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Existence and Ulam-Hyers-Rassias stability of solutions for integro-differential equations using Hilfer-Katugampola fractional derivatives

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Abstract

This paper investigates the existence and Ulam-Hyers-Rassias stability of solutions for integrodifferential equations involving the Hilfer-Katugampola fractional derivative. By employing Ulam-Hyers-Rassias stability analysis and applying Schaefer's fixed point theorem, we establish key results concerning the behavior and stability of these equations. Additionally, we provide illustrative examples to demonstrate the practical relevance of our theoretical findings.

Keywords. Hilfer-Katugampula fractional derivative (H-K FD), Ulam-Hyers-Rassias type stability (U-H-R), integrodifferential equation (IDE).
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Uncorrected Proof

1. Introduction

Derivatives and integrals of real or complex orders are integrated with classical integral calculus to form fractional calculus, which is an intriguing field of mathematical inquiry. Recently, fractional differential equations have garnered substantial attention due to their practical applications in physics, chemistry, engineering, biology, and other disciplines. This shows the value of FDEs in explaining natural phenomena. Recent developments in the analysis of partial and ordinary fractional differential integral equations are demonstrated by the following publications and their references [1–3, 5–7, 11, 17, 20–22]. For specific values of the parameters, the literature has proposed some generic formulations that provide interpolations between other difference integrals. Notable definitions include those by Riemann-Liouville (R-L), Caputo (C), Hadamard (Ha), and Hilfer (H) [11, 16, 19].
Recently, R-L fractional differential derivatives and integrals were presented in a generalized form by Katugampola. [15]. Thereafter, Oliveira et al.[18] established a more generalized form of the Katugampola fractional derivative, known as the Hilfer Katugampola (H-K). This formulation serves as an interpolation between popular definitions such as the H, H-Ha, C-Ha, and R-L FDs. We direct the reader to [18]. On the other hand, in a 1940 lecture at the University of Wisconsin, Ulam brought up the stability issue of functional equations (specifically, group homomorphisms). Ulam's inquiry concerned "What circumstances lead to the existence of an additive mapping close to an approximately additive mapping?" Hyers provided the initial response to Ulam's question (about additive mapping) in the context of Banach spaces in 1941. Rassias proved linear mapping stability for the first time in 1978 when he established the Hyers-Ulam stability of linear and nonlinear mappings. Following Rassias's result, several mathematicians from around the world

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started looking into differential equation stability issues [13, 23]. According to Vivek et al., [24] (2018), Hilfer's implicit FDEs with nonlocal conditions of the type resulting in existence and stability results

$$
{}_{\mathfrak{F}}\mathfrak{D}_{0^+}^{\mathfrak{r}_1,\mathfrak{r}_2}u(\vartheta)=\varphi\left(\vartheta,u(\vartheta),{}_{\mathfrak{F}}\mathfrak{D}_{0^+}^{\mathfrak{r}_1,\mathfrak{r}_2}u(\vartheta)\right),\qquad \vartheta\in\mathcal{J}:=[0,T],\quad 0<\mathfrak{r}_1<1,0\leqslant\mathfrak{r}_2\leqslant1, \tag{1.1}
$$

$$
\mathfrak{I}_{0^+}^{1-\mathfrak{r}}u(0)=\sum_{i=1}^{n}\xi_i u\left(\eta_i\right),\qquad \mathfrak{r}_1\leqslant\mathfrak{r}=\mathfrak{r}_1+\mathfrak{r}_2(1-\mathfrak{r}_1)<1, \tag{1.2}
$$

In 2020, the work of Harikrishnan et al.[10] studied the existence, uniqueness, and stability results for IDEs involving H-K FD of the form

$$
{}_{\mathfrak{F}}\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}_1,\mathfrak{r}_2}u(\vartheta)=\varphi(\vartheta,u(\vartheta),\int_{\mathfrak{a}}^{\vartheta}h(\vartheta,\nu,u(\nu))\mathrm{d}\nu),\qquad \vartheta\in\mathcal{J}:=(\mathfrak{a},T], \tag{1.3}
$$

$$
{}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{1-\mathfrak{r}}u(\mathfrak{a})=u_0,\qquad \mathfrak{r}=\mathfrak{r}_1+\mathfrak{r}_2-\mathfrak{r}_1\mathfrak{r}_2<1, \tag{1.4}
$$

where ${}_{\mathfrak{F}}D_{\mathfrak{a}^+}^{\mathfrak{r}_1,\mathfrak{r}_2}$ is the H-K FD of order $\mathfrak{r}_1$ and type $\mathfrak{r}_2$ and ${}_{\mathfrak{F}}\mathfrak{I}_{a^+}^{1-\mathfrak{r}}$ is generalized fractional integral of order 1- $\mathfrak{r}$, $\mathfrak{F}>0$ where $\varphi:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\theta:\Delta\times\mathbb{R}\to\mathbb{R}$ are continuous.
Similarly, in [23], the existence and uniqueness results for implicit differential equations with nonlocal conditions were discussed, involving H-K FDs of the following form

$$
{}_{\mathfrak{F}}\mathfrak{D}_{0^+}^{\mathfrak{r}_1,\mathfrak{r}_2}u(\vartheta)=\varphi\left(\vartheta,u(\vartheta),{}_{\mathfrak{F}}\mathfrak{D}_{0^+}^{\mathfrak{r}_1,\mathfrak{r}_2}u(\vartheta)\right),\qquad \vartheta\in\mathcal{J}:=[0,T],\quad 0<\mathfrak{r}_1<1,0\leqslant\mathfrak{r}_2\leqslant1, \tag{1.5}
$$

$$
{}_{\mathfrak{F}}\mathfrak{I}_{0^+}^{1-\mathfrak{r}}u(0)=\sum_{i=1}^{n}\xi_i u\left(\eta_i\right),\qquad \mathfrak{r}_1\leqslant\mathfrak{r}=\mathfrak{r}_1+\mathfrak{r}_2(1-\mathfrak{r}_1)<1, \tag{1.6}
$$

where $\mathfrak{F}>0$ and ${}_{\mathfrak{F}}\mathfrak{D}_{0^+}^{\mathfrak{r}_2,\mathfrak{r}_2}$ is the H-K-FD of order $\mathfrak{r}_2$ and type $\mathfrak{r}_2$, ${}_{\mathfrak{F}}\mathfrak{I}_{0^+}^{1-\mathfrak{r}}$ is the Katugampola fractional integral of order $1-\mathfrak{r},\eta_i\in(0,T),\xi_i\in\mathbb{R}$, for all $i=1,2,\ldots,n,\eta_1\leqslant\eta_2\leqslant\cdots\leqslant\eta_{\mathrm{n}}$ and $\varphi:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.
Motivated by the above discussions, finding solutions to IDEs using the H-K FD and demonstrating the Hyers-Ulam stability of solutions are the primary goals of this essay. In this paper, we derive sufficient conditions for the existence of solutions to the following IDE and prove the U-H-R stability results on our problem,

$$
{}_{\mathfrak{F}}\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}_1,\mathfrak{r}_2}\mathfrak{u}(\vartheta)=\varkappa(\vartheta,u(\vartheta),\int_{\mathfrak{a}}^{\vartheta}\theta(\vartheta,\nu,u(\nu))\mathrm{d}\nu),\qquad \vartheta\in\mathcal{J}=(\mathfrak{a},T], \tag{1.7}
$$

$$
{}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{1-\mathfrak{r}}u(\mathfrak{a})=u_0,\qquad \mathfrak{r}=\mathfrak{r}_1+\mathfrak{r}_2(1-\mathfrak{r}_1)<1. \tag{1.8}
$$

The functions $\varkappa:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\theta:\Delta\times\mathbb{R}\to\mathbb{R}$ are considered continuous, where ${}_{\mathfrak{F}}\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}_2,\mathfrak{r}_2}$ is the H-K-FD of order $\mathfrak{r}_2$ and type $\mathfrak{r}_2$. The generalized fractional integral of order $1-\mathfrak{r}$, with $\mathfrak{F}>0$ is noted ${}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{1-\mathfrak{r}}$. In this case, $\Delta=\{(\vartheta,\nu):\mathfrak{a}\leq\nu\leq\vartheta\leq T\}$ defines the domain Δ. To save time, let's refer to this as

$$
\Theta(\vartheta)=\int_{\mathfrak{a}}^{\vartheta}\theta(\vartheta,\nu,u(\nu))\mathrm{d}\nu. \tag{1.9}
$$

The study is summarized briefly in this section. A concise overview of the essential information necessary to reach our main findings is presented in section 2. Our principal results are substantiated by several conjectures discussed in sections 3 and 4. Specifically, we derive the necessary results employing Schaefer's fixed point theorems. Furthermore, we provide relevant instances to reinforce the proposed concept.

2. Foundational Preliminaries

Let us start this part by reviewing some fundamental definitions and important findings in fractional calculus. Sources like [14, 16, 18] are where the observations provided here come from. These observations will form the framework for our discussion in this paper. $\mathcal{J}=(\mathfrak{a},T]$ and $0<\mathfrak{a}<T$ be given. The weighted space $\mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of continuous functions on $[\mathfrak{a},T]$ for $\mathfrak{F}>0$ and $\mathfrak{r}\in[0,1)$ is defined by

$$
\mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]=\left\{\varphi:\mathcal{J}\to\mathbb{R}:\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}}\varphi(\vartheta)\in\mathfrak{C}[\mathfrak{a},T]\right\},\mathfrak{r}\in[0,1),
$$

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with the norm

$$\|\varphi\|_{\mathfrak{C}_{\mathfrak{r},\mathfrak{F}}} = \sup_{\vartheta\in\mathcal{J}} \left\| \left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}} \varphi(\vartheta) \right\|,$$

where,

$$\mathfrak{C}_{0,\mathfrak{F}}[\mathfrak{a}, T] = \mathfrak{C}[\mathfrak{a}, T].$$

Obviously,

$$\mathfrak{C}^n_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T] = \left\{ \varphi \in \mathfrak{C}^{n-1}, \varphi^{(n)} \in \mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T] \right\}, n \in \mathbb{N},$$

with the norm

$$\|\varphi\|_{\mathfrak{C}^n_{\mathfrak{r},\mathfrak{F}}} = \sum_{i=0}^{n-1} \left\| \varphi^{(i)} \right\|_{\mathfrak{C}} + \left\| \varphi^{(n)} \right\|_{\mathfrak{C}_{\mathfrak{r},\mathfrak{F}}},$$

where,

$$\mathfrak{C}^0_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T] = \mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T].$$

Let $\delta_{\mathfrak{F}} = \left(\vartheta^{1-\mathfrak{F}} \frac{d}{d\vartheta}\right)$. For any $n \in \mathbb{N}$, we use the notation $\mathfrak{C}^n_{\delta_{\mathfrak{F}},\mathfrak{r}}[\mathfrak{a}, T]$ to represent the Banach space of functions $\varkappa$ that possess continuous differentiability, with respect to the operator $\delta_{\mathfrak{F}}$, on the interval $\mathcal{J}$ up to the $(n-1)$ order. Additionally, the n-th derivative $\delta^n_{\mathfrak{F}}$ of φ must be defined on the interval $\mathcal{J}$ and belong to the space $\mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[a, b]$. In formal terms, this can be expressed as:

$$\mathfrak{C}^n_{\delta_{\mathfrak{F}},\mathfrak{r}}[\mathfrak{a}, T] = \{\varphi : \mathcal{J} \to \mathbb{R} : \delta^i_{\mathfrak{F}}\varphi \in \mathfrak{C}[\mathfrak{a}, T], i = 0, 1, \ldots, n-1, \delta^n_{\mathfrak{F}}\varphi \in \mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T]\},$$

with the norm

$$\|\varphi\|_{\mathfrak{C}^n_{\delta_{\mathfrak{F}},\mathfrak{r}}} = \sum_{i=0}^{n-1} \|\delta^i_{\mathfrak{F}}\varphi\|_{\mathfrak{C}} + \|\delta^n_{\mathfrak{F}}\varphi\|_{\mathfrak{C}_{\mathfrak{r},\mathfrak{F}}}$$

$$\|\varphi\|_{\mathfrak{C}^n_{\delta_{\mathfrak{F}},\mathfrak{r}}} = \sum_{i=0}^{n} \max_{\vartheta\in\mathcal{J}} |\delta^i_{\mathfrak{F}}\varphi(\vartheta)|.$$

If $n = 0$, we obtain,

$$\mathfrak{C}^0_{\delta_{\mathfrak{F}},\mathfrak{r}}[\mathfrak{a}, T] = \mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T]. \tag{2.1}$$

Definition 2.1. [18] Let $\mathfrak{r}_1 > 0, \vartheta > \mathfrak{a}, \mathfrak{F} > 0$ and $\varphi \in \mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T]$. The Katugampola fractional integral ${}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi$ of order $\mathfrak{r}_1$ is defined as

$${}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi(\vartheta) = \frac{1}{\mathfrak{d}(\mathfrak{r}_1)} \int_{\mathfrak{a}}^{\vartheta} \nu^{\mathfrak{F}-1} \left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}_1 - 1} \varphi(\nu) d\nu. \tag{2.2}$$

and the corresponding Katugampola fractional derivative ${}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{a}}_{\mathfrak{a}+}\varphi$ is defined as

$$\begin{aligned} {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi(\vartheta) &= \delta^n_{\mathfrak{F}} \left({}_{\mathfrak{F}}\mathfrak{I}^{n-\mathfrak{r}_1}_{\mathfrak{a}+}\varphi \right)(\vartheta) \\ &= \left(\vartheta^{1-\mathfrak{F}} \frac{d}{d\vartheta} \right)^n \frac{1}{\mathfrak{d}(n-\mathfrak{r}_1)} \int_{\mathfrak{a}}^{\vartheta} \nu^{p-1} \left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}} \right)^{n-\mathfrak{r}_1-1} \varphi(\nu) d\nu \end{aligned} \tag{2.3}$$

such that $n = [\mathfrak{r}_1] + 1$, and $\mathfrak{d}(\mathfrak{r}_1)$ is Euler Gamma function.

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Definition 2.2. [18] Given the order $\mathfrak{r}_1$ and the type $\mathfrak{r}_2$ satisfying $\mathfrak{r}_1 \in (0,1)$ and $\mathfrak{r}_2 \in [0,1]$. The H-K FD of the function $\varphi \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ with $\mathfrak{F} > 0$, $\mathfrak{r} = \mathfrak{r}_1 + \mathfrak{r}_2 - \mathfrak{r}_1\mathfrak{r}_2$ defined by

$$\begin{aligned}\left({}_{\mathfrak{F}}\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}_1,\mathfrak{r}_2}\varphi\right)(\vartheta) &= \left({}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{r}_2(1-\mathfrak{r}_1)}\left(\vartheta^{\mathfrak{F}-1}\frac{d}{d\vartheta}\right){}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{(1-\mathfrak{r}_2)(1-\mathfrak{r}_1)}\varphi\right)(\vartheta) \\ &= \left({}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{r}_2(1-\mathfrak{r}_1)}\,{}_{\mathfrak{F}}\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}}\right)\varphi(\vartheta). \end{aligned} \tag{2.4}$$

Proposition 2.3. *An interpolator of the following fractional derivatives is* ${}_{\mathfrak{F}}\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}_1,\mathfrak{r}_2}$*:*
H $\mathfrak{F} \to 1$, *(see[11]),*
H-Ha $\mathfrak{F} \to 0^+$, *(see[13]),*
generalized $\mathfrak{r}_2 = 0$, *(see[15]),*
Caputo $\mathfrak{r}_2 = 1$, *(see[18]),*
R-L $\mathfrak{r}_2 = 0, \mathfrak{F} \to 1$, *(see[16]),*
Ha $\mathfrak{r}_2 = 0, \mathfrak{F} \to 0^+$, *(see[16]),* ‘
C $\mathfrak{r}_2 = 1, \mathfrak{F} \to 1$, *(see[16]),*
C-Ha $\mathfrak{r}_2 = 1, \mathfrak{F} \to 0^+$, *(see[8]),*
Liouville $\mathfrak{r}_2 = 0, \mathfrak{F} \to 1, \mathfrak{a} = 0$, *(see[16]),*
Weyl $\mathfrak{r}_2 = 0, \mathfrak{F} \to 1, \mathfrak{a} = \infty$, *(see[12]).*

Definition 2.4. [9] Let Ξ and $\mathit{\Pi}$ be two Banach spaces, and φ a definite application from Ξ to values in $\mathit{\Pi}$. We have

- If φ is both continuous and converts every bounded set in Ξ into a relatively compact set in $\mathit{\Pi}$, then φ is completely continuous.
- If $\varphi(\Xi)$ is relatively compact in $\mathit{\Pi}$, then φ is a compact application.

Lemma 2.5. *[9]**(Ascoli-Arzela of type* $\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}$*)*
Let $\Omega \subset \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$*. We say that* Ω *is relatively compact if the following conditions hold:*

a) *The set* Ω *is uniformly bounded:*

$\exists M > 0,\ \ |\varphi(\vartheta)| < M,\ \ \forall \varphi \in \Omega,\ \vartheta \in [\mathfrak{a},T].$

b) *The set* Ω *is equicontinuous on* $[\mathfrak{a},T]$*:*

$\forall \epsilon > 0, \exists \delta > 0, |\vartheta_1 - \vartheta_2| < \delta$ *implie that* $|\varphi(\vartheta_1) - \varphi(\vartheta_2)| < \epsilon,\ \ \forall \vartheta_1, \vartheta_2 \in [\mathfrak{a},T],\ \forall \varphi \in \Omega.$

Theorem 2.6. *[9] **(Schaefer's fixed point theorem)***
Let Ξ *be a Banach space and* $\mathfrak{K} : \Xi \to \Xi$ *be a continuous and compact application. If,*

$$\Upsilon = \{\lambda \in \Xi : \lambda = \omega\mathfrak{K}(\lambda),\ \text{for some}\ \omega \in [0,1]\}, \tag{2.5}$$

is a bounded set, then $\mathfrak{K}$ *has a fixed point.*

The following parameters are taken into consideration: The spaces are defined as follows: $\mathfrak{r}_1 \in (0,1)$, $\mathfrak{r}_2 \in [0,1]$ and $\mathfrak{r} \in [0,1)$, such that $\mathfrak{r} = \mathfrak{r}_1 + \mathfrak{r}_2 - \mathfrak{r}_1\mathfrak{r}_2$.

$$\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}^{\mathfrak{r}_1,\mathfrak{r}_2} = \left\{\varphi \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T], {}_{\mathfrak{F}}\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}_1,\mathfrak{r}_2}\varphi \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]\right\}, \tag{2.6}$$

and

$$\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}^{\mathfrak{r}} = \{\varphi \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T], {}_{\mathfrak{F}}\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}}\varphi \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]\}, \tag{2.7}$$

where $\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}_1,\mathfrak{r}_2}$ and $\mathfrak{D}_{\mathfrak{a}^+}^{\mathfrak{r}}$ are the H-K FD.
Clearly,

$$\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}^{\mathfrak{r}}[\mathfrak{a},T] \subset \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}^{\mathfrak{r}_1,\mathfrak{r}_2}[\mathfrak{a},T] \subset \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]. \tag{2.8}$$

Lemma 2.7. *[12] If* $\varphi \in \mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ *and* $\mathfrak{r}_1, \mathfrak{r}_2 > 0$*, then*

$${}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{r}_1}\left({}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{r}_2}\varphi\right)(\vartheta) = {}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{r}_1+\mathfrak{r}_2}\varphi(\vartheta). \tag{2.9}$$

Lemma 2.8. *[18] For* $\mathfrak{r}_1 > 0$*,* ${}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{r}_1}$ *is maps from* $\mathfrak{C}[\mathfrak{a},T]$ *into* $\mathfrak{C}[\mathfrak{a},T]$*.*

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Lemma 2.9. *[18] Let $\mathfrak{r}_1 > 0$ and $\mathfrak{r} \in [0,1)$. Then ${}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}$ is bounded maps from $\mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ into $\mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$*

Lemma 2.10. *[18] Let $0 < \mathfrak{a} < T, \mathfrak{r} \in [0,1)$ and $\varphi \in \mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$. If $\mathfrak{r}_1 > \mathfrak{r}$ then ${}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi$ is continuous maps on $\mathcal{J}$, and we have*

$$ {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi(\mathfrak{a}) = \lim_{\vartheta\to\mathfrak{a}^+} {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi(\vartheta) = 0. \tag{2.10}$$

Lemma 2.11. *[18] Let $\nu > \mathfrak{a}$, ${}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}$ and ${}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1}_{\mathfrak{a}+}$,as defined in definition 2.1. Then for $\mathfrak{r}_1 \geqslant 0$, $\mathfrak{r}_2 > 0$, we have*

$$\left({}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\left(\frac{\nu^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_2-1}\right)(\vartheta) = \frac{\mathfrak{d}(\mathfrak{r}_2)}{\mathfrak{d}(\mathfrak{r}_1+\mathfrak{r}_2)}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1+\mathfrak{r}_2-1}, \tag{2.11}$$

and for $\mathfrak{r}_1 \in (0,1)$

$$\left({}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1}_{\mathfrak{a}+}\left(\frac{\nu^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\right)(\vartheta) = 0. \tag{2.12}$$

Lemma 2.12. *[18] Let $\mathfrak{r}_1 \in (0,1)$ and $\mathfrak{r} \in [0,1)$. If $\varphi \in \mathfrak{C}_{\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ and ${}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}_1}_{\mathfrak{a}+}\varphi \in \mathfrak{C}_{\mathfrak{r}}[\mathfrak{a},T]$, then*

$$\left({}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\,{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi\right)(\vartheta) = \varphi(\vartheta) - \frac{\left({}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}_1}_{\mathfrak{a}+}\varphi\right)(\mathfrak{a})}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}, \quad \forall\vartheta\in\mathcal{J} \tag{2.13}$$

Proof. The integration by parts is used in the proof and this choice

$$z = (\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}})^{\mathfrak{r}_1-1} \quad \text{and} \quad dz = \frac{d}{d\nu}{}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}_1}_{\mathfrak{a}+}\varphi(\nu)d\nu. \tag{2.14}$$

□

Lemma 2.13. *[15] Let $\mathfrak{r}_1 > 0$, $\mathfrak{r} \in [0,1)$ and $\varphi \in \mathfrak{C}_{\mathfrak{r}}[\mathfrak{a},T]$. Then ${}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1}_{\mathfrak{a}+}\left({}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi\right)(\vartheta) = \varphi(\vartheta)$.*

Lemma 2.14. *[18] Let $\mathfrak{r}_1 \in (0,1)$, $\mathfrak{r}_2 \in [0,1]$ and $\mathfrak{r} = \mathfrak{r}_1 + \mathfrak{r}_2(1-\mathfrak{r}_1)$. Assume that $\varkappa(\vartheta, u(\vartheta)) \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ where $\varkappa : \mathcal{J}\times\mathbb{R}\to\mathbb{R}$ be a function for any $u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$. If $u \in \mathfrak{C}^{\mathfrak{r}}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$, then u satisfies*

$$ {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}+}u(\vartheta) = \varkappa(\vartheta, u(\vartheta)), \tag{2.15}$$

$$ {}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}}_{\mathfrak{a}+}u(\mathfrak{a}) = u_0, \tag{2.16}$$

if and only if the Volterra integral equation given below is satisfied by

$$u(\vartheta) = \frac{u_0}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} + \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu))d\nu. \tag{2.17}$$

Proof. see [18] □

Lemma 2.15. *[18] Assume that $\mathfrak{r}_1 \in (0,1), \mathfrak{r}_2 \in [0,1]$, and $\mathfrak{r} = \mathfrak{r}_1 + \mathfrak{r}_2(1-\mathfrak{r}_1)$. If $\varphi \in \mathfrak{C}^{\mathfrak{r}}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$, then*

$$ {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}}_{\mathfrak{a}+}\,{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+}\varphi = {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\,{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}+}\varphi, \tag{2.18}$$

and,

$$ {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+}\,{}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varphi = {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}\varphi. \tag{2.19}$$

For our Eq. (1.7) and (1.8), we now establish an important equivalent mixed-type Volterra FIE in the following Lemma 2.16, which we will use later in the main result.

Lemma 2.16. *[18] Let $\mathfrak{r} = \mathfrak{r}_1 + \mathfrak{r}_2(1-\mathfrak{r}_1)$, where $\mathfrak{r}_1 \in (0,1)$ and $\mathfrak{r}_2 \in [0,1]$. If $\varkappa : \mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function such that $\varkappa(\cdot, u(\cdot), \Theta u(\cdot)) \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[a,b]$ for all $u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ and $\Theta u(\vartheta) = \int_{\mathfrak{a}}^{\vartheta}\theta(\vartheta,\nu,u(\nu))\mathrm{d}\nu$.*
A function $u \in \mathfrak{C}^{\mathfrak{r}}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ is the solution of fractional initial value Eq. (1.7)-(1.8) if and only if u satisfies the following Volterra integral equation

$$u(\vartheta) = \frac{u_0}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} + \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu. \tag{2.20}$$

Uncorrected Proof

Proof. We will prove the lemma in two steps: firstly, by demonstrating the necessity of condition (2.20), and subsequently by establishing its sufficiency.

- **Step 1**: Let $u \in \mathfrak{C}_{1-\mathfrak{r}}[\mathfrak{a}, T]$ be a solution of problem of Eq. (1.7)-(1.8). We will prove that u is also satisfies Eq. (2.20).
By the definition of $\mathfrak{C}^{\mathfrak{r}}_{1-\mathfrak{r}}[\mathfrak{a}, T]$, Lemma 2.9, and Definition 2.2, we have: ${}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}}_{\mathfrak{a}^+}u \in \mathfrak{C}[\mathfrak{a}, T]$ and,
$$ {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}^+}u = \delta_{\mathfrak{F}}\,{}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}}_{\mathfrak{a}^+}u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T], \tag{2.21} $$
which implies, ${}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}}_{\mathfrak{a}^+}u \in \mathfrak{C}^1_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T]$. Using Lemma 2.12, with $\mathfrak{r}_1 = \mathfrak{r}$ and the Eq. (1.8), we can write:
$$ {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}}_{\mathfrak{a}+}\,{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+}u(\vartheta) = u(\vartheta) - \frac{u_0}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}, \tag{2.22} $$
where $\vartheta \in \mathcal{J}$. By hypothesis, ${}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}^+}u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T]$. Using Lemma 2.15 with $\mathfrak{r}_1 = \mathfrak{r}$ and the Eq. (1.8), we have:
$$ \begin{aligned} {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}}_{\mathfrak{a}+}\,{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+}u(\vartheta) &= {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\,{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}+}u(\vartheta) \\ &= {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varkappa(\vartheta, u(\vartheta)). \end{aligned} \tag{2.23} $$
Comparing Eq. (2.22) and Eq. (2.23), we see that
$$ u(\vartheta) = \frac{u_0}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} + {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_1}_{\mathfrak{a}+}\varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)); \quad \vartheta \in \mathcal{J} \tag{2.24} $$
Thus, $u(\vartheta)$ satisfies Eq. (2.20).
- **Step 2**: Let $u \in \mathfrak{C}^{\mathfrak{r}}_{1-\mathfrak{r}}[\mathfrak{a}, T]$ satisfying Eq. (2.20). We will show that u also satisfies Eq. (1.7).
Apply operator ${}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+}$ on both sides of Eq. (2.20). Then, from Lemma 2.11, Lemma 2.15, and Definition 2.2, we obtain:
$$ {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+}u(\vartheta) = {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}\varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)). \tag{2.25} $$
By hypothesis, ${}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+} \in \mathfrak{C}_{1-\mathfrak{r}}[\mathfrak{a}, T]$, then, the Eq. (2.25) implies that
$$ {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+}u(\vartheta) = \delta_{\mathfrak{F}}\,{}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}u(\vartheta) = {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}u(\vartheta) \in \mathfrak{C}_{1-\mathfrak{r}}[\mathfrak{a}, T] \tag{2.26} $$
As $\varkappa(\cdot, u(\cdot)) \in \mathfrak{C}_{1-\mathfrak{r}}[\mathfrak{a}, T]$, from Lemma 2.9 , we have
$$ \mathfrak{I}^{1-\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}\varkappa \in \mathfrak{C}_{1-\mathfrak{r}}[\mathfrak{a}, T] \tag{2.27} $$
From Eq. (2.27) and Definition 2.2, we obtain:
$$ {}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}u \in \mathfrak{C}^1_{1-\mathfrak{r}}[\mathfrak{a}, T]. \tag{2.28} $$
Applying operator ${}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}$ and using Lemmas 2.12, 2.11, and 2.15, we have:
$$ \begin{aligned} {}_{\mathfrak{F}}\mathfrak{I}^{\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}\,{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}}_{\mathfrak{a}+}u(\vartheta) &= \varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)) + \frac{\mathfrak{I}^{\mathfrak{r}_2(1-\mathfrak{r}_1)}_{\mathfrak{a}+}\varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta))(\mathfrak{a})}{\mathfrak{d}(\mathfrak{r}_2(1-\mathfrak{r}_1))}\left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}_2(1-\mathfrak{r}_1)} \\ &= {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}+}u(\vartheta) \\ &= \varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)). \end{aligned} \tag{2.29} $$
Thus, the Eq. (1.7) holds.
Next, we show that if $u \in \mathfrak{C}^{\mathfrak{r}}_{1-\mathfrak{r}}[\mathfrak{a}, T]$ satisfies Eq. (2.20), it also satisfies the Eq. (1.7).
To this end, we multiply both sides of Eq. (2.24) by ${}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}}_{\mathfrak{a}+}$ and use Lemmas 2.11 and 2.7 to get:
$$ {}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}}_{\mathfrak{a}+}u(\vartheta) = u_0 + {}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}(1-\mathfrak{r}_1)}_{\mathfrak{a}+}\varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)) \tag{2.30} $$
Finally, taking $\vartheta \to a$ in Eq. (2.30), the Eq. (1.8) follows.

$\square$

Uncorrected Proof

3. Principal outcomes

Let's articulate some fundamental assumptions necessary for establishing the existence of solutions to the given problem.

($\mathbf{H_1}$) Let $\theta : \Delta \times \mathbb{R} \to \mathbb{R}$, $\Theta u(\vartheta) = \int_{\mathfrak{a}}^{\vartheta} \theta(\vartheta, \nu, u(\nu))\mathrm{d}\nu$ be continuous, and there exists a constant $\sigma > 0$ such that $|\Theta u(\vartheta)| \leq \sigma|u|$.

($\mathbf{H_2}$) Let $\varkappa$ a function from $\mathcal{J} \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, and there exist a function $p \in L^1(\mathcal{J}, \mathbb{R}^+)$ and two continuous nondecreasing functions $\phi_i : [0, \infty) \longrightarrow [0, \infty)$ such that,

$$|\varkappa(\vartheta, u, \mu)| \leq p(\vartheta)\phi_1(|u|)\phi_2(|\mu|), \vartheta \in \mathcal{J}. \tag{3.1}$$

with $p^* = \sup_{\vartheta \in \mathcal{J}} p(\vartheta)$. And we have if $\psi(u) = \phi_1(|u|)\phi_2(\sigma|u|)$,

$$\int_k^{\infty} \frac{du}{\psi(u)} = \infty, \quad \text{where} \quad k = \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})} \left(\frac{T^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}-1}. \tag{3.2}$$

Theorem 3.1. *Suppose that hypotheses ($\mathbf{H_1}$) and ($\mathbf{H_2}$) are fulfilled. Then, the problem(1.7)-(1.8) has at least one solution.*

Proof. In view of Lemma 2.16 , an operator $N : \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T] \longrightarrow \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T]$ related to the problems (1.7)-(1.8), is defined by

$$(Nu)(\vartheta) = \frac{u_0}{\mathfrak{d}(\mathfrak{r})} \left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}-1} + \frac{1}{\mathfrak{d}(\mathfrak{r}_1)} \int_{\mathfrak{a}}^{\vartheta} \left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}_1-1} \nu^{\mathfrak{F}-1} \varkappa(\nu, u(\nu), \Theta u(\nu)) d\nu. \tag{3.3}$$

Notice that the problems (1.7)-(1.8) is equivalent to the fixed point problem: $Nu(\vartheta) = u(\vartheta)$,, where N is defined in (3.3)

The proof will be presented in multiple steps.
where

Step 1: N is continuous.
Let u_n be a sequence such that $u_n \to u$ in $\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}$. Then for all $\vartheta \in \mathcal{J}$,

$$\begin{aligned} \left| ((Nu_n)(\vartheta) - (Nu)(\vartheta)) \left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{1-\mathfrak{r}} \right| &\leq \left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{1-\mathfrak{r}} \frac{1}{\mathfrak{d}(\mathfrak{r}_1)} \int_{\mathfrak{a}}^{\vartheta} \left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}_1-1} \nu^{\mathfrak{F}-1} \\ &\quad \times |\varkappa(\nu, u_n(\nu), \Theta u_n(\nu)) - \varkappa(\nu, u(\nu), \Theta u(\nu))| \, d\nu \\ &\leq \left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{1-\mathfrak{r}} \frac{1}{\mathfrak{d}(\mathfrak{r}_1)} \int_{\mathfrak{a}}^{\vartheta} \left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}_1-1} \nu^{\mathfrak{F}-1} \left(\frac{\nu^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}-1} \\ &\quad \times \left| \left(\frac{\nu^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{1-\mathfrak{r}} \varkappa(\nu, u_n(\nu), \Theta u_n(\nu)) - \varkappa(\nu, u(\nu), \Theta u(\nu)) \right| d\nu. \end{aligned} \tag{3.4}$$

From Definition 2.1, we have:

$$\left[{}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{r}_1} \left(\frac{\nu^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}-1} \right] (\vartheta) = \frac{1}{\mathfrak{d}(\mathfrak{r}_1)} \int_{\mathfrak{a}}^{\vartheta} \left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}_1-1} \nu^{\mathfrak{F}-1} \left(\frac{\nu^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}-1} d\nu. \tag{3.5}$$

Also, from Lemma 2.11, we have:

$$\left({}_{\mathfrak{F}}\mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{r}_1} \left(\frac{\nu^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}-1} \right) (\vartheta) = \frac{\mathfrak{d}(\mathfrak{r})}{\mathfrak{d}(\mathfrak{r}_1 + \mathfrak{r})} \left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}} \right)^{\mathfrak{r}_1+\mathfrak{r}-1}, \tag{3.6}$$

Uncorrected Proof

So,

$$\left|((Nu_n)(\vartheta)-(Nu)(\vartheta))\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\right| \leq \left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1+\mathfrak{r}-1} \\ B(\mathfrak{r},\mathfrak{r}_1)\|\varkappa(\cdot,u_n(\cdot),\Theta u_n(\cdot))-\varkappa(\cdot,u(\cdot),\Theta u(\cdot))\|_{\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}}, \tag{3.7}$$

which implies

$$\|Nu_n-Nu\|_{\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}} \leq B(\mathfrak{r},\mathfrak{r}_1)\left(\frac{T^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1}\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\|\varkappa(\cdot,u_n(\cdot),\Theta u_n(\cdot))-\varkappa(\cdot,u(\cdot),\Theta u(\cdot))\|_{\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}}. \tag{3.8}$$

Since $\varkappa$ is continuous, then Eq. (3.8) implies that $\|Nu_n-Nu\|_{\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}} \to 0$ as $n\to\infty$ It implies that N is continuous.
Step 2: N maps bounded sets into bounded sets in $\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$.
Actually, it suffices to demonstrate that for $\varsigma>0$, there exists $\varpi_\bullet>0$ such that if

$$u\in B_\varsigma=\left\{u\in\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]:\|u\|_{\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}}\leqslant\varsigma\right\}, \tag{3.9}$$

then $\|Nu\|_{\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}}\leqslant\varpi_\bullet$. Let $u\in B_\varsigma$. We have

$$\left|(Nu)(\vartheta)\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\right| \leq \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})}+\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}|\varkappa(\nu,u(\nu),\Theta u(\nu))|\,d\nu. \tag{3.10}$$

From $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we have:

$$\begin{aligned}|\varkappa(\nu,u(\nu),\Theta u(\nu)| &\leq p(\nu)\phi_1(|u(\nu)|)\phi_2(|\Theta u(\nu)|)\\ &\leq p^*\phi_1(|u(\nu)|)\phi_2(\sigma|u(\nu)|),\end{aligned} \tag{3.11}$$

then

$$\left|(Nu)(\vartheta)\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\right| \leq \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})}+\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\times\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}p^*\phi_1(|u(\nu)|)\phi_2(\sigma|u(\nu)|)d\nu. \tag{3.12}$$

From Definition 2.1, we have

$$\left[{}_{\mathfrak{F}}\mathfrak{J}_{\mathfrak{a}+}^{\mathfrak{r}_1}\left(\frac{\nu^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\right](\vartheta)=\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\left(\frac{\nu^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}d\nu. \tag{3.13}$$

Also from Lemma 2.11, we have

$$\left({}_{\mathfrak{F}}\mathfrak{J}_{\mathfrak{a}+}^{\mathfrak{r}_1}\left(\frac{\nu^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\right)(\vartheta)=\frac{\mathfrak{d}(\mathfrak{r})}{\mathfrak{d}(\mathfrak{r}_1+\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1+\mathfrak{r}-1} \tag{3.14}$$

Since $u\in B_\varsigma$ and $\nu,\vartheta\in\mathcal{J}$, we have $\left(\frac{\mathfrak{F}}{\vartheta^{\mathfrak{F}}-a^{\mathfrak{F}}}\right)^{1-\mathfrak{r}}<C$ and $|u(\nu)|\leq\varsigma C$. So

$$\left|(Nu)(\vartheta)\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\right| \leq \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})}+\frac{p^*}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\mathfrak{b}^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1}B(\mathfrak{r},\mathfrak{r}_1)\phi_1(\varsigma C)\phi_2(\sigma\varsigma C). \tag{3.15}$$

Put $\psi(\varsigma C)=\phi_1(\varsigma C)\phi_2(\sigma\varsigma C)$. Then, we have

$$\left|(Nu)(\vartheta)\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\right| \leq \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})}+\frac{p^*}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{T^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1}B(\mathfrak{r},\mathfrak{r}_1)\psi(\varsigma T). \tag{3.16}$$

Thus,

$$\left|(Nu)(\vartheta)\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}}\right| \leq \varpi_\bullet. \tag{3.17}$$

Uncorrected Proof

So, $\|Nu\|_{\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}} \leqslant \varpi_\bullet$, where

$$\varpi_\bullet = \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})} + \frac{p^*}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{T^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1} B(\mathfrak{r}, \mathfrak{r}_1)\psi(\varsigma C). \tag{3.18}$$

Step 3: N maps bounded sets into equicontinuous set of $\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T]$.
Let $\vartheta_1 \leqslant \vartheta_2$ for $\vartheta_1, \vartheta_2 \in \mathcal{J}$. For all $u \in B_\varsigma$ such that $B_\varsigma = \left\{u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T] : \|u\|_{\mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}} \leqslant \varsigma\right\}$,

$$\begin{aligned}
&\left|(Nu)(\vartheta_1)\left(\frac{\vartheta_1^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} - (Nu)(\vartheta_2)\left(\frac{\vartheta_2^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\right| \\
&\leq \left|\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\vartheta_1^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\int_{\mathfrak{a}}^{\vartheta_1}\left(\frac{{\vartheta_1}^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu\right. \\
&\left.-\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\vartheta_2^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\int_{\mathfrak{a}}^{\vartheta_2}\left(\frac{\vartheta_2^{\mathfrak{F}} - s^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu\right| \\
&\leq \left|\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta_1}\left(\left(\frac{{\vartheta_1}^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\left(\frac{\vartheta_1^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} - \left(\frac{\vartheta_2^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\left(\frac{\vartheta_2^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\right)\right. \\
&\times\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu \\
&\left.-\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\vartheta_1^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\int_{\vartheta_1}^{\vartheta_2}\left(\frac{\vartheta_2^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu\right| \\
&\leq \left|\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta_1}\left(\left(\frac{{\vartheta_1}^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\left(\frac{\vartheta_1^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} - \left(\frac{\vartheta_2^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\left(\frac{\vartheta_2^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\right)\right. \\
&\times\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu\Big| \\
&+\left|\frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\vartheta_2^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}\int_{\vartheta_1}^{\vartheta_2}\left(\frac{\vartheta_2^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu\right|
\end{aligned} \tag{3.19}$$

The right part of the above inequality tends to zero as $\vartheta_1 \to \vartheta_2$.
Based on the results of steps one through three and the Arzela-Ascoli theorem, we conclude that the operator N is completely continuous.
Step 4: We will prove that the set $\Upsilon = \{u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a}, T] : u = \omega(Nu), 0 \leqslant \omega \leqslant 1\}$ is a bounded.
Assume that $u \in \Upsilon; u = \omega(Nu)$ for some $0 \leqslant \omega \leqslant 1$. Then, for each $\vartheta \in \mathcal{J}$, we have

$$\begin{aligned}
|u(\vartheta)| &= \left|\omega\left(\frac{u_0}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} + \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu\right)\right| \\
&\leq \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})}\left(\frac{T^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} + \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}|\varkappa(\nu, u(\nu), \Theta u(\nu))|d\nu \\
&\leq \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})}\left(\frac{T^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1} + \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}p^*\phi_1(|u(\nu)|)\phi_2(\sigma|u(\nu)|)d\nu \\
&\leq k + \int_{\mathfrak{a}}^{\vartheta} f(\nu)\psi(|u(\nu)|)d\nu.
\end{aligned} \tag{3.20}$$

Uncorrected Proof

Where

$$f(\nu)=\frac{p^*}{\mathfrak{d}(\mathfrak{r}_1)}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}, \tag{3.21}$$

$$k=\frac{|u_0|}{\mathfrak{d}(\mathfrak{r})}\left(\frac{T^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}-1}, \tag{3.22}$$

and

$$\psi(|u(\vartheta)|)=\phi_1(|u(\vartheta)|)\phi_2(\sigma|u(\vartheta)|). \tag{3.23}$$

First of all, we calculate

$$\begin{aligned}\int_{\mathfrak{a}}^{T} f(\nu)d\nu &= \frac{p^*}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{T}\nu^{\mathfrak{F}-1}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}d\nu\\ &= \frac{p^*\mathfrak{F}}{\mathfrak{d}(\mathfrak{r}_1)(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{T}\frac{d}{d\nu}\left(-\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1}d\nu\\ &= \frac{p^*\mathfrak{F}}{\mathfrak{d}(\mathfrak{r}_1+1)}\left(\frac{T^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1}<\infty.\end{aligned} \tag{3.24}$$

Let's examine the function $\mu(\vartheta)$, which is defined by $\mu(\vartheta)=\sup_{\vartheta\in\mathcal{J}}|u(\vartheta)|$.
If $\vartheta^*\in[\mathfrak{a},\vartheta]$ such that $\mu(\vartheta)=|u(\vartheta^*)|$, and $\vartheta^*\in\mathcal{J}$, from the previous inequality, we can write:

$$\mu(\vartheta)\le k+\int_{\mathfrak{a}}^{\vartheta} f(\nu)\psi(|\mu(\nu)|)d\nu. \tag{3.25}$$

Let's consider $\varphi(\vartheta)$ as the right-hand side of the inequality above. In other words, we define:

$$\varphi(\vartheta)=\begin{cases}k, & \text{if } \ \vartheta=\mathfrak{a},\\ k+\int_{\mathfrak{a}}^{\vartheta} f(\nu)\psi(|u(\vartheta)|)d\nu, & \vartheta\in\mathcal{J}.\end{cases} \tag{3.26}$$

Additionally, by differentiating the two halves of the given equality, we get, for $\vartheta\in\mathcal{J}$,

$$\varphi'(\vartheta)=f(\vartheta)\psi(\mu(\vartheta)). \tag{3.27}$$

By utilizing the non-decreasing nature of the function ψ, we establish that

$$\mu(\vartheta)\le\varphi(\vartheta)\quad \text{implie that } \psi(\mu(\vartheta))\le\psi(\varphi(\vartheta)). \tag{3.28}$$

Therefore, we conclude:

$$\frac{\varphi'(\vartheta)}{\psi(\varphi(\vartheta))}\le f(\vartheta). \tag{3.29}$$

Integrating from a to ϑ, if $\vartheta\in\mathcal{J}$, we obtain

$$\int_{\mathfrak{a}}^{\vartheta}\frac{\varphi'(\nu)}{\psi(\varphi(\nu))}d\nu\le\int_{\mathfrak{a}}^{\vartheta} f(\nu)d\nu. \tag{3.30}$$

By changing the variable to $\varphi(\nu)=\zeta$, we get

$$\int_{k}^{\varphi(\vartheta)}\frac{d\zeta}{\psi(\zeta)}\le\int_{\mathfrak{a}}^{\vartheta} f(\nu)d\nu\le\int_{\mathfrak{a}}^{\infty}\frac{d\zeta}{\psi(\zeta)}. \tag{3.31}$$

Hence, for all $\vartheta\in\mathcal{J}$, there exists a constant η such that

$$|u(\vartheta)|\le\mu(\vartheta)\le\varphi(\vartheta)\le\eta. \tag{3.32}$$

Uncorrected Proof

Thus,

$$||u(\vartheta)||_{C_{1-\mathfrak{r},\mathfrak{F}}} \leq \eta \left(\frac{T^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{1-\mathfrak{r}} = \eta_{\bullet}. \tag{3.33}$$

Consequently, the set Υ is bounded. Hence, by Shaefer's point fixed Theorem 2.6, we conclude that N has a fixed point, which is a solution to the problem of Eqs. (1.7) and (1.8). □

4. Stability Analysis

For the problem of Eqs. (1.7) and (1.8), let's outline the definitions of Ulam-Hyers-Rassias stability (U-H-R) requirements. Suppose $\zeta \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ and $\epsilon > 0$ is a positive real number. We examine the ensuing disparities

$$|{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}+} z(\vartheta) - \varkappa(\vartheta, z(\vartheta), \Theta z(\vartheta))| \leq \epsilon, \quad \vartheta \in \mathcal{J}. \tag{4.1}$$

$$|{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}+} z(\vartheta) - \varkappa(\vartheta, z(\vartheta), \Theta z(\vartheta))| \leq \epsilon\zeta(\vartheta), \quad \vartheta \in \mathcal{J}. \tag{4.2}$$

$$|{}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}+} z(\vartheta) - \varkappa(\vartheta, z(\vartheta), \Theta z(\vartheta))| \leq \zeta(\vartheta), \quad \vartheta \in \mathcal{J}. \tag{4.3}$$

Definition 4.1. [4] The problem of Eqs. (1.7) and (1.8) is said to be U-H stable if there exists a constant $C \in \mathbb{R}^*_+$ such that for every $\epsilon > 0$ and for every solution $z \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of the inequality (4.1), there exists a solution $u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of problem (1.7)-(1.8) such that

$$|z(\vartheta) - u(\vartheta)| \leq C\epsilon, \qquad \vartheta \in \mathcal{J}. \tag{4.4}$$

Definition 4.2. [4] The Eqs. (1.7) and (1.8) is said to be generalized U-H stable with respect to $\mathrm{f}_{\varkappa} \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$, where $\mathrm{f}_{\varkappa}(\mathfrak{a}) = 0$, if for every solution $z \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of the inequality (4.1), there exists a solution $u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of (1.7)-(1.8) such that

$$|z(\vartheta) - u(\vartheta)| \leq \mathrm{f}_{\varkappa}(\epsilon) \qquad \vartheta \in \mathcal{J}. \tag{4.5}$$

Definition 4.3. [4] The Eqs. (1.7)-(1.8)is said to be generalized U-H-R stable with respect to $\zeta \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$, if there exists a real number $C_\zeta > 0$ such that for every solution $z \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of the inequaliy (4.2), there exists a corresponding solution $u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of (1.7)-(1.8)such that

$$|z(\vartheta) - u(\vartheta)| \leq C_\zeta \epsilon \zeta(\vartheta), \quad \forall \vartheta \in \mathcal{J}. \tag{4.6}$$

Definition 4.4. [4] The Eqs. (1.7)-(1.8)is said to be generalized U-H-R stable with respect to $\zeta \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$, if there exists a real number $C_\zeta > 0$ such that for every solution $z \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of the inequaliy (4.3), there exists a corresponding solution $u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ of (1.7)-(1.8)such that

$$|z(\vartheta) - u(\vartheta)| \leq C_\zeta \zeta(\vartheta), \quad \forall \vartheta \in \mathcal{J} \tag{4.7}$$

Remark 4.5. A function $z \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ is a solution of the iniquality (4.2) if and only if there exists a function $g \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[\mathfrak{a},T]$ such that

a) $|\mathrm{g}(\vartheta)| \leq \epsilon\zeta(\vartheta), \quad \vartheta \in \mathcal{J},$

b) ${}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}+} z(\vartheta) = \varkappa(\vartheta, z(\vartheta), \Theta z(\vartheta)) + \mathrm{g}(\vartheta), \quad \vartheta \in \mathcal{J}.$

Remark 4.6. Let $\mathfrak{r}_1 \in (0,1)$. If z is a solution of the inequality (4.1) , then z is also a solution of the following integral inequality:

$$|z(\vartheta) - \frac{z_0}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1} - \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}} - \nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, z(\nu), \Theta z(\nu))d\nu| \leq \epsilon\frac{T^{\mathfrak{F}\mathfrak{r}_1}}{\mathfrak{F}^{\mathfrak{r}_1}\mathfrak{d}(\mathfrak{r}_1+1)}. \tag{4.8}$$

In this section, we provide the main result, the Ulam-Hyer stability theorem.

$(\mathbf{H_3})$ There exists a constant $\mathfrak{R} > 0$ such that $\dfrac{\mathfrak{R}}{p^*\psi(\mathfrak{R})l} > 1$, where $l = \dfrac{1}{\mathfrak{d}(\mathfrak{r}_1+1)}\left(\frac{\mathfrak{b}^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1}$.

Now we put $k' = k + k_z$ where k is defined by (3.2) and $k_z = \dfrac{|z_0|}{\mathfrak{r}(\mathfrak{r})}\left(\frac{\mathfrak{b}^{\mathfrak{F}} - \mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}$.

Uncorrected Proof

Theorem 4.7. *Let's assume that hypotheses* $(\mathbf{H_1})-(\mathbf{H_3})$ *are satisfied, and there exists a function* $\mathrm{g}(\vartheta)$ *that fulfills the Lemma 4.5, where* $\vartheta \in \mathcal{J}$ *and* $2\mathfrak{R}+k' \leq \mathrm{g}(\vartheta)$*. Hence, the Eqs. (1.7)-(1.8)is U–H–R stable, and consequently, it is also generalized U-H-R stable.*

Proof. Let z be a solution of inequality (4.2), and by Theorem 3.1, there exists a solution u to the problem:

$$ {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{\mathfrak{a}^+}u(\vartheta) = \varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)), \tag{4.9} $$

$$ {}_{\mathfrak{F}}\mathfrak{I}^{1-\mathfrak{r}}_{\mathfrak{a}^+}u(\mathfrak{a}) = u_0. \tag{4.10} $$

Then, we have

$$ u(\vartheta) = \frac{u_0}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1} + \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\varkappa(\nu, u(\nu), \Theta u(\nu))d\nu, \tag{4.11} $$

$$ \begin{aligned} |z(\vartheta)-u(\vartheta)| &\leq \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\left|\mathrm{g}(\nu)\right|d\nu + \frac{|z_0|}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1} \\ &+ \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\left|\varkappa(\nu, z(\nu), \Theta z(\nu))\right|d\nu \\ &+ \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}\left|\varkappa(\nu, u(\nu), \Theta u(\nu))\right|d\nu + \frac{|u_0|}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\vartheta^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1} \\ &\leq \frac{\epsilon\zeta(\vartheta)}{\mathfrak{d}(\mathfrak{r}_1+1)}\left(\frac{\mathfrak{b}^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1} + \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}p(\nu)\phi_1(|z|)\phi_2(\sigma\left|z(\nu)\right|)d\nu \\ &+ \frac{1}{\mathfrak{d}(\mathfrak{r}_1)}\int_{\mathfrak{a}}^{\vartheta}\left(\frac{\vartheta^{\mathfrak{F}}-\nu^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}\nu^{\mathfrak{F}-1}p(\nu)\phi_1(|u(\nu)|)\phi_2(\sigma\left|u(\nu)\right|)d\nu \\ &\leq \frac{\epsilon\zeta(\vartheta)}{\mathfrak{d}(\mathfrak{r}_1+1)}\left(\frac{\mathfrak{b}^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1} + \frac{p^*\phi_1(|z|)\phi_2(\sigma\left|z\right|)}{\mathfrak{d}(\mathfrak{r}_1+1)}\left(\frac{\mathfrak{b}^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1} \\ &+ \frac{p^*\phi_1(|u|)\phi_2(\sigma\left|u\right|)}{\mathfrak{d}(\mathfrak{r}_1+1)}\left(\frac{\mathfrak{b}^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1} + \frac{|u_0|+|z_0|}{\mathfrak{d}(\mathfrak{r})}\left(\frac{\mathfrak{b}^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1-1}. \end{aligned} \tag{4.12} $$

Let $l = \frac{1}{\mathfrak{d}(\mathfrak{r}_1+1)}\left(\frac{\mathfrak{b}^{\mathfrak{F}}-\mathfrak{a}^{\mathfrak{F}}}{\mathfrak{F}}\right)^{\mathfrak{r}_1}$. Now, applying Lemma 4.5 and $(\mathbf{H_3})$, we obtain:

$$ \begin{aligned} |z(\vartheta)-u(\vartheta)| &\leq l\epsilon\zeta(\vartheta) + lp^*\phi_1(|z|)\phi_2(\sigma\left|z\right|) + lp^*\phi_1(|u|)\phi_2(\sigma\left|u\right|) + k' \\ &\leq l\epsilon\zeta(\vartheta) + \mathrm{g}(\vartheta) \\ &\leq C_\zeta\epsilon\zeta(\vartheta); \quad C_\zeta = 1+l. \end{aligned} \tag{4.13} $$

Consequently, U-H-R stability is established for Eqs. (1.7)-(1.8). Similarly, it can be demonstrated that the problem represented by Eqs. (1.7)-(1.8)being U-H-R stable implies general stability. □

5. Application

Example 5.1. Consider the IDFEs by H-K DF

$$ {}_{\mathfrak{F}}\mathfrak{D}^{\mathfrak{r}_1,\mathfrak{r}_2}_{0^+}u(\vartheta) = \frac{1}{19}(t-1)^2\left(\frac{u^2}{1+|u|}+\frac{|u|+2}{3+|u|}\right)\int_0^{\vartheta}\left(\frac{4}{3}\exp^{\frac{-1}{2}u(\nu)}-1\right)d\nu, \quad \vartheta \in \mathcal{J} = (0,1], \tag{5.1} $$

$$ {}_{\mathfrak{F}}\mathcal{J}^{1-\mathfrak{r}}_{0^+}u(0) = u_0, \qquad \mathfrak{r} = \mathfrak{r}_1+\mathfrak{r}_2-\mathfrak{r}_1\mathfrak{r}_2 < 1, \tag{5.2} $$

where $u_0 = u(0) = 0$, $\mathfrak{r}_1 = \frac{1}{2}$, $\mathfrak{r}_2 = \frac{1}{3}$, and $\mathfrak{r} = \frac{2}{3}$

$$ \varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)) = \frac{1}{19}(\vartheta-1)^2\left(\frac{u^2}{1+|u|}+\frac{|u|+2}{3+|u|}\right)\int_0^{\vartheta}\left(\frac{4}{3}\exp^{\frac{-1}{2}u(\nu)}-1\right)d\nu \tag{5.3} $$

Uncorrected Proof

Hence,

$$\vartheta^{\frac{1}{3}}\varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)) = \vartheta^{\frac{1}{3}}\frac{1}{19}(\vartheta-1)^2\left(\frac{u^2}{1+|u|}+\frac{|u|+2}{3+|u|}\right)\int_0^{\vartheta}\left(\frac{4}{3}\exp^{\frac{-1}{2}u(\nu)}-1\right)d\nu \in \mathfrak{C}[0,1] \tag{5.4}$$

So, $\varkappa(\cdot, u(\cdot), \Theta u(\cdot)) \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[0,1]$

- $(\mathbf{H_1})$ is satisfied, indeed

$$|\Theta u(\vartheta)| = \left|\int_0^{\vartheta}\left(\frac{4}{3}\exp^{\frac{-1}{2}u(\nu)}-1\right)d\nu\right| \leq \frac{2}{3}|u(\vartheta)|. \tag{5.5}$$

- $(\mathbf{H_2})$ is satisfied, indeed

$$|\varkappa(u, u(\vartheta), \theta u(\vartheta))| \leq \frac{1}{19}(\vartheta-1)^2(|u|+1)\frac{2}{3}|u|, \tag{5.6}$$

where $p(\vartheta) = \frac{1}{19}(\vartheta-1)^2$, $\phi_1(u) = u+1$ and $\phi_2(\sigma u) = \frac{2}{3}u$. For $\psi(u) = \phi_1(u)\phi_2(\sigma u)$, we have

$$\int_0^{\infty}\frac{du}{\psi(u)} = \infty. \tag{5.7}$$

- $(\mathbf{H_3})$, We can show that $\frac{\mathfrak{R}}{\frac{1}{19}\psi(\mathfrak{R})l} > 1$ where $l \simeq 1,15$ implies that $\mathfrak{R} > 23,40$. Further, by choosing $\mathrm{g}(\vartheta) = 18e^{(\vartheta^2+1)^2}$ and $\mathfrak{R} = 24$, then we have $2\mathfrak{R} < \mathrm{g}(\vartheta)$, for all $\vartheta \in (0,1]$.
Using $\zeta(\vartheta) = e^{(\vartheta^2+1)^2}$ as our initial value, we can obtain $C_\zeta \simeq 2,15$. So, by Theorem 4.7, the problem of Eqs. (5.1)-(5.2) is U-H-R stable and also generalized U-H-R stable over $[0,1]$.

Example 5.2. Consider the IDE by H-K DF

$$_{\mathfrak{F}}\mathfrak{D}_{0^+}^{\mathfrak{r}_1,\mathfrak{r}_2}u(\vartheta) = \frac{u(\vartheta)\left(\sin(\vartheta u(\vartheta))\right)}{(\vartheta+2)\,e^{\vartheta+|u(\vartheta)|}}\int_0^{\vartheta}\left(e^{\frac{-1}{2}|u(\nu)|}-1\right)d\nu, \quad \vartheta \in \mathcal{J} = (0,1], \tag{5.8}$$

$$_{\mathfrak{F}}\mathfrak{I}_{0^+}^{1-\mathfrak{r}}u(0) = u_0, \qquad \mathfrak{r} = \mathfrak{r}_1+\mathfrak{r}_2-\mathfrak{r}_1\mathfrak{r}_2 < 1, \tag{5.9}$$

where $u_0 = u(0) = 0$, $\mathfrak{r}_1 = \frac{1}{2}$, $\mathfrak{r}_2 = \frac{1}{3}$, and $\mathfrak{r} = \frac{2}{3}$

$$\vartheta^{\frac{1}{3}}\varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta)) = \vartheta^{\frac{1}{3}}\frac{u(\vartheta)\left(\sin(\vartheta u(\vartheta))\right)}{(\vartheta+2)\,e^{\vartheta+|u(\vartheta)|}}\int_0^{\vartheta}\left(e^{\frac{-1}{2}|u(\nu)|}-1\right)d\nu \tag{5.10}$$

So,

$$\varkappa(\cdot, u(\cdot), \Theta u(\cdot)) \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[0,1] \tag{5.11}$$

- $(\mathbf{H_1})$ is satisfied, indeed $|\Theta u(\vartheta)| \leq \frac{1}{2}|u(\vartheta)|$.
- $(\mathbf{H_2})$ is satisfied because, we have

$$|\varkappa(\vartheta, u(\vartheta), \Theta u(\vartheta))| \leq p(\vartheta)\phi_1(|u(\vartheta)|)\phi_2(|\Theta u(\vartheta)|), \tag{5.12}$$

such that

$$p(\vartheta) = \frac{1}{(\vartheta+2)+e^{\vartheta}}, \quad \phi_1(|u(\vartheta)|) = |u(\vartheta)|^2, \quad \phi_2(|\Theta u(\vartheta)|) = \frac{1}{2}|u(\vartheta)|. \tag{5.13}$$

For $\psi(u) = \frac{1}{2}u^3$, we have $\int_0^{\infty}\frac{du}{\psi(u)} = \infty$. Thus, the satisfaction of each requirement in Theorem 3.1 indicates that the problem of Eqs. (5.8)-(5.9) has a least one solution $u \in \mathfrak{C}_{1-\mathfrak{r},\mathfrak{F}}[0,1]$.

Uncorrected Proof

6. Conclusion

The results of this work shed important light on the existence and Ulam-Hyers-Rassias stability of solutions to integrodifferential equations involving the fractional derivative of Hilfer-Katugampola. We have established the existence and stability results for the current problem using Schaefer's fixed point theorem, which is further supported by illustrative examples. These discoveries pave the door for more investigation into the stability features of fractional differential equations and their applications in mathematical analysis.

Conflicts of interest

The authors declare that they have no conflict of interest.

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Uncorrected Proof

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Uncorrected Proof