Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-14
DOI:10.22034/cmde.2025.67447.3217

Analytical Solutions for the Generalized (2+1)-D Shallow Water Wave Equation via a Novel Generalized Abel Equation with Variable Coefficients

Mir Sajjad Hashemi[1,2], **Mustafa Bayram**[3], **Farzaneh Alizadeh**[4,*], **Samad Kheybari**[5,*], **and Kamyar Hosseini**[2,4]

[1]Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab, Iran.

[2]Research Center of Applied Mathematics, Khazar University, Baku, Azerbaijan.

[3]Computer Engineering, Biruni University, Istanbul, Turkey.

[4]Department of Mathematics, Near East University TRNC, Mersin10, Nicosia 99138, Turkey.

[5]Faculty of Art and Science, University of Kyrenia, TRNC, Mersin 10, Turkey.

Abstract

This study utilizes the generalization of the second-degree Abel equation (SDAE) method with variable coefficients, initially introduced in [5], to analyze the generalized (2+1)-D shallow water wave (SWW) equation. Unlike conventional approaches that predominantly rely on constant-coefficient ordinary differential equations (ODEs) or auxiliary ODEs, the proposed method incorporates ODEs with variable coefficients within a sub-equation framework, thereby enhancing its adaptability to nonlinear wave equations. The governing nonlinear partial differential equation (PDE) is first reduced to an ODE, which is then analyzed using this method. Subsequently, various singular and periodic wave solutions are derived, and their dynamic behavior is thoroughly examined. The efficacy of this approach is demonstrated through its successful application to the SWW equation, resulting in exact analytical solutions. This method provides a systematic and efficient framework for solving complex nonlinear PDEs, establishing it as a valuable tool in the study of wave propagation in fluid dynamics. Furthermore, its versatility suggests broad applicability to a range of mathematical physics models, thereby expanding the scope of analytical solution techniques.

Keywords. Generalized (2+1)-D shallow water wave equation, Exact solution, Generalized Abel equation.

2010 Mathematics Subject Classification. 35Q35, 35C07, 76B15, 35A22.

1. Introduction

Nonlinear PDEs serve as fundamental tools for modeling a vast array of complex phenomena across disciplines such as physics, engineering, biology, and finance. Unlike their linear versions, nonlinear PDEs capture intricate interdependencies among variables, offering a refined depiction of dynamic processes. Their importance arises from their capacity to characterize fundamentally nonlinear behaviors, such as turbulence, wave movement, and pattern development phenomena that are common in both natural and industrial environments. The search for exact solutions to such equations is crucial for gaining a deeper insight into complex systems, enabling accurate predictions and efficient control strategies. However, due to the inherent mathematical challenges of solving nonlinear PDEs, various methodologies have been developed, ranging from analytical and numerical approaches to advanced computational techniques [6, 7, 13–15, 18].

The Generalized (2+1)-D SWW equation serves as a fundamental model in fluid dynamics, specifically addressing the intricate behavior of shallow water waves in a two-dimensional space. Motivated by its applicability to real-world scenarios such as coastal regions, lakes, and ocean dynamics, the equation accounts for the interplay of nonlinearity, dispersion, and wave-structure interactions. The consideration of exact solutions for the Generalized (2+1)-D SWW equation is driven by the imperative to unravel the underlying physical phenomena and gain insights into the

Received: 24 May 2025; Accepted: 05 January 2026.
∗ Corresponding author. Email: farzan.alizadeh@neu.edu.tr, samad.kheybari@kyrenia.edu.tr.

complex dynamics of SWW. The pursuit of exact solutions is crucial for understanding wave propagation patterns, predicting potential hazards, and optimizing coastal engineering designs. Different methods, ranging from analytical approaches to numerical techniques, are employed to tackle the inherent challenges posed by the nonlinear nature of the equation. The quest for exact solutions not only enhances our comprehension of fluid dynamics but also facilitates the development of effective strategies for mitigating the impact of shallow water waves on coastal environments, making it a critical pursuit with significant implications for both theoretical research and practical applications [1, 2, 8, 10, 11, 17, 19].

In this current investigation, we examine the generalized $(2+1)$-D SWW equation, as discussed in [4, 12, 16]:

$$\alpha_1\left[3\left(u_x u_t\right)_x+u_{xxxt}\right]+\alpha_2\left[3\left(u_x u_y\right)_x+u_{xxxy}\right]+\alpha_3 u_{yt}+\alpha_4 u_{xx} \quad +\alpha_5 u_{xy}+\alpha_6 u_{xt}+\alpha_7 u_{yy}=0, \tag{1.1}$$

in which $\alpha_j,\ j=1,\ldots,7$ are free parameters.

Utilizing the wave transformation given by

$$u(x,y,t)=\Omega(\xi),\ \xi=k(x+y-ct),$$

transforms (1.1) into the following ODE:

$$k^4\left(-c\alpha_1+\alpha_2\right)\Omega''''+6k^3\left(-c\alpha_1+\alpha_2\right)\Omega'\Omega''+k^2\left(-c\alpha_3-c\alpha_6+\alpha_4+\alpha_5+\alpha_7\right)\Omega''=0. \tag{1.2}$$

2. Methodology

Let us assume the following nth-order differential equation:

$$\Xi\left(\xi,\mathcal{W},\mathcal{W}',\ldots,\mathcal{W}^{(n)}\right)=0.$$

This work introduces an approach in which the solutions of the following variable coefficient ODE also satisfy the equation shown in (1.2):

$$\mathcal{W}'=\vartheta_2(\xi)\mathcal{W}^2+\vartheta_1(\xi)\mathcal{W}+\vartheta_0(\xi), \tag{2.1}$$

where $\vartheta_i(\xi)\in\mathcal{C}^n, i\in\{0,1,2\}$.

This study builds upon a range of established methods proposed by various researchers, including:

• The extensively studied tanh method [20], which utilizes the solution $\mathcal{W}(\xi)=\tanh(\xi)$. This function satisfies the differential equation $\mathcal{W}'=1-\mathcal{W}^2$, making it a suitable candidate for the structure of Equation (2.1). In this case, the corresponding parameters are given by $\{\vartheta_0(\xi),\vartheta_1(\xi),\vartheta_2(\xi)\}=\{1,0,-1\}$.

• In the modified extended tanh function method [3] and the modified extended direct algebraic method [9], the auxiliary equation is formulated as $\mathcal{W}'=\lambda+\mathcal{W}^2$. That is, $\{\vartheta_0(\xi),\vartheta_1(\xi),\vartheta_2(\xi)\}=\{\lambda,0,1\}$.

• In the well-known simplest equation algorithm, different sets of fundamental equations are examined to determine solutions that concurrently fulfill the constraints of the specified equation.

It is important to highlight that in the majority of analytical approaches employing the auxiliary equation for obtaining exact solutions, the auxiliary equation is formulated as an ODE with coefficients that remain constant.

Generalized SDAEs. Initially, we describe a method to derive Equations (2.1) from a provided Equation (1.2). Let $\mathcal{W}=\mathcal{W}(\xi)$ represent any solution to (2.1). Upon differentiating with respect to the variable ξ, we obtain:

$$\mathcal{W}''(\xi)=\vartheta_2'\mathcal{W}^2+\vartheta_1'\mathcal{W}+\vartheta_0'+\mathcal{W}'\left[2\vartheta_2\mathcal{W}+\vartheta_1\right]. \tag{2.2}$$

Therefore from (2.1) we have

$$\mathcal{W}''(\xi)=2\vartheta_2^2\mathcal{W}^3+\left(\vartheta_2'+3\vartheta_1\vartheta_2\right)\mathcal{W}^2+\left(\vartheta_1'+\vartheta_1^2+2\vartheta_0\vartheta_2\right)\mathcal{W}+\left(\vartheta_0'+\vartheta_0\vartheta_1\right). \tag{2.3}$$

The core concept behind this novel approach is explained as follows: First, polynomial expressions involving the integer-order derivatives $\mathcal{W}',\mathcal{W}'',\ldots,\mathcal{W}^{(n)}$ are substituted into the nth-order differential equation $\Xi\left(\xi,\mathcal{W},\mathcal{W}',\ldots,\mathcal{W}^{(n)}\right)$. Such a substitution varies Ξ to a polynomial Ω in $\mathcal{W}$. Next, by equating the coefficients of $\Omega(\mathcal{W})$ to zero, a system of differential-algebraic equations is generated for $\vartheta_i(\xi),\ 0\le i\le 2$. If a solution of the form $\vartheta_0(\xi),\vartheta_1(\xi),\vartheta_2(\xi)$ exists for this system, then any solution to (2.1) will also satisfy (1.2).

3. Main framework for results

This part shows how can apply the proposed method to deal with the nonlinear ODE presented in Equation (1.2). For this aim employing (2.1) into (1.2) results in a fifth-degree polynomial with respect to $\mathcal{W}$, expressed as:

$$\sum_{i=0}^{5} S_i \mathcal{W}^i = 0,$$

where

$$\begin{aligned}
S_0 = -24k^2 \Bigg[& \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_0'''}{24} + \frac{(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_1''}{8} + \frac{(\alpha_1 c-\alpha_2)k^2\vartheta_1\vartheta_0''}{24} \\
& + \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_0^2\vartheta_2'}{4} + \left(\frac{k^2(\alpha_1 c-\alpha_2)\vartheta_0'}{8} + \frac{5(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_1}{24}\right)\vartheta_1' \\
& + \left(\frac{(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2}{4} + \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_1^2}{24} + \frac{k(\alpha_1 c-\alpha_2)\vartheta_0}{4} + \left(\frac{\alpha_3}{24}+\frac{\alpha_6}{24}\right)c - \frac{\alpha_4}{24} - \frac{\alpha_5}{24} - \frac{\alpha_7}{24}\right)\vartheta_0' \\
& + \frac{\left(\frac{2(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2}{3} + \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_1^2}{12} + \frac{k(\alpha_1 c-\alpha_2)\vartheta_0}{2} + \left(\frac{\alpha_6}{12}+\frac{\alpha_3}{12}\right)c - \frac{\alpha_7}{12} - \frac{\alpha_4}{12} - \frac{\alpha_5}{12}\right)\vartheta_1\vartheta_0}{2}\Bigg],
\end{aligned}$$

$$\begin{aligned}
S_1 = -24k^2 \Bigg[& \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_1'''}{24} + \frac{(\alpha_1 c-\alpha_2)k^2\vartheta_1\vartheta_1''}{6} + \frac{(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2''}{4} + \frac{(\alpha_1 c-\alpha_2)k^2\vartheta_2\vartheta_0''}{12} \\
& + \left(\frac{k^2(\alpha_1 c-\alpha_2)\vartheta_0'}{4} + \frac{11(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_1}{12}\right)\vartheta_2' + \frac{k^2(\alpha_1 c-\alpha_2)(\vartheta_1')^2}{8} \\
& + \left(\frac{2(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2}{3} + \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_1^2}{4} + \frac{k(\alpha_1 c-\alpha_2)\vartheta_0}{4} + \frac{(\alpha_6+\alpha_3)c}{24} - \frac{\alpha_7}{24} - \frac{\alpha_4}{24} - \frac{\alpha_5}{24}\right)\vartheta_1' \\
& + \left(\frac{5(\alpha_1 c-\alpha_2)k^2\vartheta_1\vartheta_2}{12} + \frac{(\alpha_1 c-\alpha_2)k\vartheta_1}{4}\right)\vartheta_0' \\
& + \frac{\left(\frac{2(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2}{3} + \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_1^2}{12} + \frac{k(\alpha_1 c-\alpha_2)\vartheta_0}{2} + \left(\frac{\alpha_6}{12}+\frac{\alpha_3}{12}\right)c - \frac{\alpha_7}{12} - \frac{\alpha_4}{12} - \frac{\alpha_5}{12}\right)\vartheta_1^2}{2} \\
& + \Bigg(\left(\frac{2(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2}{3} + \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_1^2}{12} + \frac{k(\alpha_1 c-\alpha_2)\vartheta_0}{2} + \left(\frac{\alpha_6}{12}+\frac{\alpha_3}{12}\right)c - \frac{\alpha_7}{12} - \frac{\alpha_4}{12} - \frac{\alpha_5}{12}\right)\vartheta_2 \\
& + \frac{\left((\alpha_1 c-\alpha_2)k^2\vartheta_1\vartheta_2 + \frac{(\alpha_1 c-\alpha_2)k\vartheta_1}{2}\right)\vartheta_1}{2}\Bigg)\vartheta_0\Bigg],
\end{aligned}$$

$$\begin{aligned}
S_2 = -24k^2 \Bigg[& \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_2'''}{24} + \frac{5(\alpha_1 c-\alpha_2)k^2\vartheta_2\vartheta_1''}{24} + \frac{7(\alpha_1 c-\alpha_2)k^2\vartheta_1\vartheta_2''}{24} \\
& + \left(\frac{3k^2(\alpha_1 c-\alpha_2)\vartheta_1'}{8} + \frac{19(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2}{12} + \frac{17k^2(\alpha_1 c-\alpha_2)\vartheta_1^2}{24} + \frac{k(\alpha_1 c-\alpha_2)\vartheta_0}{4} + \frac{(\alpha_6+\alpha_3)c}{24}\right. \\
& \left. - \frac{\alpha_7}{24} - \frac{\alpha_4}{24} - \frac{\alpha_5}{24}\right)\vartheta_2' + \left(\frac{25(\alpha_1 c-\alpha_2)k^2\vartheta_1\vartheta_2}{24} + \frac{(\alpha_1 c-\alpha_2)k\vartheta_1}{4}\right)\vartheta_1' \\
& + \left(\frac{5k^2(\alpha_1 c-\alpha_2)\vartheta_2^2}{12} + \frac{(\alpha_1 c-\alpha_2)k\vartheta_2}{4}\right)\vartheta_0' \\
& + \frac{\left(\frac{2(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2}{3} + \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_1^2}{12} + \frac{k(\alpha_1 c-\alpha_2)\vartheta_0}{2} + \left(\frac{\alpha_6}{12}+\frac{\alpha_3}{12}\right)c - \frac{\alpha_7}{12} - \frac{\alpha_4}{12} - \frac{\alpha_5}{12}\right)\vartheta_1\vartheta_2}{2} \\
& + \Bigg(\left(\frac{2(\alpha_1 c-\alpha_2)k^2\vartheta_0\vartheta_2}{3} + \frac{k^2(\alpha_1 c-\alpha_2)\vartheta_1^2}{12} + \frac{k(\alpha_1 c-\alpha_2)\vartheta_0}{2} + \left(\frac{\alpha_6}{12}+\frac{\alpha_3}{12}\right)c - \frac{\alpha_7}{12} - \frac{\alpha_4}{12} - \frac{\alpha_5}{12}\right)\vartheta_2
\end{aligned}$$

$$+\frac{\left(\left(\alpha_1 c-\alpha_2\right) k^2\vartheta_1\vartheta_2+\frac{(\alpha_1 c-\alpha_2)k\vartheta_1}{2}\right)\vartheta_1}{2}\Bigg)\vartheta_1+\Bigg(\Bigg(\left(\left(\alpha_1 c-\alpha_2\right) k^2\vartheta_1\vartheta_2+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_1}{2}\right)\vartheta_2$$

$$+\frac{\left(k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_2^2+\frac{(\alpha_1 c-\alpha_2)k\vartheta_2}{2}\right)\vartheta_1}{2}\Bigg)\vartheta_0\Bigg],$$

$$S_3=-24k^2\Bigg[\frac{\left(\alpha_1 c-\alpha_2\right) k^2\vartheta_2\vartheta_2''}{3}+\frac{k^2\left(\alpha_1 c-\alpha_2\right)\left(\vartheta_2'\right)^2}{4}+\left(\frac{13\left(\alpha_1 c-\alpha_2\right) k^2\vartheta_1\vartheta_2}{6}+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_1}{4}\right)\vartheta_2'$$

$$+\left(\frac{5k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_2^2}{6}+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_2}{4}\right)\vartheta_1'+\Bigg(\Big(\Big(\frac{2\left(\alpha_1 c-\alpha_2\right) k^2\vartheta_0\vartheta_2}{3}+\frac{k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_1^2}{12}$$

$$+\frac{k\left(\alpha_1 c-\alpha_2\right)\vartheta_0}{2}+\left(\frac{\alpha_6}{12}+\frac{\alpha_3}{12}\right)c-\frac{\alpha_7}{12}-\frac{\alpha_4}{12}-\frac{\alpha_5}{12}\Big)\vartheta_2+\frac{\left(\left(\alpha_1 c-\alpha_2\right) k^2\vartheta_1\vartheta_2+\frac{(\alpha_1 c-\alpha_2)\vartheta_1 k}{2}\right)\vartheta_1}{2}\Bigg)\vartheta_2$$

$$+\Bigg(\left(\left(\alpha_1 c-\alpha_2\right) k^2\vartheta_1\vartheta_2+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_1}{2}\right)\vartheta_2+\frac{\left(k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_2^2+\frac{(\alpha_1 c-\alpha_2)k\vartheta_2}{2}\right)\vartheta_1}{2}\Bigg)\vartheta_1$$

$$+\left(k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_2^2+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_2}{2}\right)\vartheta_2\vartheta_0\Bigg],$$

$$S_4=-24k^2\Bigg(\left(\frac{3k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_2^2}{2}+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_2}{4}\right)\vartheta_2'$$

$$+\Bigg(\left(\left(\alpha_1 c-\alpha_2\right) k^2\vartheta_1\vartheta_2+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_1}{2}\right)\vartheta_2+\frac{\left(k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_2^2+\frac{(\alpha_1 c-\alpha_2)k\vartheta_2}{2}\right)\vartheta_1}{2}\Bigg)\vartheta_2$$

$$+\left(k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_2^2+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_2}{2}\right)\vartheta_2\vartheta_1\Bigg),$$

$$S_5=-24k^2\left(k^2\left(\alpha_1 c-\alpha_2\right)\vartheta_2^2+\frac{\left(\alpha_1 c-\alpha_2\right) k\vartheta_2}{2}\right)\vartheta_2^2.$$

Considering $S_i=0,\ i=0,\dots,5$, results in a differential-algebraic system yielding the following solution families:

Family 1:

$$\vartheta_0(\xi)=\frac{-\left(R_1^2\xi^2+2R_1R_2\xi+R_2^2+4R_1\right)\left(\alpha_1 c-\alpha_2\right)k^2-\left(\alpha_6+\alpha_3\right)c+\alpha_5+\alpha_7+\alpha_4}{2k\left(\alpha_1 c-\alpha_2\right)},$$

$$\vartheta_1(\xi)=R_1\xi+R_2,\ \vartheta_2(\xi)=-\frac{1}{2k},\quad R_1,R_2\in\mathbb{R}.$$

Under these circumstances, the ODE (2.1) transforms to

$$\frac{d}{d\xi}\mathcal{W}(\xi)=-\frac{1}{2k}\mathcal{W}^2(\xi)+(R_1\xi+R_2)\mathcal{W}(\xi)$$
$$+\frac{-\left(R_1^2\xi^2+2R_1R_2\xi+R_2^2+4R_1\right)\left(\alpha_1 c-\alpha_2\right)k^2+\left(-\alpha_6-\alpha_3\right)c+\alpha_5+\alpha_7+\alpha_4}{2k\left(\alpha_1 c-\alpha_2\right)}. \tag{3.1}$$

It is completely clear that the obtained solution for (3.1) is also valid for (1.2). Therefore, the exact solution for (3.1) can be expressed as follows:

$$\mathcal{W}(\xi)=\frac{1}{\alpha_1 c-\alpha_2}\Bigg(R_1\left(\alpha_1 c-\alpha_2\right)k\xi+R_2\left(\alpha_1 c-\alpha_2\right)k$$
$$+\tan\left(\frac{\sqrt{\left(\alpha_1 c-\alpha_2\right)\left(6\left(\alpha_1 c-\alpha_2\right)k^2R_1+c\alpha_3+c\alpha_6-\alpha_4-\alpha_5-\alpha_7\right)}\left(2\left(\alpha_1 c-\alpha_2\right)kR_3-\xi\right)}{2k\left(\alpha_1 c-\alpha_2\right)}\right)$$

$$\sqrt{\left(\alpha_1 c-\alpha_2\right)\left(6\left(\alpha_1 c-\alpha_2\right)k^2 R_1+c\alpha_3+c\alpha_6-\alpha_4-\alpha_5-\alpha_7\right)}\Bigg), \tag{3.2}$$

where $R_3 \in \mathbb{R}$. Thus, based on (1.2) and (3.2), the initial exact solution, presented as a periodic wave solution, can be written as follows:

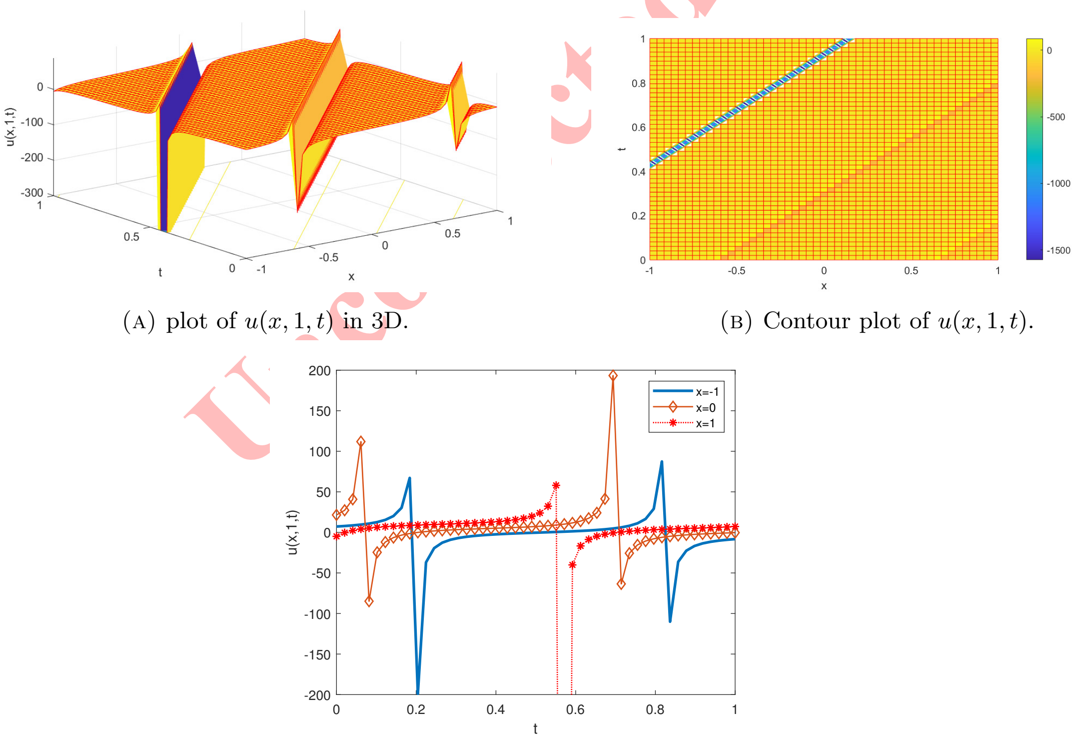
$$\begin{aligned} u(x,y,t) = {} & \frac{1}{\alpha_1 c-\alpha_2}\Bigg(R_1\left(\alpha_1 c-\alpha_2\right)k^2(x+y-ct)+R_2\left(\alpha_1 c-\alpha_2\right)k \\ & +\tan\left(\frac{\sqrt{\left(\alpha_1 c-\alpha_2\right)\left(6\left(\alpha_1 c-\alpha_2\right)k^2 R_1+c\alpha_3+c\alpha_6-\alpha_4-\alpha_5-\alpha_7\right)}\left(2\left(\alpha_1 c-\alpha_2\right)kR_3-k(x+y-ct)\right)}{2k\left(\alpha_1 c-\alpha_2\right)}\right) \\ & \sqrt{\left(\alpha_1 c-\alpha_2\right)\left(6\left(\alpha_1 c-\alpha_2\right)k^2 R_1+c\alpha_3+c\alpha_6-\alpha_4-\alpha_5-\alpha_7\right)}\Bigg). \end{aligned} \tag{3.3}$$

Family 2:

$$\vartheta_0(\xi)=\frac{-1}{R_1\xi+R_2},\ \vartheta_1(\xi)=\frac{R_1}{R_1\xi+R_2},\ \vartheta_2(\xi)=0,\quad R_1,R_2\in\mathbb{R}.$$

Here, the ODE (2.1) transforms to

$$\frac{d}{d\xi}\mathcal{W}(\xi)=\frac{R_1}{R_1\xi+R_2}\mathcal{W}(\xi)-\frac{1}{R_1\xi+R_2}. \tag{3.4}$$



(A) plot of $u(x,1,t)$ in 3D.

(B) Contour plot of $u(x,1,t)$.

(C) 2D plot of $u(x,1,t)$ for various x.

FIGURE 1. 3D, 2D and Contour plot of (3.3) in $y=1$.

Exact solution to (3.4) can be formulated as:

$$\mathcal{W}(\xi) = R_3(R_1\xi + R_2) + \frac{1}{R_1}, \quad R_3 \in \mathbb{R}. \tag{3.5}$$

Thus, deriving from (1.2) and (3.5), a periodic wave solution can be represented in the following manner:

$$u(x,y,t) = R_3(R_1k(x+y-ct) + R_2) + \frac{1}{R_1}. \tag{3.6}$$

Family 3:

$$\vartheta_0(\xi) = \sqrt{R_1k}\left(\text{JacobiSN}\left(\frac{\sqrt{-k\sqrt{R_1k}}\,\xi}{k}\,\middle|\,\frac{\sqrt{2}}{2}\right)^2 - 1\right), \ \vartheta_1(\xi) = 0, \ \vartheta_2(\xi) = 0, \quad R_1, R_2 \in \mathbb{R}.$$

Here, the ODE (2.1) transforms to

$$\frac{d}{d\xi}\mathcal{W}(\xi) = \sqrt{R_1k}\left(\text{JacobiSN}\left(\frac{\sqrt{-k\sqrt{R_1k}}\,\xi}{k}\,\middle|\,\frac{\sqrt{2}}{2}\right)^2 - 1\right). \tag{3.7}$$

This equation has the exact solution in the following form:

$$\mathcal{W}(\xi) = \sqrt{R_1k}\,\xi - \frac{2k\sqrt{R_1k}\,EllipticE\left(\text{JacobiSN}\left(\frac{\sqrt{-k\sqrt{R_1k}}\,\xi}{k}\,\middle|\,\frac{\sqrt{2}}{2}\right), \frac{\sqrt{2}}{2}\right)}{\sqrt{-k\sqrt{R_1k}}} + R_2, \quad R_2 \in \mathbb{R}. \tag{3.8}$$

So, based on Equations (1.2) and (3.8), the exact solution, presented as a singular periodic solution, can be written as follows:

$$u(x,y,t) = k\sqrt{R_1k}\,(x+y-ct) - \frac{2k\sqrt{R_1k}\,EllipticE\left(\text{JacobiSN}\left(\sqrt{-k\sqrt{R_1k}}\,(x+y-ct)\,\middle|\,\frac{\sqrt{2}}{2}\right), \frac{\sqrt{2}}{2}\right)}{\sqrt{-k\sqrt{R_1k}}} + R_2. \tag{3.9}$$

Family 4: In this family, we obtain a solution for a differential-algebraic system expressed as $c = \frac{\alpha_4+\alpha_5+\alpha_7}{\alpha_6+\alpha_3}$ and $\alpha_1 = \frac{\alpha_2(\alpha_6+\alpha_3)}{\alpha_4+\alpha_5+\alpha_7}$. Additionally, free values for $\vartheta_i(\xi),\ i = 0,1,2$ are introduced. Consequently, a diverse set of exact solutions can be generated within this family by selecting different values for $\vartheta_i(\xi),\ i = 0,1,2$. Various cases are explored as follows:

Case 1: $\vartheta_0(\xi) = \vartheta_1(\xi) = \vartheta_2(\xi) = \text{sech}(\xi)$,

In this scenario, the ODE (2.1) transforms to

$$\frac{d}{d\xi}\mathcal{W}(\xi) = \text{sech}(\xi)\left(\mathcal{W}^2(\xi) + \mathcal{W}(\xi) + 1\right). \tag{3.10}$$

Exact solution to (3.10) has the following form:

$$\mathcal{W}(\xi) = -\frac{1}{2} + \frac{\sqrt{3}\,\tan\left(\frac{\left(2\arctan\left(e^{\xi}\right)+R_2\right)\sqrt{3}}{2}\right)}{2}, \quad R_2 \in \mathbb{R}. \tag{3.11}$$

Therefore, based on the performed calculations, the following solitary wave solution is obtained using (1.2) and (3.11):

$$u(x,y,t) = -\frac{1}{2} + \frac{\sqrt{3}\,\tan\left(\frac{\left(2\arctan\left(e^{k(x+y-ct)}\right)+R_2\right)\sqrt{3}}{2}\right)}{2}. \tag{3.12}$$

Case 2: $\vartheta_0(\xi) = \vartheta_1(\xi) = \vartheta_2(\xi) = \xi$

From the Eq. (2.1) we have

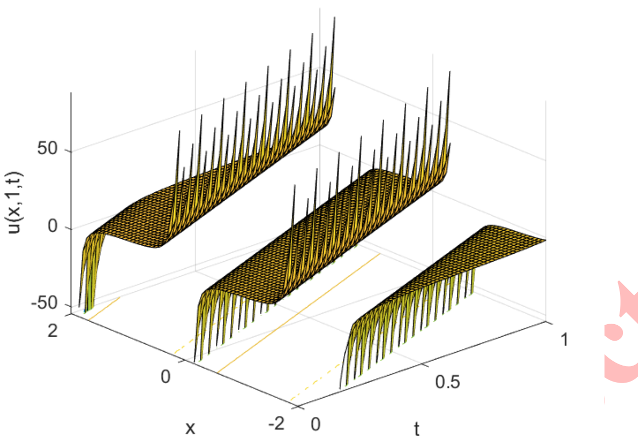
$$\frac{d}{d\xi}\mathcal{W}\left(\xi\right)=\xi\left(\mathcal{W}^{2}\left(\xi\right)+\mathcal{W}\left(\xi\right)+1\right). \tag{3.13}$$

Exact solution to (3.13) can be written as:

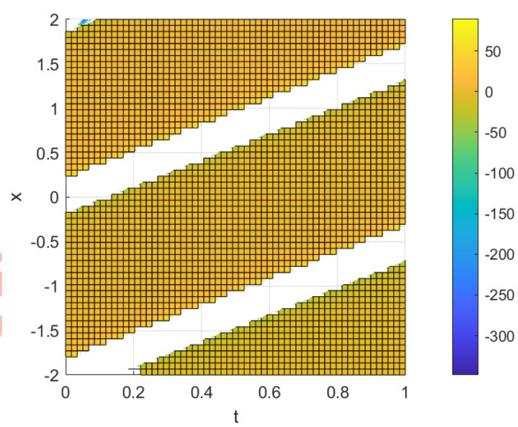
$$\mathcal{W}(\xi)=-\frac{1}{2}+\frac{\sqrt{3}\tan\left(\frac{\left(\xi^{2}+2R_{2}\right)\sqrt{3}}{4}\right)}{2},\quad R_{2}\in\mathbb{R}. \tag{3.14}$$

So, another singular periodic solution has the following form:

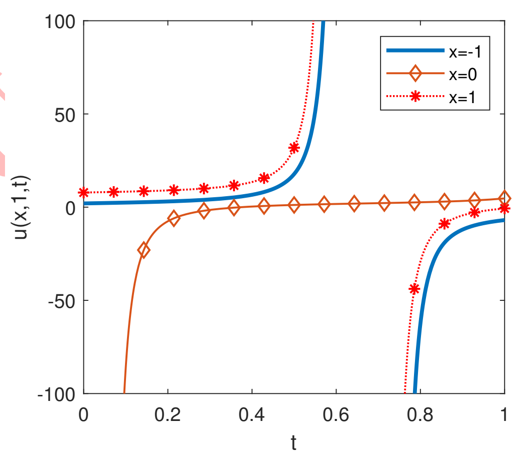
$$u(x,y,t)=-\frac{1}{2}+\frac{\sqrt{3}\tan\left(\frac{\left(k^{2}(x+y-ct)^{2}+2R_{2}\right)\sqrt{3}}{4}\right)}{2}. \tag{3.15}$$



(A) plot of $u(x,1,t)$ in 3D



(B) Contour plot of $u(x,1,t)$



(C) 2D plot of $u(x,1,t)$ for various x

Figure 2. 3D, 2D and Contour plot of (3.9) in $y=1$

Case 3: $\vartheta_0(\xi) = \vartheta_1(\xi) = \sin(\xi),\ \vartheta_2(\xi) = 0$

By these assumptions, the ODE (2.1) transforms to

$$\frac{d}{d\xi}\mathcal{W}(\xi) = \sin(\xi)\left(\mathcal{W}(\xi) + 1\right), \tag{3.16}$$

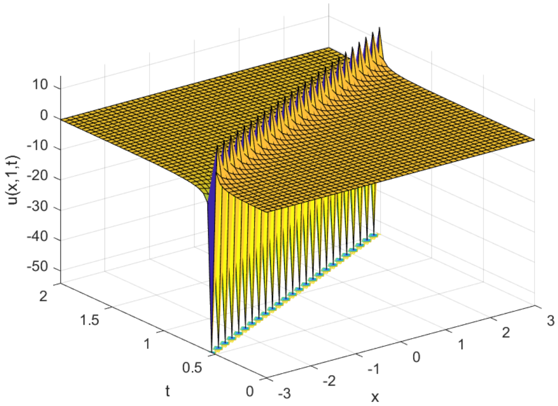
with exact solution

$$\mathcal{W}(\xi) = -1 + R_2 \mathrm{e}^{-\cos(\xi)}, \quad R_2 \in \mathbb{R}. \tag{3.17}$$

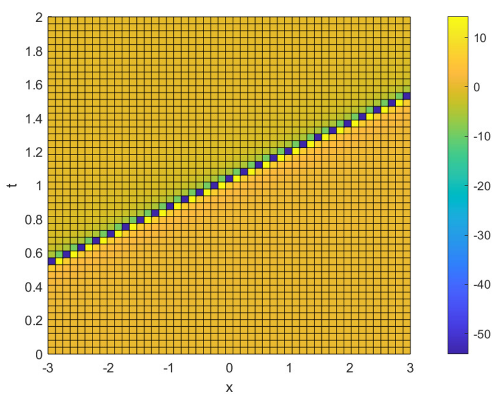
Therefore, from (1.2), and (3.17), we can derive the following periodic wave solution:

$$u(x, y, t) = -1 + R_2 \mathrm{e}^{-\cos(k(x+y-ct))}. \tag{3.18}$$

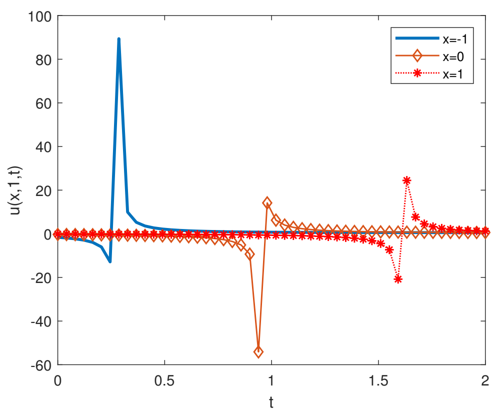
Case 4: $\vartheta_0(\xi) = \vartheta_1(\xi) = \frac{\sin(\xi)}{\xi},\ \vartheta_2(\xi) = 0$



(A) plot of $u(x, 1, t)$ in 3D.



(B) Contour plot of $u(x, 1, t)$.



(C) 2D plot of $u(x, 1, t)$ for various x.

FIGURE 3. 3D, 2D and Contour plot of (3.12) in $y = 1$.

By these assumptions, the ODE (2.1) converts into

$$\frac{d}{d\xi}\mathcal{W}(\xi) = \frac{\sin(\xi)}{\xi}\left(\mathcal{W}(\xi)+1\right). \tag{3.19}$$

Clearly, the exact solution to (3.19) can be written as[1]:

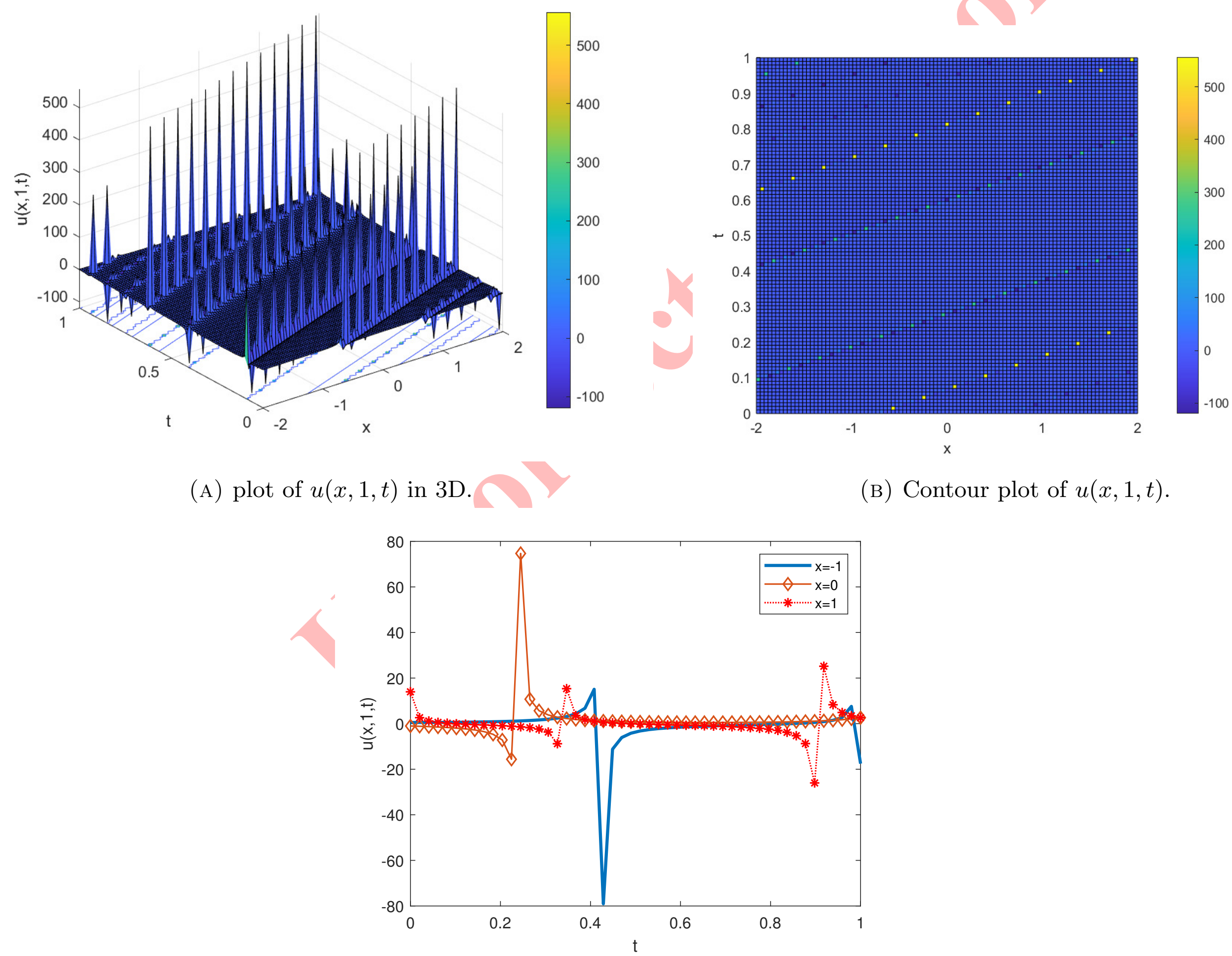
$$\mathcal{W}(\xi) = -1 + R_2 \mathrm{e}^{\mathrm{Si}(\xi)}, \quad R_2 \in \mathbb{R}. \tag{3.20}$$

Hence, from (1.2), and (3.20), a solitary wave solution can be derived as follows:

$$u(x,y,t) = -1 + R_2 \mathrm{e}^{\mathrm{Si}(k(x+y-ct))}, \tag{3.21}$$

Case 5: $\vartheta_0(\xi) = \vartheta_1(\xi) = -\vartheta_2(\xi) = \frac{-1}{\exp(\xi)}$,

[1]The $\mathrm{Si}(\xi) = \int_0^{\xi} \frac{\sin(\tau)}{\tau} d\tau$ is known as the Sine integral function



(A) plot of $u(x,1,t)$ in 3D.

(B) Contour plot of $u(x,1,t)$.

(C) 2D plot of $u(x,1,t)$ for various x.

FIGURE 4. 3D, 2D and Contour plot of (3.15) in $y = 1$.

Finally, by the mentioned assumptions, the ODE (2.1) transforms to

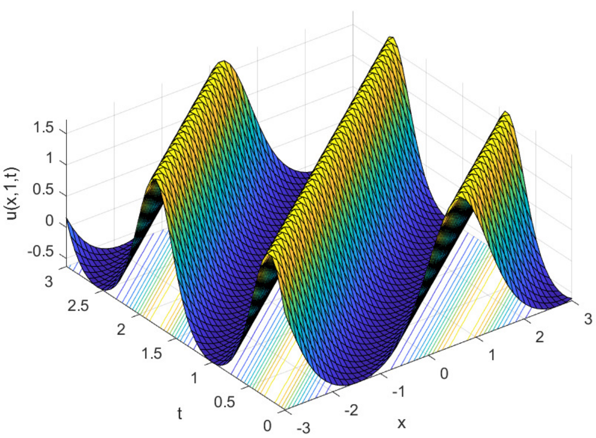
$$\frac{d}{d\xi}\mathcal{W}(\xi) = \exp(-\xi)\left(\mathcal{W}^2(\xi) - \mathcal{W}(\xi) - 1\right), \tag{3.22}$$

with exact solution

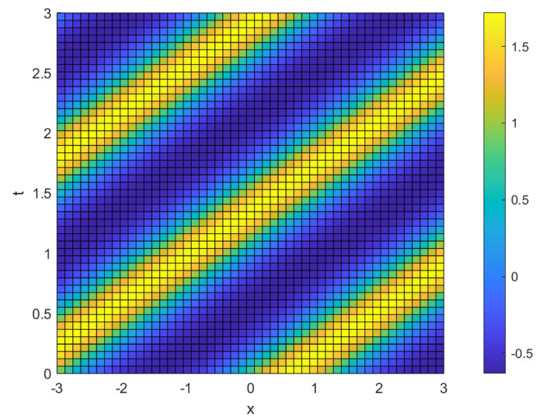
$$\mathcal{W}(\xi) = \frac{1}{2} + \frac{\sqrt{5}\,\tanh\left(\frac{\left(-R_2+\mathrm{e}^{-\xi}\right)\sqrt{5}}{2}\right)}{2}, \quad R_2 \in \mathbb{R}. \tag{3.23}$$

Thus, from (1.2), and (3.23), we can derive the following Kink soliton solution:

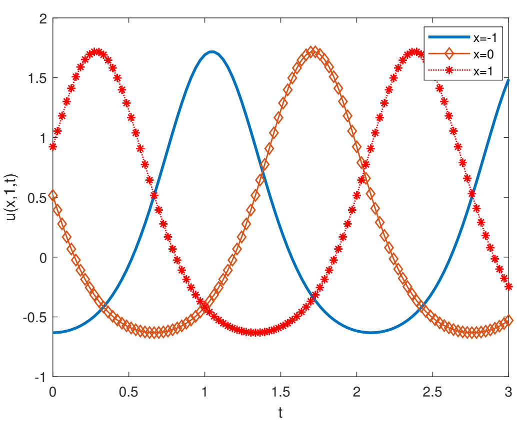
$$u(x,y,t) = \frac{1}{2} + \frac{\sqrt{5}\,\tanh\left(\frac{\left(-R_2+\mathrm{e}^{-k(x+y-ct)}\right)\sqrt{5}}{2}\right)}{2}. \tag{3.24}$$



(A) plot of $u(x,1,t)$ in 3D.



(B) Contour plot of $u(x,1,t)$.

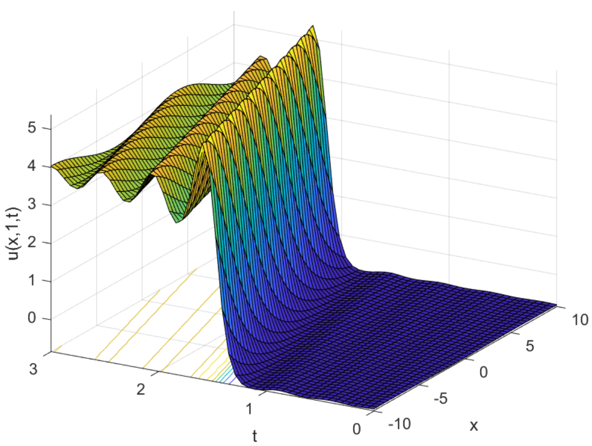


(C) 2D plot of $u(x,1,t)$ for various t.

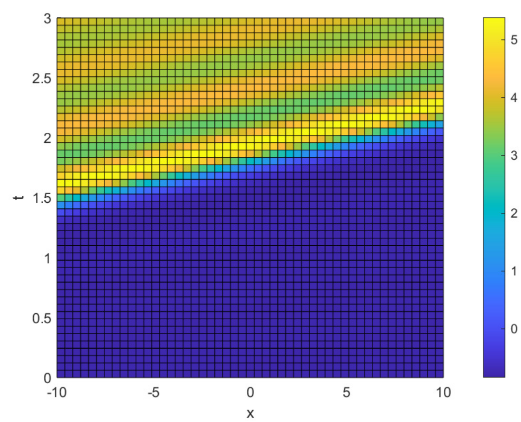
FIGURE 5. 3D, 2D and Contour plot of (3.18) in $y = 1$.

4. Results and Discussions

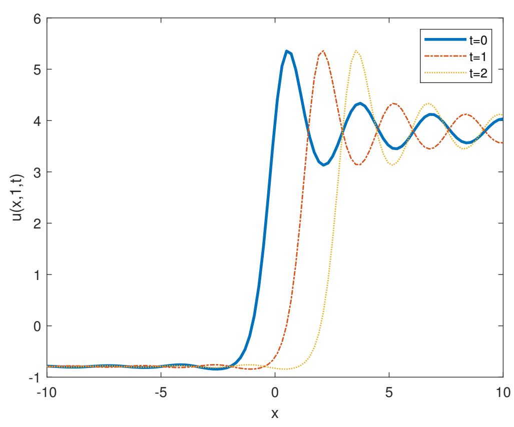
This section, accompanied by the accurate plotting of exact solutions, plays a crucial role in advancing our understanding of the generalized (2+1)-D SWW equation. This equation is a fundamental model in fluid dynamics, describing the evolution of shallow water waves in two spatial dimensions and one temporal dimension. The importance of presenting exact solutions lies in the ability to validate theoretical predictions and computational models, ensuring their reliability and applicability. Through meticulous plotting of exact solutions, researchers can visually examine the behavior of the waves, identifying patterns, anomalies, and potential applications. Furthermore, the discussions section provides a platform for interpreting these results, delving into the underlying physics and implications for real-world scenarios. This analytical discourse fosters scientific discourse, allowing researchers to refine existing theories and inspire novel approaches, ultimately advancing our understanding of complex fluid dynamics phenomena. This periodic singular solution (3.3) is plotted in Fig. 1 with respect to $c = R_2 = R_3 = k = 2,\ \alpha_i = R_1 = y = 1,\ i = 1\ldots,7$. Fig. 2 illustrates an additional singular periodic solution (3.9) concerning the parameters $k = R_2 = 2,\ R_1 = 1$, and $c = 1.5$ at the constant spatial coordinate $y = 1$. The solitary wave solution (3.12), characterized by the parameters



(A) plot of $u(x, 1, t)$ in 3D.



(B) Contour plot of $u(x, 1, t)$.



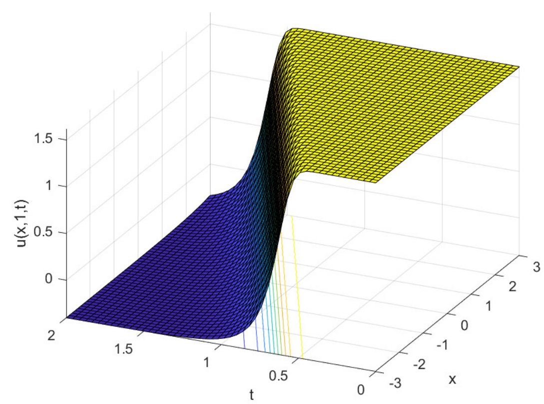
(C) 2D plot of $u(x, 1, t)$ for various t.

Figure 6. 3D, 2D and Contour plot of (3.21) in $y = 1$.

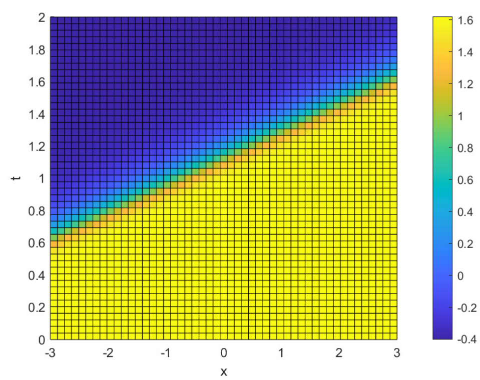
$k = 2,\ R_2 = 1$, and $c = 1.5$, is graphically represented in Figure 3 at the specific spatial location $y = 1$. The singular periodic solution given by (3.15) is visually depicted in Figure 4 using identical parameters as the preceding illustration. Moreover, we use the same parameter values in the forthcoming figures. Figure 5 displays three-dimensional, two-dimensional, and contour plots illustrating the solutions of (3.18), depicting periodic traveling wave solutions. The figure 6 showcases the solitary wave solution described by (3.21) at the constant spatial position $y = 1$. Lastly, the solution representing the Kink soliton, as expressed by (3.24), is graphically represented in three dimensions, two dimensions, and as a contour plot in Figure 7.

5. Conclusion

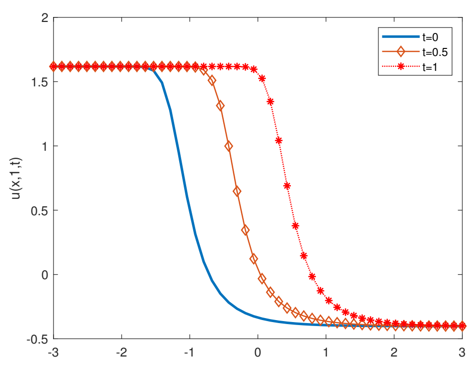
This study introduced a pioneering analytical method, the generalization of the SDAE method, designed specifically for solving the generalized (2+1)-D SWW equation. Unlike conventional approaches relying on constant coefficient ODEs and auxiliary ODEs, our proposed method stands out by incorporating variable coefficient ODEs within a sub-equation structure. The applicability and versatility of this innovative method are exemplified through its successful implementation in the context of the generalized (2+1)-D SWW equation. The analytical solutions obtained highlight the method's effectiveness and efficiency, positioning it as a valuable tool for addressing complex nonlinear PDEs inherent in wave propagation fluid and dynamics studies. Apart from the specific model under consideration, the presented method broadens the scope of available analytical techniques, contributing to the advancement of solutions



(A) plot of $u(x, 1, t)$ in 3D.



(B) Contour plot of $u(x, 1, t)$.



(C) 2D plot of $u(x, 1, t)$ for various t.

Figure 7. 3D, 2D and Contour plot of (3.24) in $y = 1$.

for diverse models within the field of mathematical physics. Overall, this research not only expands the repertoire of analytical tools but also marks a significant step forward in the quest for comprehensive solutions to intricate problems in the field.

Declarations

Ethical Approval: Not applicable.

Competing interests: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding: No funding was received for this paper.

Availability of data and materials: Not applicable.

References

[1] Y. Chen, M. Song, and Z. Liu, *Soliton and Riemann theta function quasi-periodic wave solutions for a (2+1)-dimensional generalized shallow water wave equation*, Nonlinear Dynamics, *82* (2015), 333–347.

[2] Y. Chu, M. Khater, and Y. S. Hamed, *Diverse novel analytical and semi-analytical wave solutions of the generalized (2+ 1)-dimensional shallow water waves model*, AIP Advances, *11*(1) (2021), 015223.

[3] S. A. Elwakil, S. El-Labany, M. Zahran, and R. Sabry, *Modified extended tanh-function method for solving nonlinear partial differential equations*, Physics Letters A, *299*(2-3) (2002), 179–188.

[4] Y. Gu, S. M. Zia, M. Isam, J. Manafian, A. Hajar, and M. Abotaleb, *Bilinear method and semi-inverse variational principle approach to the generalized (2+ 1)-dimensional shallow water wave equation*, Results in Physics, *45* (2023), 106213.

[5] M. S. Hashemi, *A variable coefficient third degree generalized Abel equation method for solving stochastic Schrdinger-Hirota model*, Chaos, Solitons & Fractals, *180* (2024), 114606.

[6] M. S. Hashemi, and D. Baleanu, *Lie symmetry analysis and exact solutions of the time fractional gas dynamics equation*, Journal of Optoelectronics and Advanced Materials, *8*(3-4) (2016), 383–388.

[7] M. S. Hashemi, and M. Mirzazadeh, *Optical solitons of the perturbed nonlinear Schrdinger equation using Lie symmetry method*, Optik, *281* (2023), 170816.

[8] Q.-M. Huang, and Y.-T. Gao, *Wronskian, Pfaffian and periodic wave solutions for a (2 +1)-dimensional extended shallow water wave equation*, Nonlinear Dynamics, *89* (2017), 2855–2866.

[9] M. B. Hubert, M. Justin, G. Betchewe, S. Y. Doka, A. Biswas, Q. Zhou, M. Ekici, S. P. Moshokoa, and M. Belic, *Optical solitons with modified extended direct algebraic method for quadratic-cubic nonlinearity*, Optik, *162* (2018), 161–171.

[10] D. Kumar, and S. Kumar, *Some new periodic solitary wave solutions of (3+ 1)-dimensional generalized shallow water wave equation by Lie symmetry approach*, Computers & Mathematics with Applications, *78*(3) (2019), 857–877.

[11] S. Kumar, and D. Kumar, *Analytical soliton solutions to the generalized (3+ 1)-dimensional shallow water wave equation*, Modern Physics Letters B, *36*(02) (2022), 2150540.

[12] Y.-L. Ma, and B.-Q. Li, *Soliton interactions, soliton bifurcations and molecules, breather molecules, breather-to-soliton transitions, and conservation laws for a nonlinear (3+ 1)-dimensional shallow water wave equation*, Nonlinear Dynamics, *112* (2024), 2851–2867.

[13] M. Ozisik, I. Onder, H. Esen, M. Cinar, N. Ozdemir, A. Secer, and M. Bayram, *On the investigation of optical soliton solutions of cubicquartic FokasLenells and SchrdingerHirota equations*, Optik, *272* (2023), 170389.

[14] N. Ozdemir, A. Secer, M. Ozisik, and M. Bayram, *Perturbation of dispersive optical solitons with SchrdingerHirota equation with Kerr law and spatio-temporal dispersion*, Optik, *265* (2022), 169545.

[15] H. U. Rehman, I. Iqbal, M. S. Hashemi, M. Mirzazadeh, and M. Eslami, *Analysis of cubic-quarticnonlinear Schrdingers equation with cubic-quintic-septic-nonic form of self-phase modulation through different techniques*, Optik, (2023), 171028.

[16] S. Singh, K. Sakkaravarthi, T. Tamizhmani, and K. Murugesan, *Painlev analysis and higher-order rogue waves of a generalized (3+ 1)-dimensional shallow water wave equation*, Physica Scripta *97*(5) (2022), 055204.

[17] X. Xin, L. Zhang, Y. Xia, and H. Liu, *Nonlocal symmetries and exact solutions of the (2+ 1)- dimensional generalized variable coefficient shallow water wave equation*, Applied Mathematics Letters, *94* (2019), 112–119.

[18] Y.-J. Xu, and M. S. Hashemi, *Exact solutions for porous fins under a uniform magnetic field: A novel reduction method*, Case Studies in Thermal Engineering, *45* (2023), 103013.

[19] L. Ying, and M. Li, *The dynamics of some exact solutions of the (3+ 1)-dimensional generalized shallow water wave equation*, Nonlinear Dynamics, *111*(17) (2023), 15633–15651.

[20] A.-M. Wazwaz, *The tanh method for traveling wave solutions of nonlinear equations*, Applied Mathematics and Computation, *154*(3) (2004), 713–723.