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Cardinal Pseudospectral method for solving the periodic and semiperiodic Sturm–Liouville problem

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Abstract

This paper investigates the periodic and semiperiodic Sturm–Liouville problem. We begin by examining the fundamental properties of its eigenvalues and eigenfunctions. In the first part, the eigenvalues are approximated utilizing the pseudospectral method using the Chebyshev cardinal functions. While existing literature predominantly focuses on numerical and theoretical analysis of the eigenvalues, this study, in its second part, shifts focus to the eigenfunctions themselves. For both periodic and semiperiodic cases, we demonstrate that the explicit computation of these eigenfunctions is achievable. The methodology involves formulating the problem into the Volterra integral equation, which is then solved numerically via the pseudospectral method. Finally, the numerical results are rigorously validated by comparing them with analytical solutions. Error terms at both the inception and termination of the interval are calculated and bench marked against exact results, confirming the accuracy and efficacy of the proposed method.

Keywords. Chebyshev cardinal function, Eigenvalues, Eigenfunctions, Numerical methods, Periodic and semiperiodic Sturm–Liouville Problem.
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1. Introduction

The Sturm–Liouville problem (SLP) stands as a cornerstone in modern mathematical physics and spectral theory, forming a fundamental class of boundary value problems. We often found these problems fascinating because they elegantly describe both natural and engineered systems using the second-order differential operator

$$\ell u := -\frac{d^2u}{dx^2} + q(x)u = \lambda u, \tag{1.1}$$

where $q(x) \in \mathcal{L}^2[0,\omega]$ is a real-valued potential function and λ represents the spectral parameter [8, 25]. Of particular interest are the periodic and semiperiodic Sturm–Liouville problems (1.1) with the periodic

$$\begin{aligned} &B_1(u) := u(0) - u(\omega) = 0, \\ &B_2(u) := u'(0) - u'(\omega) = 0, \end{aligned} \tag{1.2}$$

or the semiperiodic boundary conditions

$$\begin{aligned} &B_3(u) := u(0) + u(\omega) = 0, \\ &B_4(u) := u'(0) + u'(\omega) = 0, \end{aligned} \tag{1.3}$$

which are deeply connected to systems with intrinsic periodicity or band-structure phenomena. For simplicity, we use the notations L for the problem (1.1)–(1.2), and L_1 (1.1) and (1.3). The solutions of the problems L and L_1, known as eigenfunctions, along with their corresponding eigenvalues, often serve as an orthogonal basis for representing a variety of physical fields, from temperature distributions to waveforms and quantum states. Examples we encounter

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include photonic crystals, molecular vibrations, fluid oscillations in periodic media, and stability analysis of nonlinear oscillators [3, 16, 26].

From a physical perspective, it is clear that periodic and semiperiodic Sturm–Liouville problems naturally arise when modeling wave propagation in structured materials. A periodic SLP, whose eigenvalues map out the energy bands and forbidden gaps for electrons in a crystal lattice, results directly from the Schrödinger equation with a periodic potential in solid-state physics [25]. Since the eigenvalue spectrum dictates how waves are transmitted or reflected in periodic potentials, the periodic SLP is crucial for describing electromagnetic wave propagation in optics [26]. A semiperiodic Sturm–Liouville formulation can also describe mechanical and acoustic systems, such as vibrating rods or membranes with periodically varying properties [16]. The periodic SLP is essential for studying periodic media because it provides a robust mathematical framework for understanding phenomena like Bragg reflection, band gaps, and resonance.

Mathematically, periodic and semiperiodic Sturm–Liouville operators have always stuck our as particularly rich in structure. Unlike classical Dirichlet or Neumann problems, these operators exhibits unique spectral features. Their self-adjointness guarantees real eigenvalues and orthogonal eigenfunctions, while the underlying periodicity leads to double eigenvalues and indicate band spectra [4, 8, 25]. The spectral analytic functions depend on the potential $q(x)$, which connects the issue to inverse spectral theory and more general domains of functional analysis, is another fascinating feature. Any square-integrable function in periodic domains can be expanded thanks to the eigenfunctions' completeness and orthogonality, which forms the basis of the Floquet-Bloch theory, which forms the basis of a large portion of contemporary differential operator analysis [25]. For these reasons, periodic and semi-periodic SLPs are often used models to investigate spectral gaps, localization, and stability phenomena in applied mathematics and physics. Even though these problems are theoretically important, we have found that analytical solutions for periodic and semi-periodic SLPs are available only for very specific forms of the potential function $q(x)$, such as constant or piecewise constant coefficients [5]. In more real-word scenarios, where $q(x)$ can be smooth, oscillatory, or even discontinuous, we simply can't find closed-form expressions for the eigenvalues and eigenfunctions. This is why reliable numerical methods are so vital. Classical discretization techniques, such as finite difference and finite element methods, have certainly contributed a lot, but they come with their own limitations. Their polynomial convergence rates and the challenge of imposing boundary conditions accurately can be problematic, specially when high spectral resolution is needed, like in modeling high-frequency wave propagation or identifying narrow spectral gaps [7, 23].

In this regard, Pseudospectral approaches have become a more attractive option, providing spectral (exponential) convergence for smooth problems while preserving computational efficiency [6, 17, 22]. Instead of approximating the solution locally, these techniques use global interpolation bases, like Chebyshev or Legendre polynomials. Because it can provide precise imposition of boundary conditions and stable and accurate differentiation matrices, the Chebyshev cardinal pseudospectral method in particular has shown great promise [18, 22]. It produces extremely accurate representations of both eigenvalues and eigenfunctions by combining the advantages of discrete collocation and orthogonal polynomial approximations [1, 2, 7, 9, 10]. Nonetheless, the majority of earlier research has focused on calculating eigenvalues, with comparatively little focus on numerically reconstructing eigenfunctions, particularly when dealing with periodic or semiperiodic boundary conditions [4, 5, 7].

The current study aims to close this gap by implementing the pseudospectral framework for the periodic and semiperiodic Sturm–Liouville problems based on Chebyshev cardinal functions. Before reformulating the problem into an equivalent Volterra integral equation, this work first investigates the fundamental analytical properties of the Sturm–Liouville operator, including self-adjointness, real spectrum, orthogonality, and completeness. In addition to providing accurate eigenvalue approximations, the resulting algorithm makes it possible to explicitly compute eigenfunctions and their symmetry properties when the potential satisfies $q(x) = q(w - x)$. Compared to previous research, where eigenfunctions were either not approximated at all or only indirectly, this is a major improvement [3–5].

A number of examples are provided in order to validate the methodology. The calculated results show excellent agreement with analytical and previously published numerical data and exponential convergence with respect to the number of collocation points. The stability and robustness of the method are confirmed by its accurate capture of the fine structure of eigenfunctions for smooth and piecewise smooth potentials as well as the eigenvalue spectrum.

Additionally, the framework naturally adapts to Sturm–Liouville operators that are more general, such as those that involve nonlinear terms or fractional derivatives [1, 2, 14, 19, 21, 22]. For the study of periodic and semiperiodic systems in mathematical physics, engineering, and applied computation, the suggested methodology thus offers a unified spectral tool.

In conclusion, this paper advances the analysis of periodic and semiperiodic Sturm–Liouville problems by combining numerical innovation, mathematical rigor, and physical motivation. Accuracy, efficiency, and theoretical consistency are all optimally balanced through the use of Chebyshev cardinal pseudospectral techniques. The findings provide new insights into the spectral behavior of periodic operators and pave the way for their use in materials science, quantum mechanics, and wave propagation. Furthermore, the approach lays the groundwork for further studies of inverse, multidimensional, or fractional Sturm–Liouville problems, where spectral completeness and high-order accuracy continue to be crucial [13, 15, 17, 20, 23].

Part 1. **Eigenvalues**

2. The periodic and semiperiodic Sturm–Liouville problems

Let $C(x,\lambda)$ and $S(x,\lambda)$ be solutions of (1.1) under the initial conditions

$$C(0,\lambda)=1, \quad C'(0,\lambda)=0, \tag{2.1}$$

$$S(0,\lambda)=0, \quad S'(0,\lambda)=1. \tag{2.2}$$

Let us show that

$$C(x,\lambda)=\cos\rho x+\int_0^x \frac{\sin\rho(x-t)}{\rho}q(t)C(t,\lambda)dt, \quad \lambda=\rho^2, \tag{2.3}$$

and

$$S(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_0^x \frac{\sin\rho(x-t)}{\rho}q(t)S(t,\lambda)dt. \tag{2.4}$$

Indeed, the Volterra integral Equations (2.3) and (2.4) are unique solutions (for the theory of Volterra integral equations, see [11]) of (1.1) with the initial conditions (2.1) and (2.2), respectively. From the linear differential equations, we obtain that the Wronskian

$$W(f,g)=f(x)g'(x)-f'(x)g(x), \tag{2.5}$$

is constant on $x\in[0,\omega]$ for two solutions $\ell f=\lambda f$, $\ell g=\lambda g$. The general solution of (1.1) is given by

$$u(x,\lambda)=\mathbb{A}S(x,\lambda)+\mathbb{B}C(x,\lambda). \tag{2.6}$$

Applying the periodic boundary conditions (1.2) in Eq. (2.6), we get

$$\begin{aligned} &\mathbb{A}(S(0,\lambda)-S(\omega,\lambda))+\mathbb{B}(C(0,\lambda)-C(\omega,\lambda))=0,\\ &\mathbb{A}(S'(0,\lambda)-S'(\omega,\lambda))+\mathbb{B}(C'(0,\lambda)-C'(\omega,\lambda))=0. \end{aligned} \tag{2.7}$$

For the system (2.7) to have a nontrivial solution, it is necessary and sufficient that

$$\begin{vmatrix} S(\omega,\lambda) & C(\omega,\lambda)-1\\ S'(\omega,\lambda)-1 & C'(\omega,\lambda)\end{vmatrix}=0.$$

Since the Wronskian of two solutions of (1.1) is constant, then by using the initial conditions (2.1) and (2.2), we get

$$W(S,C)=S(0)C'(0)-S'(0)C(0)=S(\omega)C'(\omega)-S'(\omega)C(\omega)=-1.$$

Denote

$$\Delta(\lambda):=\begin{vmatrix} S(\omega,\lambda) & C(\omega,\lambda)-1\\ S'(\omega,\lambda)-1 & C'(\omega,\lambda)\end{vmatrix}=C(\omega,\lambda)+S'(\omega,\lambda)-2. \tag{2.8}$$

Using the semiperiodic boundary conditions (1.3) in the general solution (2.6), we get

$$\mathbb{A}(S(0,\lambda)+S(\omega,\lambda))+\mathbb{B}(C(0,\lambda)+C(\omega,\lambda))=0,$$
$$\mathbb{A}(S'(0,\lambda)+S'(\omega,\lambda))+\mathbb{B}(C'(0,\lambda)+C'(\omega,\lambda))=0.$$

Therefore, by a similar definition of $\Delta(\lambda)$, we denote

$$\delta(\lambda) := \begin{vmatrix} S(\omega,\lambda) & C(\omega,\lambda)+1 \\ S'(\omega,\lambda)+1 & C'(\omega,\lambda) \end{vmatrix} = C(\omega,\lambda)+S'(\omega,\lambda)+2. \tag{2.9}$$

The functions $\Delta(\lambda)$ and $\delta(\lambda)$ are the characteristic functions of the problems (1.1) with the boundary conditions (1.2) and (1.3), respectively. The zeros $\{\lambda_{0,1},\lambda_{n,i}\}$ and $\{\mu_{n,i}\}$, $n=1,2,\dots,\ i=1,2$, of the characteristic functions $\Delta(\lambda)$ and $\delta(\lambda)$, coincide with the eigenvalues of L and L_1, respectively, (For more details see [12, Sec. 1.4]).

In this part, we attempt to briefly state and prove several important properties of the periodic and semiperiodic Sturm–Liouville problem.

Lemma 2.1. *Let φ_i and φ_j be solutions of periodic or semiperiodic Sturm–Liouville problem on $[0,\omega]$. then $\langle \ell\varphi_i,\varphi_j\rangle = \langle \varphi_i,\ell\varphi_j\rangle$, i.e. the operators L and L_1 are self-adjoint.*

Now we will demonstrate that the eigenvalues of a periodic and semiperiodic Sturm–Liouville problem are real and that the corresponding eigenfunctions to distinct eigenvalues are orthogonal.

Lemma 2.2. *[24] The periodic and semiperiodic Sturm–Liouville problems have the following statement:*

(1) *The eigenvalues are real. Thus, we have $\lambda_{0,1}\le\lambda_{1,1}\le\lambda_{1,2}\le\lambda_{2,1}\le\lambda_{2,2}\le\cdots\le\lambda_{n,i}\le\cdots$, and $\mu_{1,1}\le\mu_{1,2}\le\mu_{2,1}\le\mu_{2,2}\le\cdots\le\mu_{n,i}\le\cdots$ and when $n\to\infty$, $i=0,1$, $\lambda_{n,i}\to\infty$ and $\mu_{n,i}\to\infty$.*
(2) *The eigenvalue $\lambda_{0,1}$ is simple and each one of the other eigenvalues $\lambda_{n,i}$ and $\mu_{n,i}$ may be simple or double.*
(3) *For each double eigenvalues $\{\lambda_{n,i}\}$ and $\{\mu_{n,i}\}$ there exists two independent eigenfunctions. It has at most $2n$ roots in the interval $(0,\omega)$.*
(4) *Eigenfunctions corresponding to eigenvalues are orthogonal. This means that*

$$\langle\varphi_{n,i},\varphi_{m,j}\rangle=\langle\varphi_{n,i},\varphi_{n,i}\rangle\delta_{mn}\delta_{ij},\quad n,m=1,2,\dots,\ i,j=1,2.$$

(5) *The eigenfunctions constitute a complete set. In other words, any function belonging to the Hilbert space $\mathcal{L}^2[0,\omega]$ can be approximated as follows*

$$f(x)\sim\sum_{i=1}^{2}\sum_{n=1}^{\infty}c_{n,i}\varphi_{n,i}(x),\quad c_{n,i}=\frac{\langle f,\varphi_{n,i}\rangle}{\langle\varphi_{n,i},\varphi_{n,i}\rangle}.$$

Corollary 2.3. *With the same method as Lemma [8, Lem. 1.1.2], as $|\rho|\to\infty$, the asymptotic form of $\Delta(\lambda)$ is*

$$\Delta(\lambda)=\sin^2\left(\frac{\rho\omega}{2}\right)+O\left(\frac{1}{\rho}e^{|\mathrm{Im}\rho|\omega}\right).$$

Then, by a straightforward calculation, we obtain the asymptotic form of the eigenvalues as follows:

$$\lambda_{n,i}=\frac{2n\pi}{\omega}+O(1).\quad n=1,2,3,\dots,\ i=1,2.$$

According to the first term of the asymptotic form of the characteristic function, $\Delta(\lambda)$, the eigenvalues of the problem may have a repetition order of at most two, which we denote by i.

3. Cardinal Chebyshev Pseudospectral Method

In this section, we determine the eigenvalues of the periodic and the semiperiodic Sturm–Liouville problems using numerical methods of cardinal Chebyshev pseudospectral techniques.
The Chebyshev polynomials on the interval $[-1,1]$ are

$$\mathcal{T}_N(\mathrm{t})=\cos\left(N\arccos\mathrm{t}\right),\quad N=0,1,2,\dots, \tag{3.1}$$

the roots of Eq. (3.1) are given by

$$\mathrm{t}_j = \cos\left(\frac{2j-1}{2N+2}\pi\right), \qquad j = 1,2,3,\ldots,N.$$

We can express the Chebyshev polynomial in any desired interval $[0,\omega]$ by changing the variable $x = \frac{\omega}{2}(\mathrm{t}+1)$ as

$$\mathcal{T}^*_{N+1}(x) = \mathcal{T}_{N+1}\left(\frac{2x}{\omega}-1\right), \qquad x \in [0,\omega].$$

Then its roots are written as follows:

$$x_j = \frac{\omega}{2}(\mathrm{t}_j+1), \qquad j = 1,2,3,\ldots,N. \tag{3.2}$$

To construct the Chebyshev cardinal functions, it is sufficient to expand the Chebyshev function around its roots. In the vicinity of the j-th root, Taylor expansion gives

$$\mathcal{T}_{N+1}(x) \approx \mathcal{T}_{N+1}(x_j) + \mathcal{T}_{N+1,x}(x_j)(x-x_j) + O((x-x_j)^2).$$

The value of the cardinal function can be normalized to unity at 1 at $x = x_j$ through the following definition:

$$C_j(x) = \frac{\mathcal{T}_{N+1}(x)}{\mathcal{T}_{N+1,x}(x_j)(x-x_j)}, \qquad j = 1,2,3,\ldots,N,$$

where

$$\mathcal{T}_{N+1,x}(x_j) = \frac{d}{dx}\mathcal{T}_{N+1}(x_j).$$

The function $C_j(x)$ is called the Chebyshev cardinal functions. Any function $f(x)$ defined on $[0,\omega]$ can now be approximated as

$$f(x) \approx \sum_{j=1}^{N+1} f(x_j)C_j(x) = F^T\Phi_N(x). \tag{3.3}$$

where

$$F = [f(x_1), f(x_2), \ldots, f(x_{N+1})]^T,$$

$$\Phi_N(x) = [C_1(x), C_2(x), \ldots, C_{N+1}(x)]^T,$$

it can be easily checked that

$$C_i(x_j) = \delta_{ij} = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

3.1. **Operational matrix of integration.** Let $\Phi_N(x)$ be the shifted Chebyshev cardinal vector, then we can write

$$\mathcal{I}(u)(x) := \int_0^x \Phi_N(z)dz = P\Phi_N(x)$$

where P is the $(N+1)\times(N+1)$ operational matrix of integration for Chebyshev cardinal functions is defined as follows:

$$P = \begin{bmatrix} \Omega(0,0) & \Omega(0,1) & \cdots & \Omega(0,N) \\ \Omega(1,0) & \Omega(1,1) & \cdots & \Omega(1,N) \\ \vdots & \vdots & \ldots & \vdots \\ \Omega(i,0) & \Omega(i,1) & \cdots & \Omega(n,i) \\ \vdots & \vdots & \ldots & \vdots \\ \Omega(N,0) & \Omega(N,1) & \cdots & \Omega(N,N) \end{bmatrix}.$$

As we know, and using (3.3) any integral of the function $C_i(x)$ can be approximated as

$$\mathcal{I}(C_i)(x) = \sum_{j=1}^{N+1} \Omega(i,j) C_j(x),$$

where

$$\Omega(i,j) = \int C_i(x_j)dx = \frac{2^{2N+1}}{\mathcal{T}_{N+1,x}(tj)} \sum_{k=0}^{N} c_{i,k} \frac{x_j^{N-k+1}}{(N-k+1)!}.$$

Then, the coefficients $c_{i,k}$ is given as follows:

$$c_{i,0} = 1, \; c_{i,k} = -\frac{1}{k} \sum_{s=1}^{k} s_{i,s} c_{i,k-s}, \quad k = 1,2,\ldots,N, \quad i = 1,2,\ldots,N+1,$$

and

$$s_{i,k} = \sum_{j=1, j\neq i}^{N+1} t_j^k, \quad k = 1,2,\ldots,N, \quad i = 1,2,\ldots,N+1,$$

(see, for example, [1, 18]).

3.2. **Formulas and method.** To develop the collocation method for the Equation (1.1) over the domain $[0,\omega]$. We apply the integral operator to both sides of (1.1), i.e.,

$$-u(x) + u'(0)x + u(0) = \int_0^x \left(\int_0^s (\lambda u(z) - q(z)u(z))\, dz \right) ds. \tag{3.4}$$

Equivalently, in the abstract form, one can write

$$-u(x) + u'(0)x + u(0) = \mathcal{I}^2 \left(\lambda u(z) - q(z)u(z) \right)(x).$$

The equivalence of the solutions of Equations (1.1) and (3.4) can be easily verified. Here we should point out that two new unknowns, $u(0)$ and $u'(0)$, must be considered.

To utilize the collocation method for the integral Equation (3.4), the unknown solution $u(x)$ should be mapped into the approximation space $P_N(I)$ via the projection operator $\mathcal{P}_N$, i.e.,

$$u(x) \approx \mathcal{P}_N(u)(x) = U^T \Phi_N(x) := u_N(x),$$

where $U \in \mathbb{R}^{N+1}$ whose elements are unknown.

Substituting the approximate solution $u_N(x)$ into the integral Equation (3.4) yields

$$-u_N(x) + u'(0)x + u(0) = \mathcal{I}^2 \left(\lambda u_N(z) - q(z)u_N(z) \right)(x).$$

Now, we use the matrix form of the integral operator to obtain the residual

$$r_N(x) := \left(-U^T + \bar{U}^T - \lambda U^T P^2 + U^T \Upsilon P^2 \right) \Phi_N(x),$$

where

$$\begin{aligned} u'(0)x + u(0) &\approx \mathcal{P}_N(u'(0)x + u(0)) = \bar{U}^T \Phi_N(x), \\ q(x)u_N(x) &\approx \mathcal{P}_N(q(x)u_N(x)) = U^T \Upsilon \Phi_N(x), \end{aligned} \tag{3.5}$$

in which $\bar{U} \in \mathbb{R}^{N+1}$ and $\Upsilon \in \mathbb{R}^{(N+1)\times(N+1)}$.

By choosing the nodes $\{x_i\}_{i=1}^{N+1}$ as collocation points, one obtains the following system of linear equations:

$$r_N(x_i) = 0, \qquad i = 1,\ldots,N+1. \tag{3.6}$$

Equation (3.6) involves $N+3$ unknowns but provides only $N+1$ linear equations. To address this, we incorporate either the periodic boundary conditions (1.2) or the semiperiodic boundary conditions (1.3). Let $\bar{U} \in \mathbb{R}^{N+3}$ denote the augmented vector consisting of the vector U, along with the additional components $u(0)$ and $u'(0)$. Using (3.5) and (3.6) in the point $x = x_i$, we have

$$\Theta(\lambda)\bar{U} = 0,$$

where $\Theta(\lambda)$ is a matrix function of λ. Since the vector $\bar{U}$ is non-zero, we must have

$$\det(\Theta(\lambda)) = 0. \tag{3.7}$$

Equation (3.7) is a polynomial of λ of degree $N + 1$, whose roots provide approximate eigenvalues.

Remark 3.1. Since the proof of convergence is analogous to that of Lemma 2.1 in [2], it is omitted here for the sake of brevity. We refer interested readers to [2] for further details.

4. Numerical Examples 1

In this section, we provide examples to illustrate the method's accuracy and the algorithm's efficiency. All computations were performed using Maple 2022 and MATLAB R2022a on a personal computer equipped with an Intel Core i7 processor and 16 GB RAM.

Example 4.1. Suppose that $q(x) = x^2(\pi - x)$ in (1.1) with $\omega = \pi$ in (1.2) and (1.3) by using the Cardinal Chebyshev pseudo-spectral method, the following eigenvalues are obtained (See Table 1).

Table 1. The 8 first eigenvalues obtained with $N = 10, 20, 30$ for Example 4.1.

n	Λ_n[7]	Λ_n^{10}	Λ_n^{20}	Λ_n^{30}	$\lvert\Lambda_n^{30} - \Lambda_n^{20}\rvert$
1	2.0294	2.029456869	2.029416154	2.029416154	2.2×10^{-11}
2	6.5005	6.500556862	6.500490703	6.500490703	1.3×10^{-10}
3	7.0151	7.015104493	7.015056858	7.015056858	2.4×10^{-11}
4	18.5848	18.58574589	18.58477218	18.58477218	3.6×10^{-09}
5	18.6655	18.67964139	18.66548151	18.66548151	9.6×10^{-10}
6	38.5816	41.48593837	38.58162693	38.58162795	1.0×10^{-06}
7	38.6215	42.15370807	38.62154167	38.62154248	8.1×10^{-07}
8	66.5821	–	66.58195548	66.58204791	9.2×10^{-05}
n	μ_n^{10}	μ_n^{20}	μ_n^{30}	$\lvert\mu_n^{30} - \mu_n^{10}\rvert$	$\lvert\mu_n^{30} - \mu_n^{20}\rvert$
1	2.334047546	2.334300060	2.334300062	2.5×10^{-04}	5.2×10^{-10}
2	4.523951423	4.523740633	4.523740633	2.1×10^{-04}	6.2×10^{-10}
3	11.61269252	11.60243575	11.60243569	2.5×10^{-04}	5.1×10^{-08}
4	11.72350628	11.71988689	11.71988686	3.6×10^{-03}	2.7×10^{-08}
5	28.26939012	27.58201038	27.58200548	6.8×10^{-01}	4.9×10^{-06}
6	28.26939012	27.63739062	27.63738694	6.3×10^{-01}	3.7×10^{-06}
7	66.31360746	51.58201176	51.58178493	$1.5 \times 10^{+01}$	2.3×10^{-04}
8	67.59344479	51.61214122	51.61178524	$1.6 \times 10^{+01}$	3.6×10^{-04}

Remark 4.2. The values of Λ_n^k and μ_n^k are approximate eigenvalues for the Sturm–Liouville Equation (1.1) with the boundary conditions (1.2) and (1.3) with $N = k$, respectively.

Example 4.3. Let us consider

$$q(x) = \begin{cases} 0, & 0 < x < \frac{1}{2}, \\ \frac{1}{1+x^2}, & \frac{1}{2} < x < \frac{7}{4}, \\ 0, & \frac{7}{4} < x < 2, \end{cases}$$

in Eq. (1.1) and $\omega = 2$ in (1.2) and (1.3) by using the Cardinal Chebyshev pseudo-spectral method, the following eigenvalues are obtained (See Table 2).

TABLE 2. The 6 first eigenvalues obtained with $N = 10, 20, 30$ for example 4.3.

n	Λ_n^{19}[5]	Λ_n^{10}	Λ_n^{20}	Λ_n^{30}	$\lvert\Lambda_n^{30} - \Lambda_n^{20}\rvert$
1	0.537638999	0.577155720	0.536623512	0.536820577	1.9×10^{-04}
2	3.141592654	3.180554855	3.177669124	3.177669124	1.3×10^{-05}
3	3.198665145	3.209641795	3.198265684	3.198265685	2.9×10^{-04}
4	6.283185307	6.308892793	6.302711104	6.303028830	3.2×10^{-04}
5	6.309892873	6.317588485	6.309963928	6.309982349	1.5×10^{-05}
6	–	9.905823066	9.438541706	9.438948133	4.1×10^{-04}
n	μ_n^{10}	μ_n^{20}	μ_n^{30}	$\lvert\mu_n^{30} - \mu_n^{10}\rvert$	$\lvert\mu_n^{30} - \mu_n^{20}\rvert$
1	1.629551299	1.607624011	1.614291428	1.5×10^{-02}	6.7×10^{-03}
2	1.719770193	1.700096058	1.706385319	1.3×10^{-02}	6.3×10^{-03}
3	4.742256233	4.736043524	4.739537204	2.7×10^{-03}	3.5×10^{-03}
4	4.751142624	4.745973814	4.747465562	3.7×10^{-03}	1.5×10^{-03}
5	8.001137263	7.869582838	7.871280542	1.3×10^{-01}	1.7×10^{-03}
6	8.012094247	7.872731833	7.874046684	1.4×10^{-01}	1.3×10^{-03}

Part 2. Eigenfunctions

5. Eigenfunctions for the symmetric potential function

Definition 5.1. A function $f(x)$ is referred to as symmetric or antisymmetric with respect to the point $\omega/2$ if, for all x, it satisfies the corresponding condition,

$$f(x) = f(\omega - x) \text{ or } f(x) = -f(\omega - x),$$

respectively.

Theorem 5.2. *[6, P. 164] Let us consider an ordinary differential equation of the form*

$$\sum_{n=0}^{N} a_n(x) \frac{d^n u}{dx^n} = f(x). \tag{5.1}$$

The function $u(x)$ is symmetric if and only if the boundary conditions are compatible with symmetry and one of the following conditions is satisfied:

- **1:** *The coefficients associated with even-order derivatives in (5.1) are even functions, those associated with odd-order derivatives are odd functions, and $f(x)$ is symmetric; or*
- **2:** *The coefficients corresponding to even-order derivatives in (5.1) are odd functions, those corresponding to odd-order derivatives are even functions, and $f(x)$ is antisymmetric.*

Similarly, the function $u(x)$ is antisymmetric if and only if the boundary conditions change sign under the transformation $x \to (\omega - x)$, and one of the following conditions holds:

- **3:** *The coefficients of the even-order derivatives in (5.1) are odd functions, those of the odd-order derivatives are even functions, and $f(x)$ is symmetric; or*
- **4:** *The coefficients corresponding to even-order derivatives in (5.1) are even functions, those corresponding to odd-order derivatives are odd functions, and $f(x)$ is antisymmetric.*

We demonstrate that if $q(x) = q(\omega - x)$, then

$$C(x, \lambda_{n,1}), \quad n = 0, 1, 2, \dots,$$
$$S(x, \lambda_{n,2}), \quad n = 1, 2, 3, \dots,$$

defined in (2.3) and (2.4) are eigenfunctions for the periodic or

$$C(x, \mu_{n,1}), \quad S(x, \mu_{n,2}), \qquad n = 1, 2, 3, \dots,$$

are eigenfunctions for the semiperiodic Sturm–Liouville problems (1.1) under boundary conditions (1.2) or (1.3). To perform this task, the functions (2.3) and (2.4) must satisfy the Eq. (1.1) and the boundary conditions (1.2) or (1.3). It is easy to see that the functions $C(x, \lambda_{n,1})$ and $S(x, \lambda_{n,2})$ satisfy in the (1.1) with $\lambda = \lambda_{n,i}$. To achieve these purposes, we have the following Lemmas:

Lemma 5.3. *Let $L = L(q(x))$ and $\tilde{L} := L(q(\omega - x))$. If $q(x) = q(\omega - x)$, then $\Delta(\lambda) = \tilde{\Delta}(\lambda)$, i.e., the spectra of operators L and $\tilde{L}$ are the same.*

Proof. By changing x with $\omega - x$ in (1.1), (1.2), and using $q(x) = q(\omega - x)$, we get $\Delta(\lambda) = \tilde{\Delta}(\lambda)$. Then all roots (eigenvalues) of the two problems are the same. □

Lemma 5.4. *If $C(x, \lambda)$ and $S(x, \lambda)$ be solutions of (1.1) under the initial conditions (2.1) and (2.2), respectively and $q(x) = q(\omega - x)$, then $S(0, \lambda_{n,2}) = S(\omega, \lambda_{n,2})$ and $C(0, \lambda_{n,1}) = C(\omega, \lambda_{n,1})$, i.e. $C(x, \lambda_{n,1})$ and $S(x, \lambda_{n,2})$ are eigenfunctions of the problem (1.1)-(1.2).*

Proof. From [8, Thms. 1.3.1 and 1.3.2], the solutions $C(x, \lambda)$ and $S(x, \lambda)$ with the initial conditions (2.1) and (2.2) are as follows:

$$C(x, \lambda) = \cos \rho x + \int_0^x k(x, s) \cos \rho s \, ds, \tag{5.2}$$

and

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x p(x, s) \frac{\sin \rho s}{\rho} \, ds, \tag{5.3}$$

where $k(x, s)$ and $p(x, s)$ are real continuous functions, and

$$k(x, x) = p(x, x) = \frac{1}{2} \int_0^x q(t) dt.$$

By changing x to $t = \omega - x$, we get

$$\tilde{C}(t, \lambda) = \cos \rho t + \int_0^t k(t, \tau) \cos \rho \tau \, d\tau, \tag{5.4}$$

and

$$\tilde{S}(t, \lambda) = \frac{\sin \rho t}{\rho} + \int_0^t p(t, \tau) \frac{\sin \rho \tau}{\rho} d\tau. \tag{5.5}$$

Replacing $\lambda = \lambda_{n,i}$, $x = \omega$, and $t = \omega$, in (5.2)–(5.5), we get

$$S(\omega, \lambda_{n,2}) = \tilde{S}(\omega, \lambda_{n,2}), \quad C(\omega, \lambda_{n,1}) = \tilde{C}(\omega, \lambda_{n,1}). \tag{5.6}$$

Note that $\tilde{S}(\omega, \lambda_{n,2}) = S(0, \lambda_{n,2})$ and $\tilde{C}(\omega, \lambda_{n,1}) = C(0, \lambda_{n,1})$. So, we get

$$S(\omega, \lambda_{n,2}) = S(0, \lambda_{n,2}) \text{ and } C(\omega, \lambda_{n,1}) = C(0, \lambda_{n,1}).$$

By using the derivative of functions $S(x, \lambda)$, $\tilde{S}(t, \lambda)$, $C(x, \lambda)$, and $\tilde{C}(t, \lambda)$, we get

$$S'(\omega, \lambda_{n,2}) = S'(0, \lambda_{n,2}) \text{ and } C'(\omega, \lambda_{n,1}) = C'(0, \lambda_{n,1}).$$

□

Lemma 5.5. *If $q(x) = q(\omega - x)$ in (1.1), then*

$$C(x, \lambda_{n,1}) = C(\omega - x, \lambda_{n,1}) \text{ and } S(x, \lambda_{n,2}) = -S(\omega - x, \lambda_{n,2}). \tag{5.7}$$

Proof. From Lemma 5.4, the functions $C(x,\lambda_{n,1})$ and $S(x,\lambda_{n,2})$ are eigenfunctions of (1.1)-(1.2). By changing x to $\omega - x$ the initial conditions (2.1) and (2.2) are reduced to

$$C(\omega,\lambda_{n,1}) = 1, \quad C'(\omega,\lambda_{n,1}) = 0, \tag{5.8}$$

and

$$S(\omega,\lambda_{n,2}) = 0, \quad S'(\omega,\lambda_{n,2}) = -1. \tag{5.9}$$

From (5.6), (5.8) and (5.9), we conclude that $C(\omega - x,\lambda_{n,1}) = C(x,\lambda_{n,1})$ and $S(\omega - x,\lambda_{n,2}) = -S(x,\lambda_{n,2})$. □

Lemma 5.6. *The eigenfunctions $S(x,\lambda_{n,2})$ and $C(x,\lambda_{n,1})$ are orthogonal, i.e.,*

$$\langle S(x,\lambda_{n,2}), C(x,\lambda_{n,1})\rangle = 0.$$

Proof. Using Lemma 5.5 and the straightforward calculations, we have

$$\begin{aligned}
\langle S(x,\lambda_{n,2}), C(x,\lambda_{n,1})\rangle &= \int_0^{\omega} S\left(x,\lambda_{n,2}\right) C\left(x,\lambda_{n,1}\right) dx \\
&= \int_0^{\frac{\omega}{2}} S\left(x,\lambda_{n,2}\right) C\left(x,\lambda_{n,1}\right) dx + \int_{\frac{\omega}{2}}^{\omega} S\left(x,\lambda_{n,2}\right) C\left(x,\lambda_{n,1}\right) dx \\
&= \int_0^{\frac{\omega}{2}} S\left(x,\lambda_{n,2}\right) C\left(x,\lambda_{n,1}\right) dx - \int_{\frac{\omega}{2}}^{\omega} S\left(\omega - x,\lambda_{n,2}\right) C\left(\omega - x,\lambda_{n,1}\right) dx \\
&= \int_0^{\frac{\omega}{2}} S\left(x,\lambda_{n,2}\right) C\left(x,\lambda_{n,1}\right) dx - \int_0^{\frac{\omega}{2}} S\left(x,\lambda_{n,2}\right) C\left(x,\lambda_{n,1}\right) dx = 0.
\end{aligned}$$

□

Corollary 5.7. *With a simple modification, Lemmas 5.3–5.6 can be expressed and proved for the semiperiodic problem (1.1) and (1.3).*

In some examples, $\lambda_{n,1}$ and $\lambda_{n,2}$ may be equal, in which case functions $C(x,\lambda_{n,1})$ and $S(x,\lambda_{n,1})$ will be orthogonal eigenfunctions of the problem.

6. Eigenfunction approximation

As mentioned earlier, the eigenfunctions can be obtained from (2.3) and (2.4). It should be noted that the approximation of the eigenfunctions can be obtained by substituting the approximate value of the eigenvalues, $\Lambda_{n,i} = \varrho_{n,i}^2$, into Eqs. (2.3) and (2.4). So, two Volterra integral equations of the second kind must be solved to find the approximation of the eigenfunctions. To this end, we again employ the collocation method. Suppose the unknown functions $C(x,\Lambda_{n,1})$ and $S(x,\Lambda_{n,2})$ are approximated as follows:

$$\begin{aligned}
C(x,\Lambda_{n,1}) &\approx \mathcal{P}_N(C(x,\Lambda_{n,1})) = C_{\Lambda_{n,1}}^T \Phi(x) = C_N(x), \\
S(x,\Lambda_{n,2}) &\approx \mathcal{P}_N(S(x,\Lambda_{n,2})) = S_{\Lambda_{n,2}}^T \Phi(x) = S_N(x),
\end{aligned}$$

where $C_{\Lambda_{n,1}}, S_{\Lambda_{n,2}} \in \mathbb{R}^{N+1}$, and their elements need to be identified. Substituting these approximations into (2.3) and (2.4), the residuals r_1 and r_2 can be specified as

$$\begin{aligned}
r_1(x) &= C_N(x) - \cos(\varrho_{n,1}x) - \mathcal{I}\left(\frac{\sin(\varrho_{n,1}(x-t))}{\varrho_{n,1}} q(t) C_N(t)\right), \\
r_2(x) &= S_N(x) - \frac{\sin(\varrho_{n,2}x)}{\varrho_{n,2}} - \mathcal{I}\left(\frac{\sin(\varrho_{n,2}(x-t))}{\varrho_{n,2}} q(t) S_N(t)\right).
\end{aligned}$$

The goal of the collocation method is to minimize these residuals at specific points. Using the matrix form of the integral operator, and choosing the nodes $\{x_i\}_{i=1}^{N+1}$ as collocation points, we obtain

$$r_1(x_i) = 0, \quad \Rightarrow \quad C^T_{\Lambda_{n,1}} - G^T_1 - F^T_1 = 0,$$
$$r_2(x_i) = 0, \quad \Rightarrow \quad S^T_{\Lambda_{n,2}} - G^T_2 - F^T_2 = 0,$$

in which

$$\mathcal{I}\left(\frac{\sin(\varrho_{n,1}(x-t))}{\varrho_{n,1}} q(t) C_N(t)\right) \approx \mathcal{I}\left(\Phi^T(x) K_1 \Phi(t)\right) \approx \Phi^T(x) K_1 P \Phi(x) \approx F^T_1 \Phi(x),$$
$$\mathcal{I}\left(\frac{\sin(\varrho_{n,2}(x-t))}{\varrho_{n,2}} q(t) S_N(t)\right) \approx \mathcal{I}\left(\Phi^T(x) K_2 \Phi(t)\right) \approx \Phi^T(x) K_2 P \Phi(x) \approx F^T_2 \Phi(x),$$

and

$$\cos(\varrho_{n,1} x) \approx G^T_1 \Phi(x),$$
$$\frac{\sin(\varrho_{n,2} x)}{\varrho_{n,2}} \approx G^T_2 \Phi(x).$$

Solving these equations leads to the solution of (2.3) and (2.4), with $\lambda = \Lambda_{n,i}$.

7. Numerical Examples 2

In this section, we present the proposed eigenfunctions corresponding to several Sturm–Liouville problems. It is worth mentioning that the associated eigenvalues are computed using the Chebyshev cardinal pseudospectral method.

Example 7.1. Suppose that $q(x) = 0$ in (1.1) with $\omega = 1$ in (1.2) and (1.3) by using the Cardinal Chebyshev pseudo-spectral method, the following eigenvalues are obtained(See Table 3) and the difference of approximation of the eigenfunctions in the end points (See Table 4).

Table 3. The square root of 6 eigenvalues obtained from the cardinal Chebyshev pseudo-spectral method ($i = 1, 2$), for Example 7.1

n	$\lambda_{n,i}$	$\Lambda^{10}_{n,i}$	$\Lambda^{20}_{n,i}$	$\Lambda^{30}_{n,i}$	$\lvert\lambda_{n,i} - \Lambda^{30}_{n,i}\rvert$
0	$\lambda_0 = 0$	2.2×10^{-50}	1.7×10^{-50}	3.6×10^{-51}	3.6×10^{-51}
1	6.2831853071	6.2831829493	6.2831853071	6.2831853072	3.7×10^{-29}
2	12.566370614	12.573256004	12.566370614	12.566370614	6.1×10^{-20}
3	18.849555922	19.794686556	18.849556258	18.849555922	1.1×10^{-14}
4	25.132741229	–	25.132717535	25.132741229	4.6×10^{-11}
5	31.415926536	–	31.431633680	31.415926516	2.0×10^{-08}
n	$\mu_{n,i}$	$\mu^{10}_{n,i}$	$\mu^{20}_{n,i}$	$\mu^{30}_{n,i}$	$\lvert\mu_{n,i} - \mu^{30}_{n,i}\rvert$
1	3.1415926536	3.1415927454	3.1415926536	3.1415926536	1.1×10^{-35}
2	9.4247779608	9.4217948834	9.4247779608	9.4247779608	1.8×10^{-21}
3	15.707963268	15.976436539	15.707962724	15.707963268	5.9×10^{-15}
4	21.991148575	–	21.991345109	21.991148575	8.4×10^{-11}
5	28.274333882	–	28.266359035	28.274333959	7.7×10^{-08}

Example 7.2. Consider the periodic and the semiperiodic Sturm-Liouville problem with $q(x) = 2x(1-x)$ and $\omega = 1$ in the conditions (1.2) and (1.3), respectively (See Tables 5 and 6 and Figure 1).

Example 7.3. Consider the periodic and the semiperiodic Sturm–Liouville problem with $q(x) = e^{x(1-x)} - \cos 2\pi x$ and $\omega = 1$ (See Tables 7 and 8 and Figure 2).

Table 4. The difference of approximation of the eigenfunctions in the points $x = 0$ and $x = \omega$ using the boundary conditions (1.2) and (1.3), in the Example 7.1.

n	$\lvert C(0,\Lambda^{10}_{n,1}) - C(\omega,\Lambda^{10}_{n,1})\rvert$	$\lvert C(0,\Lambda^{20}_{n,1}) - C(\omega,\Lambda^{20}_{n,1})\rvert$	$\lvert C(0,\Lambda^{30}_{n,1}) - C(\omega,\Lambda^{30}_{n,1})\rvert$
0	2×10^{-100}	2×10^{-100}	6×10^{-100}
1	2.8×10^{-12}	4.5×10^{-35}	6.8×10^{-58}
2	2.4×10^{-05}	8.9×10^{-21}	1.9×10^{-39}
3	4.1×10^{-01}	5.7×10^{-14}	6.5×10^{-29}
n	$\lvert S(0,\Lambda^{10}_{n,2}) - S(\omega,\Lambda^{10}_{n,2})\rvert$	$\lvert S(0,\Lambda^{20}_{n,2}) - S(\omega,\Lambda^{20}_{n,2})\rvert$	$\lvert S(0,\Lambda^{30}_{n,2}) - S(\omega,\Lambda^{30}_{n,2})\rvert$
1	2.4×10^{-06}	9.5×10^{-17}	3.7×10^{-29}
2	6.9×10^{-03}	1.3×10^{-10}	6.1×10^{-20}
3	8.1×10^{-01}	3.4×10^{-07}	1.1×10^{-14}
n	$\lvert C(0,\mu^{10}_{n,1}) + C(\omega,\mu^{10}_{n,1})\rvert$	$\lvert C(0,\mu^{20}_{n,1}) + C(\omega,\mu^{20}_{n,1})\rvert$	$\lvert C(0,\mu^{30}_{n,1}) + C(\omega,\mu^{30}_{n,1})\rvert$
1	4.2×10^{-15}	8.1×10^{-41}	5.5×10^{-71}
2	4.4×10^{-06}	3.8×10^{-22}	1.7×10^{-42}
3	3.5×10^{-02}	1.5×10^{-13}	1.7×10^{-29}
n	$\lvert S(0,\mu^{10}_{n,2}) + S(\omega,\mu^{10}_{n,2})\rvert$	$\lvert S(0,\mu^{20}_{n,2}) + S(\omega,\mu^{20}_{n,2})\rvert$	$\lvert S(0,\mu^{30}_{n,2}) + S(\omega,\mu^{30}_{n,2})\rvert$
1	9.2×10^{-08}	1.3×10^{-20}	1.1×10^{-25}
2	3.0×10^{-03}	3.4×10^{-11}	1.8×10^{-21}
3	2.7×10^{-01}	5.4×10^{-07}	5.8×10^{-15}

Table 5. The eigenvalues obtained from the Chebyshev cardinal pseudospectral method for Example 7.2.

n,i	$\lambda_{n,i}$	$\Lambda^{20}_{n,i}$[5]	$\Lambda^{10}_{n,i}$	$\Lambda^{20}_{n,i}$	$\Lambda^{30}_{n,i}$
0,1	0.57691941	0.576891408	0.576891414	0.576891414	0.576891414
1,1	6.31198463	6.307681115	6.307678451	6.307680286	6.307680286
1,2	6.31204087	6.311655825	6.311654838	6.311656793	6.311656793
2,1	–	–	12.58613317	12.57937709	12.57937709
2,2	–	–	12.58669500	12.57987900	12.57987900
3,1	–	–	19.80069112	18.85832203	18.85832170
3,2	–	–	19.80218514	18.85847107	18.85847074
n,i	$\mu^{10}_{n,i}$	$\mu^{20}_{n,i}$	$\mu^{30}_{n,i}$	$\lvert\mu^{10}_{n,i} - \mu^{30}_{n,i}\rvert$	$\lvert\mu^{20}_{n,i} - \mu^{30}_{n,i}\rvert$
1,1	3.1782709953	3.1782710806	3.1782710806	8.5×10^{-08}	4.8×10^{-19}
1,2	3.2100131466	3.2100132073	3.2100132073	6.1×10^{-08}	3.4×10^{-19}
2,1	9.4389758007	9.4418558599	9.4418558599	2.9×10^{-03}	2.4×10^{-11}
2,2	9.4401168581	9.4430421832	9.4430421831	2.9×10^{-03}	2.6×10^{-11}
3,1	15.985842853	15.718441580	15.718442104	2.7×10^{-01}	5.2×10^{-07}
3,2	15.986096201	15.718698914	15.718699441	2.7×10^{-01}	5.3×10^{-07}

7.1. **Discussion of Numerical Results.** This section discusses the numerical results presented in Tables 1–8 and Figures 1 and 2, which validate the performance of the proposed Chebyshev cardinal pseudospectral method for periodic and semiperiodic Sturm–Liouville problems.

In the first set of experiments (Examples 4.1 and 4.3), the method was applied to approximate eigenvalues. The numerical values reported in Tables 1 and 2 show excellent agreement with previously published results, see [5, 7]. In the absence of exact analytical eigenvalues, accuracy was assessed by comparing the results for $N = 10$ and $N = 20$ against the reference values computed with $N = 30$. The consistent reduction of error confirms spectral convergence with respect to the number of collocation points.

In the second part of the study, a symmetric potential function was considered. Theoretical results established that $C(x, \lambda_{n,1})$ and $S(x, \lambda_{n,2})$ are eigenfunctions of the periodic problem, and $C(x, \mu_{n,1})$ and $S(x, \mu_{n,2})$ corresponds to the semiperiodic case. By using the calculated eigenvalues to solve the related Volterra integral equations, numerical approximations of these eigenfunctions were produced. While Tables 4, 6, and 8 show that the resulting eigenfunctions

TABLE 6. The difference of approximation of the eigenfunctions in the points $x = 0$ and $x = \omega$ using the boundary conditions (1.2) and (1.3), in the Example 7.2.

n	$\lvert C(0,\Lambda^{10}_{n,1}) - C(\omega,\Lambda^{10}_{n,1})\rvert$	$\lvert C(0,\Lambda^{20}_{n,1}) - C(\omega,\Lambda^{20}_{n,1})\rvert$	$\lvert C(0,\Lambda^{30}_{n,1}) - C(\omega,\Lambda^{30}_{n,1})\rvert$
0	1.3×10^{-12}	5.6×10^{-22}	9.1×10^{-35}
1	1.4×10^{-08}	7.6×10^{-15}	5.4×10^{-24}
2	1.1×10^{-04}	2.5×10^{-11}	5.0×10^{-17}
3	7.4×10^{-02}	1.8×10^{-10}	1.3×10^{-13}
n	$\lvert S(0,\Lambda^{10}_{n,2}) - S(\omega,\Lambda^{10}_{n,2})\rvert$	$\lvert S(0,\Lambda^{20}_{n,2}) - S(\omega,\Lambda^{20}_{n,2})\rvert$	$\lvert S(0,\Lambda^{30}_{n,2}) - S(\omega,\Lambda^{30}_{n,2})\rvert$
1	7.1×10^{-06}	1.5×10^{-11}	2.0×10^{-20}
2	2.9×10^{-02}	1.9×10^{-08}	6.6×10^{-13}
3	1.4×10^{-01}	1.1×10^{-05}	1.0×10^{-09}
n	$\lvert C(0,\mu^{10}_{n,1}) + C(\omega,\mu^{10}_{n,1})\rvert$	$\lvert C(0,\mu^{20}_{n,1}) + C(\omega,\mu^{20}_{n,1})\rvert$	$\lvert C(0,\mu^{30}_{n,1}) + C(\omega,\mu^{30}_{n,1})\rvert$
1	5.7×10^{-9}	1.2×10^{-19}	2.4×10^{-31}
2	1.8×10^{-06}	1.2×10^{-12}	8.8×10^{-20}
3	1.2×10^{-02}	3.6×10^{-12}	4.7×10^{-15}
n	$\lvert S(0,\mu^{10}_{n,2}) + S(\omega,\mu^{10}_{n,2})\rvert$	$\lvert S(0,\mu^{20}_{n,2}) + S(\omega,\mu^{20}_{n,2})\rvert$	$\lvert S(0,\mu^{30}_{n,2}) + S(\omega,\mu^{30}_{n,2})\rvert$
1	1.5×10^{-06}	8.4×10^{-17}	3.4×10^{-28}
2	6.1×10^{-05}	6.0×10^{-09}	9.3×10^{-16}
3	9.6×10^{-02}	1.4×10^{-06}	9.6×10^{-11}

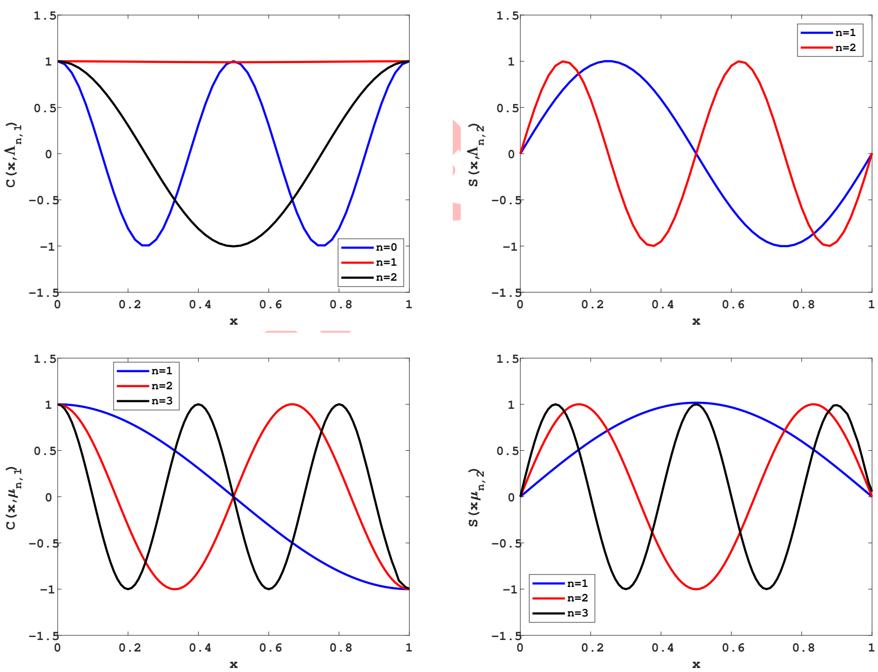

FIGURE 1. Approximation of the eigenfunctions of Example 7.2, using periodic boundary conditions $a: \ C(x,\Lambda_{n,1}),\ n = 0,1,2$ and $b: \ S(x,\Lambda_{n,2}),\ n = 1,2$ and semiperiodic boundary conditions $c: \ C(x,\mu_{n,1}),\ n = 1,2,3$ and $d: \ S(x,\mu_{n,2}),\ n = 1,2,3$.

satisfy the necessary boundary conditions to high precision, the values in Tables 3, 5, and 7 validate that the method provides accurate eigenvalue estimates.

Table 7. The eigenvalues obtained from the Chebyshev cardinal pseudospectral method for Example 7.3.

n,i	$\Lambda_{n,i}^{10}$	$\Lambda_{n,i}^{20}$	$\Lambda_{n,i}^{30}$	$\lvert\Lambda_{n,i}^{10}-\Lambda_{n,i}^{30}\rvert$	$\lvert\Lambda_{n,i}^{20}-\Lambda_{n,i}^{30}\rvert$
0,1	1.081090040	1.081089178	1.081089178	8.6×10^{-07}	2.4×10^{-13}
1,1	6.376773613	6.376759793	6.376759793	1.3×10^{-05}	4.5×10^{-12}
1,2	6.377588004	6.377578942	6.377578942	9.1×10^{-06}	1.6×10^{-12}
2,1	12.61886217	12.61333307	12.61333307	5.5×10^{-03}	8.7×10^{-10}
2,2	12.62046898	12.61358271	12.61358271	6.9×10^{-03}	6.3×10^{-10}
3.1	19.81011859	18.88092708	18.88092688	9.3×10^{-01}	2.1×10^{-07}
3,2	19.82170848	18.88100150	18.88100127	9.4×10^{-01}	2.3×10^{-07}
n,i	$\mu_{n,i}^{10}$	$\mu_{n,i}^{20}$	$\mu_{n,i}^{30}$	$\lvert\mu_{n,i}^{10}-\mu_{n,i}^{30}\rvert$	$\lvert\mu_{n,i}^{20}-\mu_{n,i}^{30}\rvert$
1,1	3.238982227	3.238982432	3.238982432	2.1×10^{-07}	1.4×10^{-11}
1,2	3.407266517	3.407266617	3.407266617	1.0×10^{-07}	4.0×10^{-12}
2,1	9.487216075	9.487224988	9.487224988	8.9×10^{-06}	8.6×10^{-10}
2,2	9.487809717	9.487812592	9.487812592	2.9×10^{-06}	8.9×10^{-10}
3,1	15.77054406	15.74558177	15.74558170	2.5×10^{-02}	7.2×10^{-08}
3,2	15.77287232	15.74570988	15.74571001	2.7×10^{-02}	1.4×10^{-07}

Table 8. The difference of approximation of the eigenfunctions in the points $x=0$ and $x=\omega$ using the boundary conditions (1.2) and (1.3), in the Example 7.3.

n	$\lvert C(0,\Lambda_{n,1}^{10})-C(\omega,\Lambda_{n,1}^{10})\rvert$	$\lvert C(0,\Lambda_{n,1}^{20})-C(\omega,\Lambda_{n,1}^{20})\rvert$	$\lvert C(0,\Lambda_{n,1}^{30})-C(\omega,\Lambda_{n,1}^{30})\rvert$
0	9.3×10^{-08}	1.2×10^{-13}	1.8×10^{-20}
1	2.1×10^{-08}	1.6×10^{-13}	5.4×10^{-20}
3	8.5×10^{-05}	1.3×10^{-10}	1.4×10^{-15}
5	6.8×10^{-02}	5.3×10^{-11}	8.8×10^{-13}
n	$\lvert S(0,\Lambda_{n,2}^{10})-S(\omega,\Lambda_{n,2}^{10})\rvert$	$\lvert S(0,\Lambda_{n,2}^{20})-S(\omega,\Lambda_{n,2}^{20})\rvert$	$\lvert S(0,\Lambda_{n,2}^{30})-S(\omega,\Lambda_{n,2}^{30})\rvert$
2	5.1×10^{-04}	2.1×10^{-09}	9.6×10^{-16}
4	2.6×10^{-03}	8.3×10^{-07}	3.2×10^{-11}
6	1.3×10^{-01}	8.6×10^{-05}	4.1×10^{-09}
n	$\lvert C(0,\mu_{n,1}^{10})+C(\omega,\mu_{n,1}^{10})\rvert$	$\lvert C(0,\mu_{n,1}^{20})+C(\omega,\mu_{n,1}^{20})\rvert$	$\lvert C(0,\mu_{n,1}^{30})+C(\omega,\mu_{n,1}^{30})\rvert$
1	1.3×10^{-07}	2.0×10^{-12}	2.6×10^{-19}
2	1.7×10^{-07}	1.7×10^{-11}	4.8×10^{-19}
3	1.1×10^{-03}	4.1×10^{-10}	5.8×10^{-14}
n	$\lvert S(0,\mu_{n,2}^{10})+S(\omega,\mu_{n,2}^{10})\rvert$	$\lvert S(0,\mu_{n,2}^{20})+S(\omega,\mu_{n,2}^{20})\rvert$	$\lvert S(0,\mu_{n,2}^{30})+S(\omega,\mu_{n,2}^{30})\rvert$
1	3.3×10^{-05}	8.5×10^{-11}	2.2×10^{-17}
2	5.9×10^{-04}	1.3×10^{-07}	4.5×10^{-14}
3	5.0×10^{-02}	1.7×10^{-05}	1.8×10^{-09}

The approximated eigenfunctions for $N=20$ in both periodic and semiperiodic cases are shown in Figures 1 and 2. The plots clearly show the theoretical Sturm–Liouville properties: the symmetry or antisymmetry behavior is accurately captured for the selected potential function, and the nth eigenfunction shows at most $2n$ zeros in $(0,\omega)$. These findings confirm the numerical method s alignment with the analytical theory.

8. Conclusion

This work has developed a Chebyshev cardinal pseudospectral method for computing both eigenvalues and eigenfunctions for periodic and semiperiodic Sturm–Liouville problems. To obtain high-accuracy numerical approximations, the technique uses cardinal collocation and reformulates the differential system into a Volterra integral equation. The suggested approach maintains high accuracy even when piecewise-continuous coefficients are present, as shown by the numerical experiments. Excellent agreement between eigenvalue approximations and published numerical benchmarks is demonstrated, and the convergence behavior with respect to the number of collocation points is evident. Furthermore, the theoretical symmetry properties, boundary behaviors, and nodal structure predicted by Sturm–Liouville

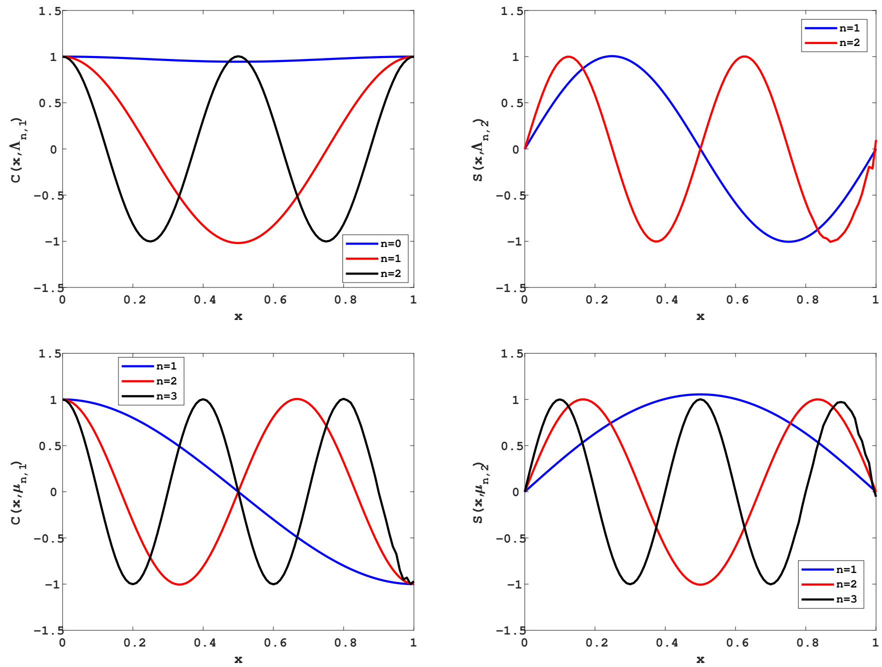

FIGURE 2. Approximation of the eigenfunctions of Example 7.3, using periodic boundary conditions $a:\ C(x,\Lambda_{n,1}),\ n=0,1,2$ and $b:\ S(x,\Lambda_{n,2}),\ n=1,2$ and semiperiodic boundary conditions $c:\ C(x,\mu_{n,1}),\ n=1,2,3$ and $d:\ S(x,\mu_{n,2}),\ n=1,2,3$.

theory are confirmed by the numerical reconstruction of eigenfunctions. For periodic spectral problems, the approach offers a cohesive and effective computational framework. Future work could focus on applications to fractional, nonlinear, and inverse Sturm–Liouville problems, extension to higher-dimensional periodic domains, and adaptive node selection techniques.

It is noteworthy that in Part 2 of the paper, the authors address the approximation of eigenfunctions for symmetric potentials. This is achieved by employing Chebyshev cardinal functions and the associated Volterra integral equations. To the best of our knowledge, this manuscript is the first to investigate the computation of eigenfunctions for this problem.

References

[1] A. Afarideh, F. Dastmalchi Saei, and B. Nemati Saray, *Eigenvalue problem with fractional differential operator: Chebyshev cardinal spectral method*, Journal of Mathematical Modeling, *11*(2) (2023), 343–355.

[2] A. Afarideh, F. D. Saei, M. Lakestani, and B. N. Saray, *Pseudospectral method for solving fractional Sturm–Liouville problem using Chebyshev cardinal functions*, Physica Scripta, *96*(12) (2021), 125267.

[3] K. Aydemir and O. S. Mukhtarov, *Spectrum of periodic Sturm–Liouville problems involving additional transmission conditions*, Rendiconti del Circolo Matematico di Palermo Series 2, *72*(1) (2023), 553–564.

[4] P. A. Binding and B. P. Rynne, *Half-eigenvalues of periodic Sturm–Liouville problems*, Journal of Differential Equations, *206*(2) (2004), 280–305.

[5] A. Boumenir, *Eigenvalues of periodic Sturm–Liouville problems by the Shannon-Whittaker sampling theorem*, Mathematics of computation, *68*(227) (1999), 1057–1066.

[6] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*. Courier Corporation, (2001).

[7] I. Çelik, and G. Gokmen, *Approximate solution of periodic Sturm–Liouville problems with Chebyshev collocation method*, Applied mathematics and computation, *170*(1) (2005), 285–295.

[8] G. Freiling and V. A. Yurko, *Inverse Sturm–Liouville Problems and Their Applications* (p. 305). Huntington: NOVA Science Publishers, (2001).

[9] M. Izadi and M. Afshar, *Solving the Basset equation via Chebyshev collocation and LDG methods*, Journal of Mathematical Modeling, *9*(1) (2021), 61–79.

[10] M. Izadi, Ş. Yüzbaşi, and D. Baleanu, *A Taylor Chebyshev approximation technique to solve the 1D and 2D nonlinear Burgers equations*, Mathematical Sciences, *16*(4) (2021), 459–471.

[11] M. G. Krein, *Solution of the Inverse Sturm–Liouville problem*, Dokl. Akad. Nauk SSSR, *76* (1951), 21–24. .

[12] B. M. Levitan, I. S. Sargsjan, and M. Hazewinkel, *Sturm-Liouville and dirac operators (Vol. 59)*. Dordrecht: Kluwer Academic, (1991).

[13] T. Liu, B. Ding, B. N. Saray, D. A. Juraev, and E. E. Elsayed, *On the pseudospectral method for solving the fractional Klein Gordon equation using Legendre cardinal functions*, Fractal and Fractional, *9*(3) (2025), 177.

[14] T. Liu, R. Xue, B. Ding, D. A. Juraev, B. N. Saray, and F. Soleymani, *A novel and effective scheme for solving the fractional telegraph problem via the spectral element method*, Fractal and Fractional, *8*(12) (2024), 711.

[15] H. Mirzaei, M. Emami, K. Ghanbari, and M. Shahriari, *An efficient algorithm for computing the eigenvalues of conformable Sturm-Liouville problem*, Computational Methods for Differential Equations, *12*(3) (2024), 471–483.

[16] S. N. Öztrk, O. Mukhtarov, and K. Aydemir, *Non-classical periodic boundary value problems with impulsive conditions*, Journal of New Results in Science, *12*(1) (2023), 1–8.

[17] M. Sahabi and A.Yazdani Cherati, *Fractional pseudospectral schemes with applications to fractional optimal control problems*, Journal of Mathematics, *2024*(1) (2024), 9917116.

[18] K. Sayevand and H. Arab, *An efficient extension of the Chebyshev cardinal functions for differential equations with coordinate derivatives of non-integer order* Computational Methods for Differential Equations, *6*(3) (2018), 339–352.

[19] L. Shi, B. N. Saray, and F. Soleymani, *Sparse wavelet Galerkin method: Application for fractional Pantograph problem*, Journal of Computational and Applied Mathematics, *451* (2024), 116081.

[20] M. Shahriari and H. Mirzaei, *Inverse Sturm-Liouville problem with conformable derivative and transmission conditions*, Hacettepe Journal of Mathematics and Statistics, *52*(3) (2023), 1–15.

[21] M. Shahriari, B. N. Saray, and J. Manafian, *Reconstruction of the Sturm-Liouville differential operators with discontinuity conditions and a constant delay*, Indian Journal of Pure and Applied Mathematics, *51*(2) (2020), 659–668.

[22] M. Shahriari, B. N. Saray, B. Mohammadalipour, and S. Saeidian, *Pseudospectral method for solving the fractional one-dimensional Dirac operator using Chebyshev cardinal functions*, Physica Scripta, *98(5)* (2023), 055205.

[23] J. Virieux, H. Calandra, ande R. Plessix, *A review of the spectral, pseudo spectral, finite difference and finite element modelling techniques for geophysical imaging*, Geophysical Prospecting, *59*(5) (2011), 794–813.

[24] Y. Yuan, J. Sun, and A. Zettl, *Eigenvalues of periodic Sturm–Liouville problems*, Linear Algebra and its Applications, *517* (2017), 148–166.

[25] A. Zettl, *Recent Developments in Sturm–Liouville Theory*, Walter de Gruyter GmbH and Co KG, (2021).

[26] K. Zhao, *Coincidence theory of a nonlinear periodic Sturm–Liouville system and its applications*, Axioms, *11*(12) (2022), 726.