.98860E-06	7.64990E-06	6.97870E-06	5.37590E-06

Example 4.2. Consider the Fisher's Equation (1.1) for $\beta = 2$ and $\alpha = \nu = 1$,

$$u_t = u_{xx} + u^2(1-u), \qquad 0 \le x \le 1, \tag{4.5}$$

with the initial condition

$$u(x,0) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}, \tag{4.6}$$

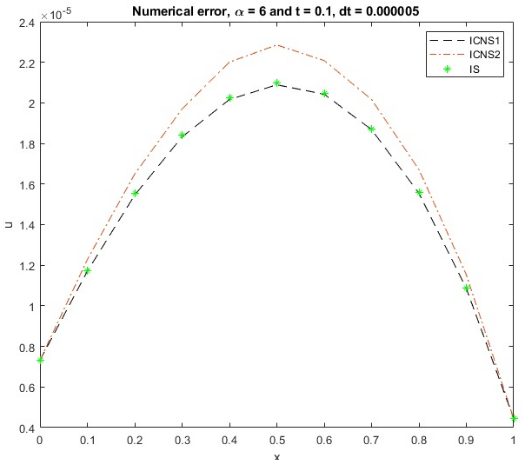

FIGURE 7. Absolute errors of ICNS-1, ICNS-2, and IS of Example 4.1.

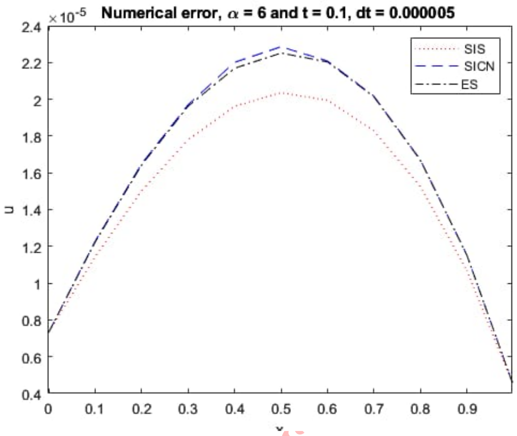

FIGURE 8. Absolute errors of SIS, SICN, and ES of Example 4.1.

TABLE 3. Numerical and exact solution of Example 4.1 at $\Delta t = 0.000005$ and $T = 0.1$ for $\alpha = 1$.

	Computed solution						**Exact solution**
x	ES	SIS	IS	SICNS	ICNS-1	ICNS-2	
0	0.271254811	0.271254811	0.271254811	0.271254811	0.271254811	0.271254811	0.271254811
0.1	0.260738237	0.260738258	0.260738251	0.260738236	0.260738252	0.260738703	0.260738428
0.2	0.250420762	0.250420799	0.250420787	0.250420761	0.250420789	0.250421159	0.250421096
0.3	0.240311256	0.240311303	0.240311288	0.240311254	0.240311290	0.240311594	0.240311688
0.4	0.230417894	0.230417946	0.230417930	0.230417892	0.230417933	0.230418190	0.230418385
0.5	0.220748135	0.22074819	0.220748174	0.220748133	0.220748176	0.220748410	0.220748648
0.6	0.211308702	0.211308755	0.211308740	0.211308700	0.211308742	0.211308976	0.211309201
0.7	0.202105564	0.202105610	0.202105598	0.202105562	0.202105599	0.202105857	0.202106010
0.8	0.193143927	0.193143963	0.193143953	0.193143925	0.193143954	0.193144255	0.193144276
0.9	0.184428227	0.184428248	0.184428243	0.184428226	0.184428243	0.184428602	0.184428430
1	0.175962132	0.175962132	0.175962132	0.175962132	0.175962132	0.175962132	0.175962132

In this case the exact solution is given [23] by

$$u(x,t) = \frac{1}{1+e^{\frac{(x-vt)}{\sqrt{2}}}}, \tag{4.7}$$

where, $v = \frac{1}{\sqrt{2}}$.

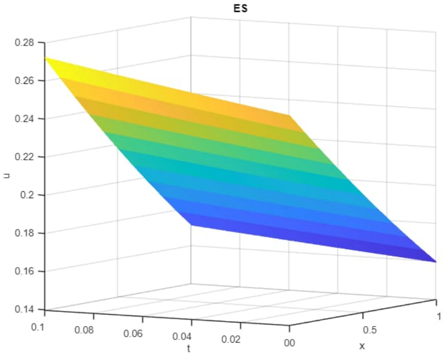

FIGURE 9. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

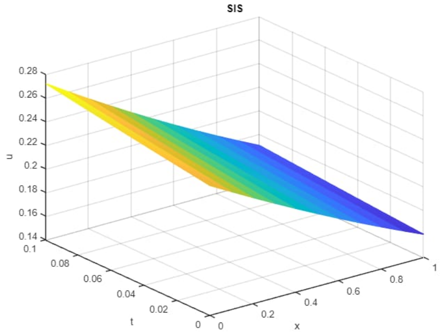

FIGURE 10. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

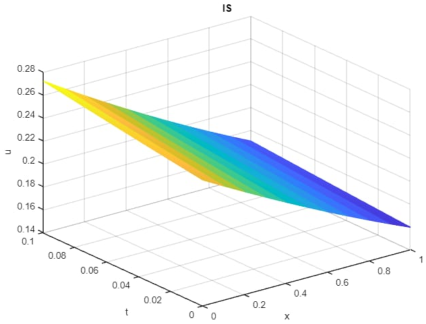

FIGURE 11. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

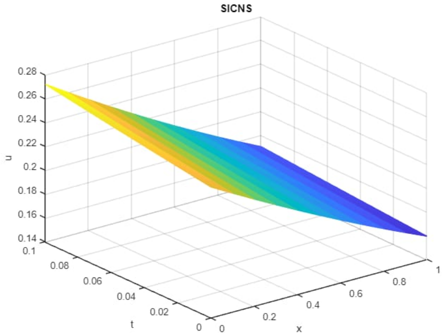

FIGURE 12. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

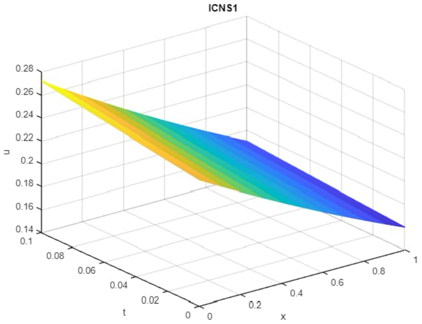

FIGURE 13. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

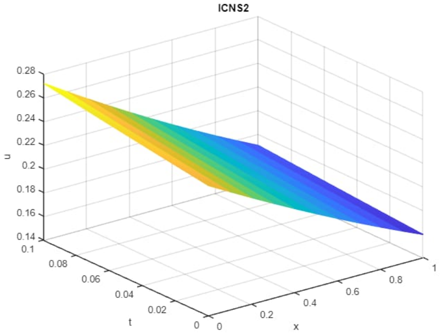

FIGURE 14. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.1.

Example 4.3. Consider the generalized Fisher's equation specified in [23]

$$u_t = u_{xx} + u(1 - u^{\alpha}), \tag{4.8}$$

TABLE 4. Absolute errors of Example 4.1 at $\Delta t = 0.000005$ and $T = 0.1$ for $\alpha = 1$.

x	ES	SIS	IS	SICNS	ICNS-1	ICNS-2
0.1	1.913E-07	1.700E-07	1.772E-07	1.924E-07	1.761E-07	2.752E-07
0.2	3.335E-07	2.971E-07	3.090E-07	3.354E-07	3.072E-07	6.280E-08
0.3	4.320E-07	3.853E-07	4.001E-07	4.342E-07	3.979E-07	9.420E-08
0.4	4.909E-07	4.385E-07	4.545E-07	4.933E-07	4.521E-07	1.949E-07
0.5	5.130E-07	4.588E-07	4.747E-07	5.153E-07	4.725E-07	2.386E-07
0.6	4.987E-07	4.463E-07	4.612E-07	5.007E-07	4.592E-07	2.249E-07
0.7	4.456E-07	3.991E-07	4.119E-07	4.473E-07	4.102E-07	1.527E-07
0.8	3.494E-07	3.131E-07	3.229E-07	3.507E-07	3.216E-07	2.110E-08
0.9	2.035E-07	1.824E-07	1.879E-07	2.041E-07	1.872E-07	1.712E-07

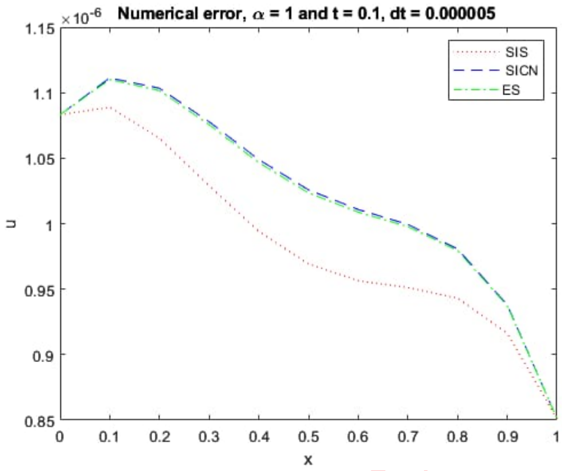

FIGURE 15. Absolute errors of SIS, SICN and ES of Example 4.1.

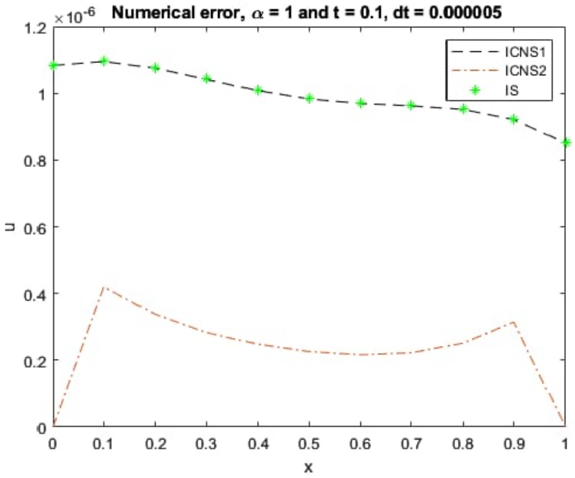

FIGURE 16. Absoluate errors of ICNS-1, SICN-2, and IS of Example 4.1.

with intial condition

$$u(x,0) = \{\frac{1}{2}\tanh(-\frac{\alpha}{2\sqrt{2\alpha+4}}x) + \frac{1}{2}\}^{\frac{2}{\alpha}}, \tag{4.9}$$

The exact solution is specified in [23] by

$$u(x,t) = \{\frac{1}{2}\tanh(-\frac{\alpha}{2\sqrt{2\alpha+4}}(x - \frac{\alpha+4}{\sqrt{2\alpha+4}}t)) + \frac{1}{2}\}^{\frac{2}{\alpha}}, \tag{4.10}$$

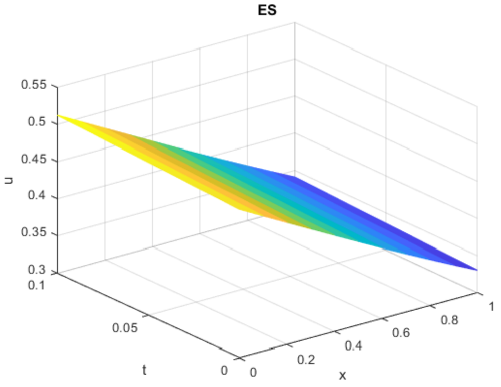

FIGURE 17. Solution at $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.2.

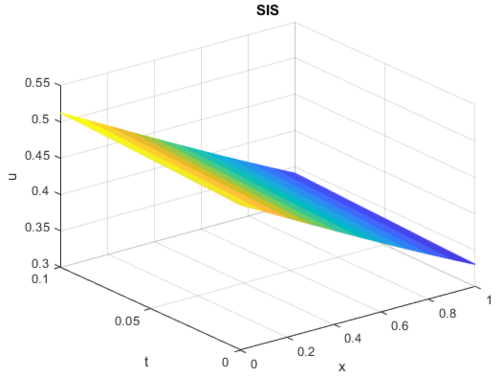

FIGURE 18. Solution at $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.2.

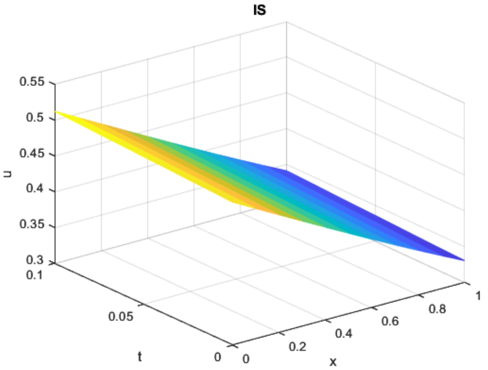

FIGURE 19. Solution at $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.2.

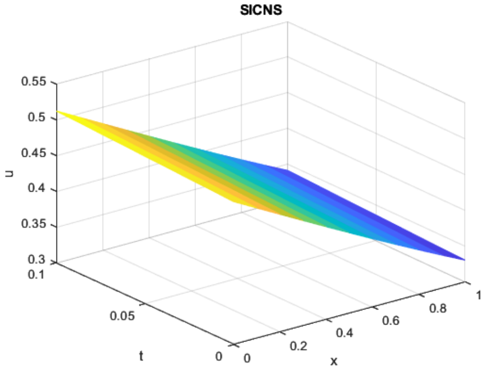

FIGURE 20. Solution at $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.2.

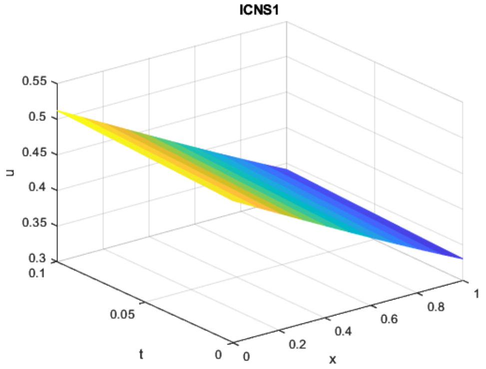

FIGURE 21. Solution at $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.2.

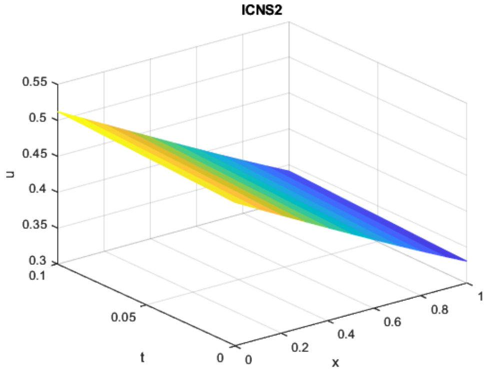

FIGURE 22. Solution at $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.2.

TABLE 5. Numerical and exact solution of Example 4.2 at $\Delta t = 0.000005$ and $T = 0.1$ for $v = 1/\sqrt{2}$.

	Computed solution						**Exact solution**
x	ES	SIS	IS	SICNS	ICNS-1	ICNS-2	
0	0.512497397	0.512497397	0.512497397	0.512497397	0.512497397	0.512497397	0.512497397
0.1	0.494822242	0.494822239	0.494822248	0.494822241	0.494822250	0.494822248	0.494822516
0.2	0.477160017	0.477160010	0.477160026	0.477160014	0.477160030	0.477160027	0.477160566
0.3	0.459554740	0.459554732	0.459554752	0.459554737	0.459554757	0.459554753	0.459555549
0.4	0.442049869	0.442049861	0.442049883	0.442049866	0.442049888	0.442049883	0.442050898
0.5	0.424687871	0.424687862	0.424687885	0.424687868	0.424687891	0.424687886	0.424689058
0.6	0.407509820	0.407509812	0.407509834	0.407509817	0.407509839	0.407509834	0.407511076
0.7	0.390555022	0.390555015	0.390555035	0.390555020	0.390555039	0.390555034	0.390556227
0.8	0.373860676	0.373860671	0.373860686	0.373860674	0.373860689	0.373860685	0.373861678
0.9	0.357461570	0.357461567	0.357461576	0.357461569	0.357461578	0.357461576	0.357462182
1	0.341389832	0.341389832	0.341389832	0.341389832	0.341389832	0.341389832	0.341389832

TABLE 6. Absolute errors of of Example 4.2 at $\Delta t = 0.000005$ and $T = 0.1$ for $v = 1/\sqrt{2}$.

x	ES	SIS	IS	SICNS	ICNS-1	ICNS-2
0.1	2.7310E-07	2.7700E-07	2.6780E-07	2.7480E-07	2.6550E-07	2.6720E-07
0.2	5.4960E-07	5.5610E-07	5.4030E-07	5.5230E-07	5.3650E-07	5.3950E-07
0.3	8.0890E-07	8.1680E-07	7.9680E-07	8.1200E-07	7.9200E-07	7.9610E-07
0.4	1.0292E-06	1.0377E-06	1.0154E-06	1.0324E-06	1.0102E-06	1.0149E-06
0.5	1.1872E-06	1.1956E-06	1.1727E-06	1.1902E-06	1.1674E-06	1.1725E-06
0.6	1.2559E-06	1.2635E-06	1.2418E-06	1.2585E-06	1.2368E-06	1.2419E-06
0.7	1.2052E-06	1.2118E-06	1.1926E-06	1.2074E-06	1.1883E-06	1.1930E-06
0.8	1.0022E-06	1.0071E-06	9.9230E-07	1.0038E-06	9.8900E-07	9.9270E-07
0.9	6.1210E-07	6.1480E-07	6.0630E-07	6.1290E-07	6.0440E-07	6.0660E-07

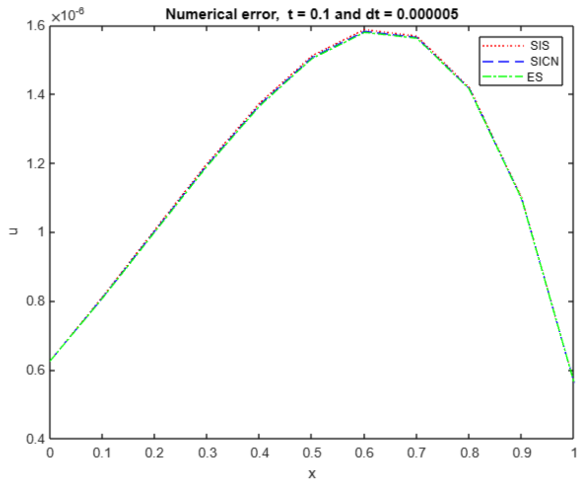

FIGURE 23. Absolute errors of SIS, SICN, and ES of Example 4.2.

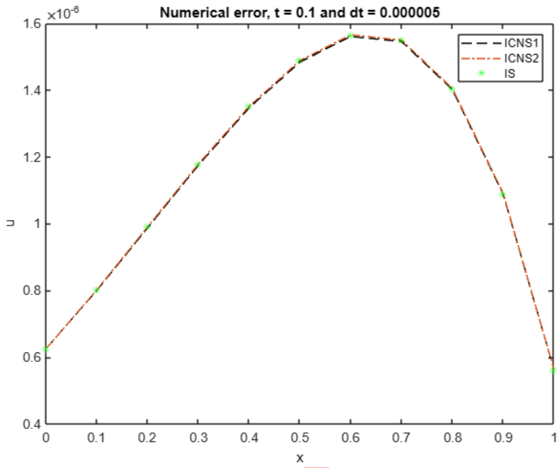

FIGURE 24. Absoluate errors of ICNS-1, ICNS-2, and IS of Example 4.2.

TABLE 7. Numerical and exact solution of Example 4.3 at $\Delta t = 0.000005$ and $T = 0.1$ for $\alpha = 1$.

	Computed solution						**Exact solution**
x	ES	SIS	IS	SICNS	ICNS-1	ICNS-2	
0	0.271254811	0.271254811	0.271254811	0.271254811	0.271254811	0.271254811	0.271254811
0.1	0.260738237	0.260738258	0.260738251	0.260738236	0.260738252	0.260738703	0.260738428
0.2	0.250420762	0.250420799	0.250420787	0.250420761	0.250420789	0.250421159	0.250421096
0.3	0.240311256	0.240311303	0.240311288	0.240311254	0.240311290	0.240311594	0.240311688
0.4	0.230417894	0.230417946	0.230417930	0.230417892	0.230417933	0.230418190	0.230418385
0.5	0.220748135	0.22074819	0.220748174	0.220748133	0.220748176	0.220748410	0.220748648
0.6	0.211308702	0.211308755	0.211308740	0.211308700	0.211308742	0.211308976	0.211309201
0.7	0.202105564	0.202105610	0.202105598	0.202105562	0.202105599	0.202105857	0.202106010
0.8	0.193143927	0.193143963	0.193143953	0.193143925	0.193143954	0.193144255	0.193144276
0.9	0.184428227	0.184428248	0.184428243	0.184428226	0.184428243	0.184428602	0.184428430
1	0.175962132	0.175962132	0.175962132	0.175962132	0.175962132	0.175962132	0.175962132

Table 8. Absolute errors of Example 4.3 at $\Delta t = 0.000005$ and $T = 0.1$ for $\alpha = 1$.

x	ES	SIS	IS	SICNS	ICNS-1	ICNS-2
0.1	1.913E-07	1.700E-07	1.772E-07	1.924E-07	1.761E-07	1.752E-07
0.2	3.335E-07	2.971E-07	3.090E-07	3.354E-07	3.072E-07	6.280E-08
0.3	4.320E-07	3.853E-07	4.001E-07	4.342E-07	3.979E-07	9.420E-08
0.4	4.909E-07	4.385E-07	4.545E-07	4.933E-07	4.521E-07	1.949E-07
0.5	5.130E-07	4.588E-07	4.747E-07	5.153E-07	4.725E-07	2.386E-07
0.6	4.987E-07	4.463E-07	4.612E-07	5.007E-07	4.592E-07	2.249E-07
0.7	4.456E-07	3.991E-07	4.119E-07	4.473E-07	4.102E-07	1.527E-07
0.8	3.494E-07	3.131E-07	3.229E-07	3.507E-07	3.216E-07	2.110E-08
0.9	2.035E-07	1.824E-07	1.879E-07	2.041E-07	1.872E-07	1.712E-07

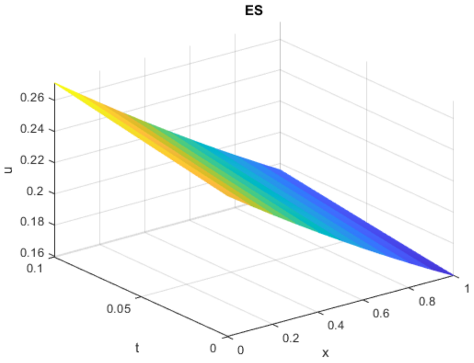

Figure 25. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.3.

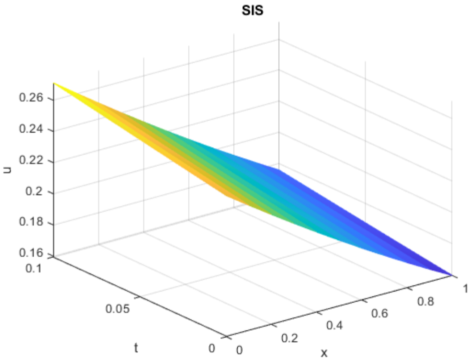

Figure 26. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.3.

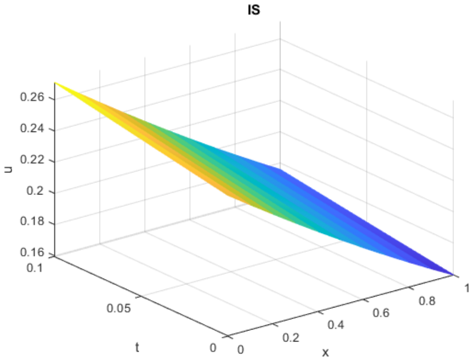

Figure 27. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.3.

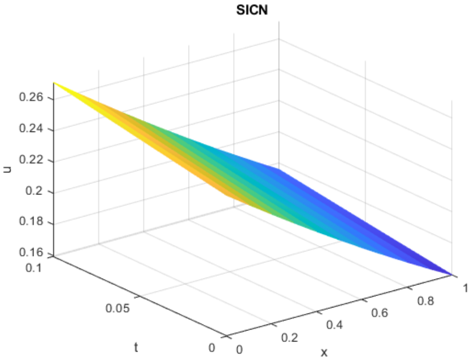

Figure 28. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.3.

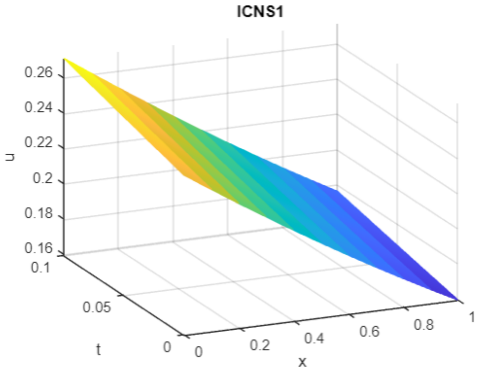

FIGURE 29. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.3.

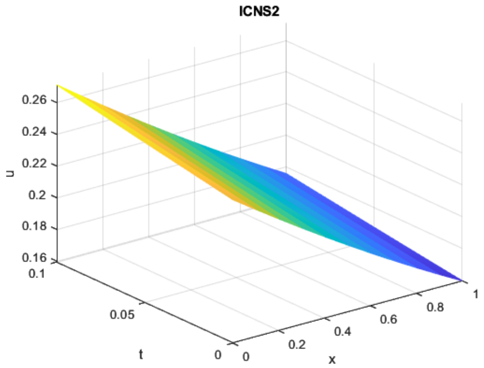

FIGURE 30. Solution at $\alpha = 1$, $\Delta t = 0.000005$ and $T = 0.1$ for Example 4.3.

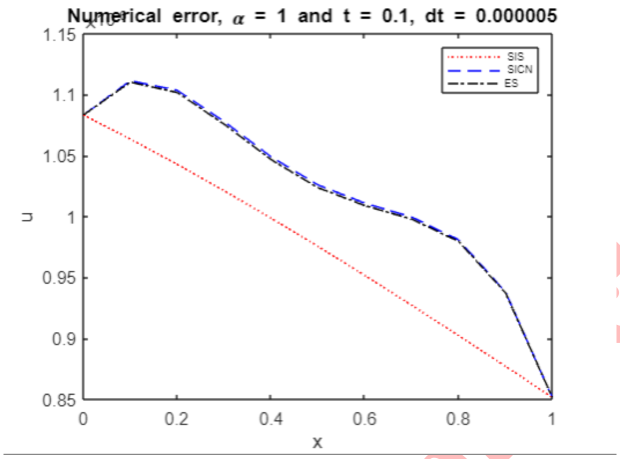

FIGURE 31. Absolute errors of SIS, SICN, and ES of Example 4.3.

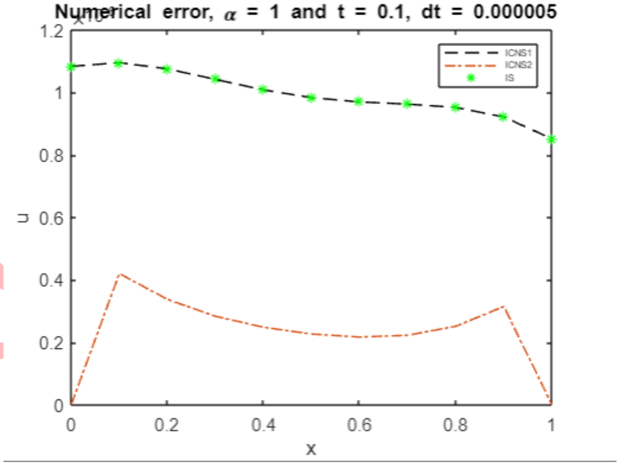

FIGURE 32. Absoluate errors of ICNS-1, ICNS-2, and IS of Example 4.3.

Table 9. Comparison of numerical and exact solutions for Example 4.1 at different $\Delta t = 0.0001$ and $\Delta t = 0.00005$ for $\alpha = 6$.

x	T	$\Delta t = 0.0001$	Present solution at $\Delta t = 0.0001$			Exact solution	$\Delta t = 0.00005$	Present solution at $\Delta t = 0.00005$			Exact solution
x	T	R.Jiwari[23]	(ICNS1)	SIS	SICN	Exact	R.Jiwari[23]	(ICNS1)	SIS	SICN	Exact
0.25	0.5	0.81847	0.818386	0.818403	0.818418	0.818393	0.81843	0.818390	0.818399	0.818406	0.818393
	1.0	0.98293	0.982916	0.982921	0.982921	0.982919	0.98292	0.982917	0.982920	0.982920	0.982919
	2.0	0.99988	0.999883	0.999883	0.999883	0.999883	0.99988	0.999883	0.999883	0.999883	0.999883
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000	1.00000	1.000000	1.000000	1.000000	1.000000
0.5	0.5	0.77580	0.775796	0.775817	0.775840	0.775803	0.77585	0.775801	0.775812	0.775823	0.775803
	1.0	0.97816	0.978143	0.978150	0.978151	0.978150	0.97815	0.978145	0.978148	0.978149	0.978147
	2.0	0.99985	0.999850	0.999850	0.999850	0.999850	0.99985	0.999850	0.999850	0.999850	0.999850
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000	1.00000	1.000000	1.000000	1.000000	1.000000
0.75	0.5	0.72594	0.725818	0.725834	0.725853	0.725824	0.72588	0.725822	0.725830	0.725839	0.725824
	1.0	0.97209	0.972067	0.972073	0.972074	0.972071	0.97208	0.972069	0.972072	0.972072	0.972071
	2.0	0.99981	0.999808	0.999808	0.999808	0.999808	0.99981	0.999808	0.999808	0.999808	0.999808
	5.0	1.00000	1.000000	1.000000	1.000000	1.000000	1.00000	1.000000	1.000000	1.000000	1.000000

Table 10. Comparison of numerical and exact solutions of Example 4.1 at $\Delta t = 0.0001$ and $\Delta t = 0.00005$ for $\alpha = 1$.

x	T	$\Delta t = 0.0001$	Present solution at $\Delta t = 0.0001$			Exact solution	$\Delta t = 0.00005$	Present solution at $\Delta t = 0.00005$			Exact solution
x	T	R.Jiwari [23]	ICNS1	SIS	SICN	Exact solution	R.Jiwari [23]	ICNS1	SIS	SICN	Exact solution
0.25	0.5	0.33412	0.3340945	0.3340950	0.3340936	0.3340942	0.33411	0.3340943	0.3340945	0.3340938	0.3340942
	1.0	0.45576	0.4557389	0.4573002	0.4557384	0.4557387	0.45575	0.4557388	0.4557392	0.4557386	0.4557387
	2.0	0.68397	0.6839505	0.6839573	0.6839510	0.6839507	0.68395	0.6839506	0.6839512	0.6839509	0.6839507
	5.0	0.96653	0.9665249	0.9665252	0.9665251	0.9665250	0.96653	0.9665250	0.9665251	0.9665251	0.9665250
0.5	0.5	0.30576	0.3057392	0.3057399	0.3057380	0.3057388	0.30575	0.3057389	0.3057393	0.3057383	0.3057388
	1.0	0.42553	0.4255088	0.4275871	0.4255082	0.4255085	0.42552	0.4255086	0.4255092	0.4255083	0.4255085
	2.0	0.65924	0.6592159	0.6592255	0.6592166	0.6592161	0.65922	0.6592161	0.6592169	0.6592164	0.6592161
	5.0	0.96303	0.9630282	0.9630286	0.9630286	0.9630284	0.96303	0.9630283	0.9630285	0.9630285	0.9630284
0.75	0.5	0.27838	0.2783537	0.2783542	0.2783528	0.2783534	0.27837	0.2783535	0.2783537	0.2783530	0.2783534
	1.0	0.39544	0.3954117	0.3969715	0.3954112	0.3954114	0.39542	0.3954115	0.3954119	0.3954113	0.3954114
	2.0	0.63338	0.6333576	0.6333645	0.6333581	0.6333578	0.63336	0.6333577	0.6333583	0.6333580	0.6333578
	5.0	0.95918	0.9591780	0.9591783	0.9591783	0.9591782	0.95918	0.9591781	0.9591782	0.9591782	0.9591782

5. CONCLUSION

In this study, we tackled the Fisher equation using various schemes and demonstrated the convergence of the difference schemes. Comparisons were made between the solutions obtained from these schemes and the exact solution. Particularly, in Table 9, 10, and 11, the implicit Crank-Nicolson discretization of type -1 (ICNS-1) scheme exhibited superior accuracy for Example 4.1, while the implicit Crank-Nicolson scheme of type-2 (ICNS-2) performed exceptionally well for Example 4.2. These findings were also compared with a few existing results. It was observed that both ICNS-1 and ICNS-2 consistently provided more accurate results.

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Table 11. Comparison of numerical and exact solutions of Example 4.2 at $\Delta t = 0.0001$ and $\Delta t = 0.00005$ for $v = 1/\sqrt{2}$.

x	T	$\Delta t = 0.0001$	Present solution at $\Delta t = 0.0001$			Exact solution	$\Delta t = 0.00005$	Present solution at $\Delta t = 0.00005$			Exact solution
x	T	R.Jiwari [23]	ICNS1	SIS	SICN	Exact solution	R.Jiwari [23]	ICNS1	SIS	SICN	Exact solution
0.25	0.5	0.51831	0.51829770	0.51829697	0.51829711	0.51829765	0.51830	0.51829762	0.51829726	0.51829733	0.51829765
	1.0	0.58012	0.58010989	0.58010917	0.58010931	0.58010959	0.58012	0.58010983	0.58010947	0.58010954	0.58010959
	2.0	0.69493	0.69492070	0.69492014	0.69492023	0.69492013	0.69492	0.69492070	0.69492042	0.69492046	0.69492013
	5.0	0.91078	0.91078213	0.91078209	0.91078205	0.91078221	0.91078	0.91078219	0.91078217	0.91078215	0.91078221
0.5	0.5	0.47415	0.47413474	0.47413377	0.47413396	0.47413476	0.47414	0.47413463	0.47413415	0.47413424	0.47413476
	1.0	0.53656	0.53654668	0.53654570	0.53654590	0.53654636	0.53655	0.53654659	0.53654610	0.53654620	0.53654636
	2.0	0.65622	0.65621003	0.65620924	0.65620937	0.65620927	0.65621	0.65621001	0.65620962	0.65620968	0.65620927
	5.0	0.89534	0.89533618	0.89533612	0.89533606	0.89533626	0.89534	0.89533626	0.89533623	0.89533620	0.89533626
0.75	0.5	0.43039	0.43037278	0.43037206	0.43037220	0.43037285	0.43038	0.43037269	0.43037233	0.43037240	0.43037285
	1.0	0.49243	0.49241825	0.49241751	0.49241765	0.49241806	0.49243	0.49241817	0.49241780	0.49241788	0.49241806
	2.0	0.61532	0.61530618	0.61530557	0.61530567	0.61530563	0.61531	0.61530616	0.61530586	0.61530591	0.61530563
	5.0	0.87758	0.87757561	0.87757556	0.87757552	0.87757565	0.87758	0.87757568	0.87757565	0.87757563	0.87757565

Table 12. Comparison of errors in L_2 norm and L_∞ norm for different values of $\alpha = 6$, $T = 0.1$, and $\Delta t = 0.000005$, corresponding to Example 4.1.

	ES	SIS	IS	SICN	ICNS-1	ICNS-2
L_2	1.77644E-05	1.61992E-05	1.66452E-05	1.78162E-05	1.65935E-05	1.54478E-05
L_∞	2.25148E-05	2.03599E-05	2.09603E-05	2.25845E-05	2.08905E-05	2.01267E-05

Table 13. Comparison of errors in L_2 norm and L_∞ norm for different values of $\alpha = 1$, $T = 0.1$, and $\Delta t = 0.000005$, corresponding to Example 4.1.

	ES	SIS	IS	SICN	ICNS-1	ICNS-2
L_2	1.07203E-06	2.34867E-01	2.34867E-01	1.07357E-06	1.04588E-06	2.72177E-07
L_∞	1.11000E-06	2.71255E-01	2.71255E-01	1.11100E-06	1.09500E-06	4.20000E-07

Table 14. Comparison of errors in L_2 norm and L_∞ norm for different values of $\nu = \frac{1}{\sqrt{2}}$, $T = 0.1$, and $\Delta t = 0.000005$, corresponding to Example 4.2.

	ES	SIS	IS	SICN	ICNS-1	ICNS-2
L_2	1.26614E-06	1.27207E-06	1.25582E-06	1.26831E-06	1.25204E-06	1.25571E-06
L_∞	1.57920E-06	1.58690E-06	1.56510E-06	1.58190E-06	1.56010E-06	1.56520E-06

Table 15. Comparison of errors in L_2 norm and L_∞ norm for different values of $\alpha = 1$, $T = 0.1$, and $\Delta t = 0.000005$, corresponding to Example 4.3.

	ES	SIS	IS	SICN	ICNS-1	ICNS-2
L_2	1.07212E-06	1.02243E-06	2.34867E-01	2.34867E-01	2.34867E-01	2.72107E-07
L_∞	1.11010E-06	1.08320E-06	2.71255E-01	2.71255E-01	2.71255E-01	4.19600E-07

TABLE 16. Comparision of numerical solution for Example 4.1 at $\Delta t = 0.0005$, $T = 0.1$ and $\alpha = 1$

x	BDF1 [31]	BDF3 [31]	SICN-1	Exact solution
0.1	0.260555346	0.260752490	0.260739058	0.260738428
0.2	0.250268689	0.250447850	0.250422186	0.250421096
0.3	0.240184531	0.240348491	0.240313089	0.240311688
0.4	0.230309036	0.230461597	0.230419968	0.230418385
0.5	0.220648412	0.220794013	0.220750295	0.220748648
0.6	0.211208986	0.211352270	0.211310796	0.211309201
0.7	0.201997201	0.202142581	0.202107433	0.202106010
0.8	0.193019543	0.193170799	0.193145392	0.193144276
0.9	0.184282411	0.184442349	0.184429081	0.184428430

TABLE 17. Comparision of numerical solution for Example 4.3 at $\Delta t = 0.000005$, $T = 0.1$ and $\alpha = 1$

x	BDF1 [31]	BDF1 [20]	SICN-2	Exact solution
0.1	0.260736413	0.260737784	0.260738703	0.260738428
0.2	0.250419251	0.250420391	0.250421159	0.250421096
0.3	0.240310001	0.240310950	0.240311594	0.240311688
0.4	0.230416819	0.230417634	0.230418190	0.230418385
0.5	0.220747153	0.220747899	0.220748410	0.220748648
0.6	0.211307719	0.211308466	0.211308976	0.211309201
0.7	0.202104493	0.202105305	0.202105857	0.202106010
0.8	0.193142693	0.193143625	0.193144255	0.193144276
0.9	0.184426775	0.184427868	0.184428602	0.184428430

TABLE 18. ROC of Example 4.1 for $\alpha = 1$.

	SICN-1 at $\Delta t = 0.000005$				**SICN-1 at** $N = 10$			
N	Δx	L_2	L_∞	$ROC(space)$	Δt	L_2	L_∞	$ROC(time)$
10	0.1	3.50567E-07	4.72E-07		0.005	1.55559E-05	2.0924E-05	
20	0.05	7.59970E-08	1.02E-07	2.2	0.0005	1.22493E-06	1.6470E-06	1.1
40	0.025	7.15367E-09	100E-08	3.4	0.00005	2.07442E-07	2.8000E-07	0.8

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