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Implicit numerical approach for nonlinear fractional differential equations with a time non-singular kernel and mixed boundary conditions

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Abstract

This study focuses on the numerical solution of the time-fractional nonlinear Cable equation with the Caput-oFabrizio derivative using an implicit CrankNicolson scheme. To demonstrate the versatility and robustness of the proposed method, we investigate the problem under both Dirichlet and Neumann boundary condition. The Stability analysis confirms that the scheme is unconditionally stable. To further evaluate the robustness of the difference scheme, the same numerical framework is applied to the fractional Burgers equation under identical settings. Numerical experiments are conducted to verify the stability and accuracy of the method, and to illustrate its applicability in simulating both signal propagation in nerve fibers (cable equation) and viscous transport (Burgers equation).

Keywords. Implicit Crank-Nicholson scheme, Caputo-Fabrizio derivative, nonlinear Cable equation, Fourier method, Stability, Dirichlet and neumann boundary conditions.

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1. Introduction

In nature, numerous physical phenomena exhibit intrinsic fractional-order behavior. For this reason, fractional calculus has become an important tool due to its efficiency in accurately explaining real-world processes. It provides a powerful framework for modeling memory and hereditary characteristics of various materials and processes. This is one of the significant advantages of fractional calculus compared to classical (integer-order) calculus [29]. Fractional calculus is an old branch of applied mathematics that has been widely employed in physical and engineering sciences, especially in areas where classical methods fall short. In recent years, it has played a vital role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Fractional differential equations, which involve fractional derivatives or integrals, are used to model such systems. These derivatives are applied in solving fractional FokkerPlanck equations, fractional wave equations, nonlinear systems, linear systems, fractional Burgers equations, fractional cable equations, distributed-order reactiondiffusion equations, fractional integro-differential equations, and many other related models [14, 20, 25, 30].

Several definitions have been presented for the fractional derivatives. The GrnwaldLetnikov, RiemannLiouville, and Caputo fractional derivatives are designed with singular kernels, and many researchers have utilized these derivatives for solving problems [4–6, 13, 15]. However, these derivatives cannot accurately represent the entire memory effect of a given system because they involve singular kernels, which limit their effectiveness in modeling certain problems. To overcome this limitation, Caputo and Fabrizio proposed a new definition of the fractional-order derivative that employs a non-singular kernel instead of a singular one. This derivative is a direct modification of the classical Caputo derivative and offers improved characteristics. It better captures structural features and heterogeneities across different scales that cannot be effectively described by fractional models with singular kernels or by classical local derivatives.

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Therefore, the CaputoFabrizio derivative with a non-singular kernel is considered more suitable for modeling such complex systems [2, 8, 28, 37].

There are many analytical approaches for solving fractional differential equations; however, fractional differential equations cannot always be solved analytically [7, 19, 34]. Finding an analytical solution is often difficult due to the presence of initial conditions, boundary conditions, and source terms. Therefore, researchers commonly use numerical methods to solve these equations. Examples include the finite-difference method, compact iterative method, CrankNicolson method, and others [1, 9, 10, 22–24, 36]. The CrankNicolson method is a finite-difference technique used for the numerical solution of partial differential equations. It is implicit in time and numerically stable. The method was developed by John Crank and Phyllis Nicholson in the mid-20th century. Finite-difference methods have been applied in several studies for numerically solving differential equations [3, 12, 16, 21, 26, 33, 37].

The cable equation plays a major role in many fields of electrophysiology and neurophysiology and in modeling neuronal dynamics [11, 31, 35]. In neurophysiology, the cable equation illustrates the trans-membrane potential $U_m(x,t)$ along the axial direction x of an approximately cylindrical segment of a nerve cell. The membrane is modeled by an electrical circuit with axial internal resistance r_i, transmembrane capacitance c_m, and transmembrane resistance r_m in parallel, connecting the inner part to the outside. The cable equation is derived from the NernstPlanck equation for the electro-diffusive motion of ions. The resulting differential equation takes the form of a standard diffusion equation with an extra term for the trans-membrane potential, resulting in a decay of the electric signal:

$$\lambda^2 \frac{\partial^2 U_m(x,t)}{\partial x^2} - \gamma \frac{\partial U_m(x,t)}{\partial t} - U_m(x,t) = 0, \tag{1.1}$$

where $\gamma = r_m c_m$ is the time constant and $\lambda = \sqrt{r_m/r_i}$ is the space constant related to the membrane resistance and capacitance. Sometimes the dimensionless variables $X = \frac{x}{\lambda}, \quad T = \frac{t}{\gamma}$ are introduced, so that

$$\frac{\partial^2 U_m(X,T)}{\partial X^2} - \frac{\partial U_m(X,T)}{\partial T} - U_m(X,T) = 0. \tag{1.2}$$

This can help shorten certain handwork and simplify the mathematical formulation of the solutions. For situations where applied electric fields, synaptic excitation, or active membrane properties are given, a more general cable equation is considered:

$$\frac{\partial^2 U_m(x,t)}{\partial x^2} - \frac{\partial U_m(x,t)}{\partial t} - U_m(x,t) = G, \tag{1.3}$$

Eq. (1.3) includes an external source term G. If $G = 0$, Eq. (1.2) represents a homogeneous partial differential equation. When current or voltage is applied only at a few discrete points along a passive cable, then $G = 0$ for the cable segments lying between these points. When an extended electrode or complex electrode array is applied along the length of the cable, we obtain Eq. (1.3) with G as a function of X; if the applied field varies with time, G becomes a function of both X and T. Furthermore, when the input disturbance is a synaptic conductance change, the resultant synaptic current depends on U, as well as on the spatiotemporal distribution of synaptic input. In this case, G is a function of U, X, and T.

The fractional cable equation model is superior to the integer-order model because the fractional derivative can describe the history of the state over all time intervals. By replacing the first-order time derivative in Eq. (1.2) with a fractional derivative of order $\alpha \in (0,1)$, we obtain

$$\frac{\partial^2 U_m(X,T)}{\partial X^2} - \frac{\partial^\alpha U_m(X,T)}{\partial T^\alpha} - U_m(X,T) = 0.$$

From a mathematical point of view, this model is a simple extension of the neuronal fractional cable model.

In this work, we consider the following general fractional cable model with given initial and boundary conditions:

$$\frac{\partial^2 U(x,t)}{\partial x^2} - \frac{\partial^\alpha U(x,t)}{\partial t^\alpha} - U(x,t) = G(x,t,u), \tag{1.4}$$

subject to the Dirichlet boundary conditions

$$u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad 0 \le t \le T,$$

and the initial condition

$$u(x,0) = r(x), \quad 0 \leq x \leq L.$$

We also adopt the Neumann boundary conditions. where the spatial derivative of the function is prescribed at both the left and right boundaries:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = h_1(t), \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = h_2(t), \quad 0 \leq t \leq T.$$

The Neumann boundary condition, which describes the rate of change of a quantity at the boundary, is frequently applied in many physical contexts, particularly in neural and thermal modeling.

The mathematical structure of the cable equationparticularly in its nonlinear and fractional formsmakes it a suitable framework for modeling various transport processes in media with memory and heterogeneity. In traffic dynamics, for example, the nonlinear term in the cable equation allows for a more accurate representation of vehicle interactions, such as congested conditions or nonlinear speed responses to changes in traffic density. Therefore, the nonlinear fractional cable equation can effectively describe the propagation of density or shock waves in traffic flow, especially in situations where the system dynamics depend on temporal memory or nonlinear behavior.

A simplified form of the Navier–Stokes equation is the Burgers equation. By the Navier–Stokes equation, the physics of several engineering and scientific phenomena is described. The Burgers equation frequently appears in various fields of applied mathematics and engineering, such as acoustic transmission, fluid dynamics, traffic flow, shock waves, and gas dynamics [17, 27]. It is the most straightforward nonlinear parabolic equation displaying phenomena characterized by an equilibrium between linear diffusion and nonlinear advection, related to conservation laws or dissipation as in the heat equation. These two effects often act in opposition: nonlinearity tends to steepen wave fronts into discontinuous shocks, while diffusion smooths them out.

If we assume that the external force is zero and that the density ρ is constant, the one-dimensional Navier–Stokes equation reduces to the viscous Burgers equation, written as

$$u_t + u\,u_x = \nu\,u_{xx},$$

where $\nu = \frac{\mu}{\rho}$ is the *kinematic viscosity*.

When the viscosity ν is zero, the above equation becomes the following inviscid Burgers equation, shown as

$$u_t + u\,u_x = 0.$$

Recent researchers have proposed and analyzed numerous models based on fractional Burgers equations. In the present work, we consider the following time-fractional Burgers equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u\,\frac{\partial u(x,t)}{\partial x} - \nu\,\frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t), \tag{1.5}$$

where $\alpha \in (0,1)$ denotes the fractional order of the Caputo–Fabrizio derivative.

The problem is supplied with boundary conditions

$$u(a,t) = h_1(t), \qquad u(b,t) = h_2(t), \qquad t \geq 0,$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 \leq x \leq L,$$

where ν is the viscosity parameter.

The remainder of this paper is structured as follows. Section 2 provides two basic definitions. Section 3 presents the numerical method for the fractional cable equation and its stability analysis. Section 4 describes the numerical approximation for the fractional Burgers equation. Three numerical experiments are reported in Section 5. Finally, Section 6 concludes and discusses .

2. PRELIMINARIES

In this section, we introduce two essential definitions of fractional calculus, namely the Caputo and Caputo–Fabrizio fractional time derivatives.

Definition 2.1. Let $u(x,t)$ be a sufficiently smooth function. The Caputo fractional time derivative of order α $(n-1 \le \alpha \le n)$ is defined as

$$ {}_{a}^{C}D_{t}^{\alpha}u(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}\frac{\mathrm{d}^{n}u(x,\tau)}{\mathrm{d}\tau^{n}}\,\mathrm{d}\tau, \tag{2.1}$$

where $a \in (-\infty, t)$ and $\Gamma(\cdot)$ denotes the Gamma function.

By changing the kernel $(t-\tau)^{-\alpha}$ with the exponential function $\exp\big(-\frac{\alpha}{1-\alpha}(t-\tau)\big)$ and substituting the coefficient $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$,

Caputo and Fabrizio proposed a new definition of the fractional derivative that avoids the singular kernel.

Definition 2.2. Let $u(\cdot,t) \in H^{1}(a,b)$, with $b>a$ and $\alpha \in (0,1)$. The Caputo–Fabrizio fractional derivative is defined as

$$ {}_{a}^{CF}D_{t}^{\alpha}u(x,t) = \frac{M(\alpha)}{1-\alpha}\int_{a}^{t}\frac{\partial u(x,\tau)}{\partial \tau}\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\mathrm{d}\tau, \tag{2.2}$$

where $M(\alpha)$ is a normalization function satisfying $M(0)=M(1)=1$.

If the function u does not belong to $H^{1}(a,b)$ but $u \in L^{1}(-\infty,b)$, then the Caputo–Fabrizio derivative can be equivalently redefined as

$$ {}_{a}^{CF}D_{t}^{\alpha}u(x,t) = \frac{\alpha\, M(\alpha)}{1-\alpha}\int_{a}^{t}\big[u(x,t)-u(x,\tau)\big]\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\mathrm{d}\tau.$$

3. FORMATION AND ANALYSIS OF THE NUMERICAL SCHEME FOR SOLVING THE CABLE EQUATION

3.1. **Numerical approximation of the fractional derivative.** We begin discretization the problem by assuming $h = 1/M$ and $\tau = T/N$ that M is the grid sizes in space and N is the grid sizes in time. Also we consider $x_i = ih, i = 0,1,\dots,M$, in interval $[0,X]$ and $t_n = n\tau, n = 0,1,\dots,N$, in interval $[0,T]$. Fractional derivative in point $(x_i, t_{n+\frac{1}{2}})$

$$ {}_{0}^{CF}D_{t}^{\alpha}u\Big(x_i, t_{n+\frac{1}{2}}\Big) = \frac{M(\alpha)}{1-\alpha}\int_{0}^{t_{n+\frac{1}{2}}}\frac{\partial u(x_i,\tau)}{\partial \tau}\exp\left(\frac{-\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-\tau\right)\right)d\tau. \tag{3.1}$$

We approximate the fractional derivative at $(x_i, t_{n+\frac{1}{2}})$:

$$\begin{aligned}
{}_{0}^{CF}D_{t}^{\alpha}u(x_i,t_{n+\frac{1}{2}}) &= \frac{M(\alpha)}{1-\alpha}\int_{0}^{t_{n+\frac{1}{2}}}\frac{\partial u(x_i,\tau)}{\partial \tau}\exp\left[-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}}-\tau)\right]d\tau \\
&= \int_{0}^{t_n}\frac{\partial u(x_i,\tau)}{\partial \tau}\exp\left[-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}}-\tau)\right]d\tau + \int_{t_n}^{t_{n+\frac{1}{2}}}\frac{\partial u(x_i,\tau)}{\partial \tau}\exp\left[-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}}-\tau)\right]d\tau \\
&= \frac{M(\alpha)}{1-\alpha}\left[\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left(\frac{u_i^{j}-u_i^{j-1}}{\Delta t} + (\tau - t_{j-\frac{1}{2}})u_{tt}(x_i,c_j)\right)\exp\left[-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}}-\tau)\right]d\tau\right] \\
&+ \int_{t_n}^{t_{n+\frac{1}{2}}}\left(\frac{u_i^{n+\frac{1}{2}}-u_i^{n}}{\Delta t/2} + O(\Delta t)\right)\exp\left[-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}}-\tau)\right]d\tau.
\end{aligned}$$

where $c_j \in (t_{j-1}, t_j)$. Then,

$$
\begin{aligned}
{}_0^{CF}D_t^{\alpha} u(x_i, t_{n+\frac{1}{2}}) &= \frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{u_i^j - u_i^{j-1}}{\Delta t} \exp\Big(-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}} - \tau)\Big)\, d\tau \\
&+ \frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big(\tau - t_{j-\frac{1}{2}}\Big)\, u_{tt}(x_i, c_j) \exp\Big(-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}} - \tau)\Big)\, d\tau \\
&+ \frac{M(\alpha)}{1-\alpha} \int_{t_n}^{t_{n+\frac{1}{2}}} \Big(\frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)\Big) \exp\Big(-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}} - \tau)\Big)\, d\tau \\
&= \frac{M(\alpha)}{\alpha\, \Delta t} \Bigg\{ (u_i^{n+1} - u_i^n) \Big[1 - \exp\Big(-\frac{\alpha}{1-\alpha}\,(\tfrac{1}{2}\Delta t)\Big)\Big] \\
&+ \sum_{j=1}^{n} \Big(u_i^j - u_i^{j-1}\Big) \Big[\exp\Big(-\frac{\alpha}{1-\alpha}\,(n - j + \tfrac{1}{2})\,\Delta t\Big) - \exp\Big(-\frac{\alpha}{1-\alpha}\,(n - j + \tfrac{3}{2})\,\Delta t\Big)\Big] \Bigg\} + R_1 + R_2,
\end{aligned}
$$

which

$$
R_1 = \frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (\tau - t_{j-\frac{1}{2}})\, u_{tt}(x_i, c_j) \exp\Big(-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}} - \tau)\Big)\, d\tau,
$$

and

$$
R_2 = \frac{M(\alpha)}{\alpha} \Big[1 - \exp\Big(-\frac{\alpha}{1-\alpha}\,(\tfrac{1}{2}\Delta t)\Big)\Big]\, O(\Delta t).
$$

$$
\begin{aligned}
R_1 &= \frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (\tau - t_{j-\frac{1}{2}})\, u_{tt}(x_i, c_j) \exp\Big(-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}} - \tau)\Big)\, d\tau \\
&= \frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} u_{tt}(x_i, c_j) \int_{t_{j-1}}^{t_j} (\tau - t_{j-\frac{1}{2}}) \exp\Big(-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}} - \tau)\Big)\, d\tau \\
&\leq \frac{\max_{1\leq j\leq n} \big|u_{tt}(x_i, c_j)\big|\, \Delta t}{2(1-\alpha)} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \exp\Big(-\frac{\alpha}{1-\alpha}(t_{n+\frac{1}{2}} - \tau)\Big)\, d\tau \\
&= \frac{\max_{1\leq j\leq n} \big|u_{tt}(x_i, c_j)\big|\, \Delta t}{2\alpha} \sum_{j=1}^{n} \Big[\exp\Big(-\frac{\alpha}{1-\alpha}\Big(n - j + \tfrac{1}{2}\Big)\Delta t\Big) - \exp\Big(-\frac{\alpha}{1-\alpha}\Big(n - j + \tfrac{3}{2}\Big)\Delta t\Big)\Big] \\
&= \frac{\max_{1\leq j\leq n} \big|u_{tt}(x_i, c_j)\big|\, \Delta t}{2\alpha} \Big[\exp\Big(-\frac{\alpha}{1-\alpha} \tfrac{1}{2}\Delta t\Big) - \exp\Big(-\frac{\alpha}{1-\alpha}\Big(n + \tfrac{1}{2}\Big)\Delta t\Big)\Big] \\
&= \frac{\max_{1\leq j\leq n} \big|u_{tt}(x_i, c_j)\big|\, \Delta t}{2\alpha} \cdot \frac{\alpha}{1-\alpha}\, n\Delta t \\
&= \frac{\max_{1\leq j\leq n} \big|u_{tt}(x_i, c_j)\big|\, \Delta t}{2(1-\alpha)}\, n\Delta t \\
&= \frac{\max_{1\leq j\leq n} \big|u_{tt}(x_i, c_j)\big|\, T\, \Delta t}{2(1-\alpha)} \\
&= O(\Delta t)
\end{aligned}
$$

We assume

$$d_{n,j} = \exp\left(-\frac{\alpha}{1-\alpha}\left(n-j+\tfrac{1}{2}\right)\Delta t\right) - \exp\left(-\frac{\alpha}{1-\alpha}\left(n-j+\tfrac{3}{2}\right)\Delta t\right), \qquad d_1 = 1-\exp\left(-\frac{\alpha}{1-\alpha}\tfrac{1}{2}\Delta t\right).$$

Therefore, the following approximation is obtained

$$ {}_{0}^{CF}D_t^{\alpha} u\left(x_i, t_{n+\frac{1}{2}}\right) = \frac{M(\alpha)}{\alpha\,\Delta t}\left[d_1\, u_i^{n+1} + \left(d_{n,n}-d_1\right) u_i^n - \sum_{j=1}^{n-1}\left(d_{n,j+1}-d_{n,j}\right) u_i^j - d_{n,1}\, u_i^0\right] + O(\Delta t). \tag{3.2}$$

3.2. **Crank-Nicolson scheme for nonlinear Cable equation.** In this Section, the following fractional nonlinear Cable equation be considered:

$$ {}_{0}^{CF}D_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) - g(x,t,u(x,t))\,. \tag{3.3}$$

The fractional derivative is approximated as follows

$$ {}_{0}^{CF}D_t^{\alpha} u\left(x_i, t_{n+\frac{1}{2}}\right) = \frac{M(\alpha)}{\alpha\,\Delta t}\left(d_1\, u_i^{n+1} + \left(d_{n,n}-d_1\right) u_i^n - \sum_{j=1}^{n-1}\left(d_{n,j+1}-d_{n,j}\right) u_i^j - d_{n,1}\, u_i^0\right) + O(\Delta t). \tag{3.4}$$

And, we also have a second-order space derivative of the Crank-Nicholson method:

$$\frac{\partial^2 u(x_i,t_n)}{\partial x^2} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2\Delta x^2} + \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{2\Delta x^2} + O(\Delta x^2). \tag{3.5}$$

We also assume that $g(x,t,u(x,t))$ has a first-order continuous partial derivative $\dfrac{\partial g(x,t,u)}{\partial t}$, and that $g(x,t,u)$ satisfies a Lipschitz condition with respect to u:

$$\forall\, \tilde{u}, \bar{u}, \qquad \left|g(x,t,\tilde{u}) - g(x,t,\bar{u})\right| \le D\left|\tilde{u}-\bar{u}\right|. \tag{3.6}$$

where D is a Lipschitz constant.

Substituting Eqs. (3.4) and (3.5) into Eq. (3.3) leads to

$$\begin{aligned} &\frac{M(\alpha)}{\alpha\,\Delta t}\left(d_1 u_i^{n+1} + (d_{n,n}-d_1)u_i^n - \sum_{j=1}^{n-1}(d_{n,j+1}-d_{n,j})u_i^j - d_{n,1}u_i^0\right) \\ &\qquad = \frac{1}{2\Delta x^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right) \\ &\qquad - \frac{1}{2}\left(u_i^{n+1} + u_i^n\right) + \frac{1}{2}\left(g(x_i,t_{n+1},u_i^{n+1}) + g(x_i,t_n,u_i^n)\right). \end{aligned} \tag{3.7}$$

Then,

$$\begin{aligned} (-\lambda_1)u_{i-1}^{n+1} + \left(2\lambda_1+\lambda_2+d_1\right)u_i^{n+1} - \lambda_1 u_{i+1}^{n+1} &= \lambda_1 u_{i-1}^n + \left(-2\lambda_1 - \lambda_2 + d_1 - d_{n,n}\right)u_i^n \\ &+ \lambda_1 u_{i+1}^n + \sum_{j=1}^{n-1}\left(d_{n,j+1}-d_{n,j}\right)u_i^j + d_{n,1}u_i^0 \\ &+ \lambda_2\left(g(x_i,t_{n+1},u_i^{n+1}) + g(x_i,t_n,u_i^n)\right), \end{aligned} \tag{3.8}$$

where

$$\lambda_1 = \frac{\alpha\,\Delta t}{2\Delta x^2 M(\alpha)}, \qquad \lambda_2 = \frac{\alpha\,\Delta t}{2M(\alpha)}.$$

The CrankNicolson discretization yields a linear system of equations at each time step. The Lanczos algorithm has been widely recognized as a fundamental and efficient technique for solving such linear systems by researchers [32].

3.3. **Stability analysis.** We analysis the stability of discrete problem Eq. (3.8) using the Fourier method. Let U_i^n be approximate solution of problem and we define $\rho_i^n = u_i^n - U_i^n$; $n = 0, 1, \dots, N$ and $i = 0, 1, \dots, M$. So, we have

$$\begin{aligned}
&(-\lambda_1)\rho_{i-1}^{n+1} + (2\lambda_1 + \lambda_2 + d_1)\rho_i^{n+1} - \lambda_1\rho_{i+1}^{n+1} - \lambda_2\big(g(x_i, t_{n+1}, u_i^{n+1}) - g(x_i, t_{n+1}, U_i^{n+1})\big)\\
&= \lambda_1\rho_{i-1}^n + (-2\lambda_1 - \lambda_2 + d_1 - d_{n,n})\rho_i^n + \lambda_1\rho_{i+1}^n + \sum_{j=1}^{n-1}(d_{n,j+1} - d_{n,j})\rho_i^j + d_{n,1}\rho_i^0\\
&+ \lambda_2(g(x_i, t_n, u_i^n) - g(x_i, t_n, U_i^n)).
\end{aligned}$$

According to Eq. (3.6) , we have

$$\begin{aligned}
&\Big|(-\lambda_1)\rho_{i-1}^{n+1} + (2\lambda_1 + \lambda_2 + d_1)\rho_i^{n+1} - \lambda_1\rho_{i+1}^{n+1}\Big|\\
&\le \Big|\lambda_1\rho_{i-1}^n + (-2\lambda_1 - \lambda_2 + d_1 - d_{n,n})\rho_i^n + \lambda_1\rho_{i+1}^n + \sum_{j=1}^{n-1}(d_{n,j+1} - d_{n,j})\rho_i^j + d_{n,1}\rho_i^0\Big|\\
&+ \lambda_2 D\big|u_i^{n+1} - U_i^{n+1}\big| + \lambda_2 D\big|u_i^n - U_i^n\big|.
\end{aligned}$$

We have

$$\begin{aligned}
&\Big|(-\lambda_1)\rho_{i-1}^{n+1} + (2\lambda_1 + \lambda_2 + d_1)\rho_i^{n+1} - \lambda_1\rho_{i+1}^{n+1}\Big|\\
&\le \Big|\lambda_1\rho_{i-1}^n + (-2\lambda_1 - \lambda_2 + d_1 - d_{n,n})\rho_i^n + \lambda_1\rho_{i+1}^n + \sum_{j=1}^{n-1}(d_{n,j+1} - d_{n,j})\,\rho_i^j + d_{n,1}\rho_i^0\Big|\\
&+ \lambda_2 D\,|\rho_i^{n+1}| + \lambda_2 D\,|\rho_i^n|.
\end{aligned} \tag{3.9}$$

where $\rho_0^n = \rho_M^n = 0$. For $n = 0, 1, \dots, N$, we define the following grid function

$$\rho^n(x) = \begin{cases} \rho_i^n, & \text{when } x_{i-\frac{h}{2}} < x < x_{i+\frac{h}{2}}, \quad i = 1, 2, \dots, M-1,\\ 0, & \text{when } 0 \le x \le \dfrac{h}{2} \text{ or } L - \dfrac{h}{2} < x \le L. \end{cases}$$

Here, $\rho^n(x)$ can be expanded in Fourier series as follows:

$$\rho^n(x) = \sum_{l=-\infty}^{\infty} d_n(l)e^{2kl\pi x}, \qquad n = 1, 2, \dots, N,$$

where

$$d_n(l) = \frac{1}{L}\int_0^L \rho^n(x)e^{2kl\pi x}\,dx.$$

We now let

$$\rho^n = [\rho_1^n, \rho_2^n, \dots, \rho_{M-1}^n]^T, \qquad n = 1, 2, \dots, N.$$

From the definition of the l^2 norm:

$$\|\rho^n\|_2 = \left(\sum_{i=1}^{M-1} h|\rho_i^n|^2\right)^{\frac{1}{2}} = \left(\int_0^L |\rho_i^n|^2 dx\right)^{\frac{1}{2}}, \qquad n = 1, 2, \dots, N.$$

Using the Parseval equality:

$$\int_0^L |\rho_i^n|^2\,dx = \sum_{l=-\infty}^{\infty} |d_n(l)|^2, \qquad n = 1, 2, \dots, N, \tag{3.10}$$

we have

$$\|\rho^n\|_2^2 = \sum_{l=-\infty}^{\infty} |d_n(l)|^2, \qquad n = 1, 2, \dots, N. \tag{3.11}$$

Assume that ρ_i^n has the form $\rho_i^n = d_n e^{rih\beta}$, where $\beta = \frac{2\pi l}{L}$, and $L = 1$ .However, substituting the above expression into Eq. (3.9), we obtain:,

$$\begin{aligned}
&\Big|(-\lambda_1)d_{n+1}e^{r(i-1)h\beta} + (2\lambda_1 + \lambda_2 + d_1)d_{n+1}e^{rih\beta} - \lambda_1 d_{n+1}e^{r(i+1)h\beta}\Big| - \lambda_2 D\,|d_{n+1}e^{rih\beta}| \\
&\quad \le \Big|\lambda_1 d_n e^{r(i-1)h\beta} + \big(-2\lambda_1 - \lambda_2 + d_1 - d_{n,n}\big)d_n e^{rih\beta} + \lambda_1 d_n e^{r(i+1)h\beta} \\
&\quad + \sum_{j=1}^{n-1}(d_{j+1,n} - d_{j,n})\, d_j e^{rih\beta} + d_{n,1}d_0 e^{rih\beta}\Big| + \lambda_2 D\,|d_n e^{rih\beta}|.
\end{aligned}$$

After simplifications, we have

$$\begin{aligned}
\Big|d_{n+1}\Big[-\lambda_1\big(e^{rh\beta} + e^{-rh\beta} - 2\big) + d_1 + \lambda_2 - \lambda_2 D\Big]\Big| \le \Big|&d_n\Big[\lambda_1\big(e^{rh\beta} + e^{-rh\beta} - 2\big) - \lambda_2 + d_1 - d_{n,n} + \lambda_2 D\Big] \\
&+ \sum_{j=1}^{n-1}\big(d_{j+1,n} - d_{j,n}\big)d_j + d_{n,1}d_0\Big|.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\Big|d_{n+1}\Big(4\lambda_1\sin^2\frac{\beta h}{2} + d_1 + \lambda_2 - \lambda_2 D\Big)\Big| \le \Big|&d_n\Big(-4\lambda_1\sin^2\frac{\beta h}{2} - \lambda_2 + d_1 - d_{n,n} + \lambda_2 D\Big) \\
&+ \sum_{j=1}^{n-1}(d_{j+1,n} - d_{j,n})\, d_j + d_{n,1}d_0\Big|.
\end{aligned} \tag{3.12}$$

Lemma 3.1. *If $d_1 \ge d_{n,n}$, then $|d_n| \le |d_0|$ for all $n \ge 1$, and d_n satisfies Eq. (3.12).*

Proof. We apply mathematical induction. Take $n = 0$:

$$|d_1| \le \left|\frac{-4\lambda_1\sin^2\frac{\beta h}{2} - \lambda_2 + d_1 - d_{n,n} + \lambda_2 D}{4\lambda_1\sin^2\frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D}\right| |d_0| \ \le \ |d_0|.$$

Then, $|d_1| \le |d_0|$. We assume that $|d_k| \le |d_0|$, $k = 1, 2, \dots, n$. We will prove for$k = n + 1$. Therefore,

$$\begin{aligned}
|d_{n+1}| \ &\le \ \left(\left|\frac{-4\lambda_1\sin^2\frac{\beta h}{2} - \lambda_2 + d_1 - d_{n,n} + \lambda_2 D}{4\lambda_1\sin^2\frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D}\right|\right. \\
&\quad \left. + \frac{1}{4\lambda_1\sin^2\frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D}\left|\sum_{j=1}^{n-1}(d_{j+1,n} - d_{j,n}) + d_{n,1}\right|\right) |d_0| \\
&= \left(\left|\frac{-4\lambda_1\sin^2\frac{\beta h}{2} - \lambda_2 + d_1 - d_{n,n} + \lambda_2 D}{4\lambda_1\sin^2\frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D}\right| + \frac{d_{n,n}}{4\lambda_1\sin^2\frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D}\right) |d_0|\,.
\end{aligned}$$

If

$$-4\lambda_1\sin^2\frac{\beta h}{2} - \lambda_2 + d_1 - d_{n,n} + \lambda_2 D > 0,$$

then

$$
\begin{aligned}
|d_{n+1}| &\leq \left(\frac{-4\lambda_1 \sin^2 \frac{\beta h}{2} - \lambda_2 + d_1 - d_{n,n} + \lambda_2 D + d_{n,n}}{4\lambda_1 \sin^2 \frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D} \right) |d_0| \,, \\
|d_{n+1}| &\leq \left(\frac{-4\lambda_1 \sin^2 \frac{\beta h}{2} - \lambda_2 + d_1 + \lambda_2 D}{4\lambda_1 \sin^2 \frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D} \right) |d_0| \,. \qquad (3.13)
\end{aligned}
$$

If

$$
-4\lambda_1 \sin^2 \left(\frac{\beta h}{2} \right) - \lambda_2 + d_1 - d_{n,n} + \lambda_2 D \leq 0,
$$

then

$$
|d_{n+1}| \leq \left(\frac{4\lambda_1 \sin^2 \frac{\beta h}{2} + \lambda_2 - d_1 + 2d_{n,n} - \lambda_2 D}{4\lambda_1 \sin^2 \frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D} \right) |d_0| \ .
$$

Therefore,

$$
\frac{4\lambda_1 \sin^2 \frac{\beta h}{2} + \lambda_2 - d_1 + 2d_{n,n} - \lambda_2 D}{4\lambda_1 \sin^2 \frac{\beta h}{2} + \lambda_2 + d_1 - \lambda_2 D} \leq 1,
$$

that is,

$$
|d_{n+1}| \leq |d_0| \,.
$$

This completed the proof by mathematical induction method. □

Theorem 3.2. *For $\alpha \in (0,1)$,the finite difference scheme in Eq. (3.8) is stable.*

Proof. By using Lemma 3.1 and Eq. (3.10), we get

$$
\|\rho^n\|_2 \leq \|\rho^0\|_2, \qquad n = 1, 2, \ldots, k,
$$

which means that the CrankNicolson difference scheme in Eq. (3.8) is unconditionally stable. □

4. Formation and analysis of the numerical scheme for solving Burgers equation

The finite difference technique is used for numerical approximations. The derivatives are substituted by difference equations utilizing a uniform mesh.

4.1. **Numerical approximation of the fractional derivative.** Discretization of the problem is begun with assuming $h = 1/M$ and $k = T/N$ where M is the grid size in space and N is the grid size in time, and also we consider $x_i = ih, \ i = 0, 1, \ldots, M,$ in interval $[0, X]$, and $t_n = nk, \ n = 0, 1, \ldots, N,$ in interval $[0, T]$.

$$
{}_{0}^{CF}D_t^{\alpha} u\left(x_i, t_n\right) = \frac{M(\alpha)}{1-\alpha} \int_0^{t_n} \frac{\partial u\left(x_i, \tau\right)}{\partial \tau} \exp\left(\frac{-\alpha}{1-\alpha} \left(t_n - \tau\right) \right) d\tau. \qquad (4.1)
$$

Using the CrankNickolson scheme, we approximate the above relationship as follows at (x_i, t_n):

$$
\begin{aligned}
{}_{0}^{CF}D_t^{\alpha} u\left(x_i, t_n\right) &= \frac{M(\alpha)}{1-\alpha} \int_0^{t_n} \frac{\partial u(x_i, \tau)}{\partial \tau} \exp\left(\frac{-\alpha}{1-\alpha} (t_n - \tau) \right) d\tau \\
&= \frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{\partial u(x_i, \tau)}{\partial \tau} \exp\left(\frac{-\alpha}{1-\alpha} (t_n - \tau) \right) d\tau \qquad (4.2) \\
&\approx \frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} \left(\frac{u_i^j - u_i^{j-1}}{\Delta t} + O(\Delta t) \right) \int_{t_{j-1}}^{t_j} \exp\left(\frac{-\alpha}{1-\alpha} (t_n - \tau) \right) d\tau. \qquad (4.3)
\end{aligned}
$$

After solving the integral, we have

$$\int_{t_{j-1}}^{t_j} \exp\left(\frac{-\alpha}{1-\alpha}(t_n-\tau)\right) d\tau = \left(\frac{1-\alpha}{\alpha}\right)\left[\exp\left(-\frac{\alpha}{1-\alpha}(t_n-t_j)\right) - \exp\left(-\frac{\alpha}{1-\alpha}(t_n-t_{j-1})\right)\right]. \tag{4.4}$$

Thus, Equation (4.3) becomes

$$\begin{aligned} {}_0^{CF}D_t^{\alpha}u(x_i,t_n) = &\left[\frac{M(\alpha)}{(1-\alpha)\,\Delta t}\sum_{j=1}^{n}\left(u_i^j-u_i^{j-1}\right)+O(\Delta t)\right] \\ &\times\frac{(1-\alpha)}{\alpha}\left[\exp\left(-\frac{\alpha}{1-\alpha}(t_n-t_j)\right)-\exp\left(-\frac{\alpha}{1-\alpha}(t_n-t_{j-1})\right)\right]. \end{aligned} \tag{4.5}$$

From the above, we obtain

$$\begin{aligned} {}_0^{CF}D_t^{\alpha}u(x_i,t_n) = &\frac{M(\alpha)}{\alpha\,\Delta t}\sum_{j=1}^{n}\left(u_i^j-u_i^{j-1}\right)\left[\exp\left(-\frac{\alpha}{1-\alpha}\Delta t(n-j)\right)-\exp\left(-\frac{\alpha}{1-\alpha}\Delta t(n-j+1)\right)\right] \\ &+O(\Delta t)\,\frac{M(\alpha)}{\alpha\,\Delta t}\sum_{j=1}^{n}\left[\exp\left(-\frac{\alpha}{1-\alpha}\Delta t(n-j)\right)-\exp\left(-\frac{\alpha}{1-\alpha}\Delta t(n-j+1)\right)\right]. \end{aligned} \tag{4.6}$$

This can be rewritten as

$${}_0^{CF}D_t^{\alpha}u(x_i,t_n) = \frac{M(\alpha)}{\alpha\,\Delta t}\left[d_{n,n}u_i^n-\sum_{j=1}^{n-1}\left(d_{j+1,n}-d_{j,n}\right)u_i^j-d_{1,n}u_i^0\right]+O(\Delta t), \tag{4.7}$$

where

$$d_{j,n} = \left[\exp\left(-\frac{\alpha}{1-\alpha}\Delta t(n-j)\right)-\exp\left(-\frac{\alpha}{1-\alpha}\Delta t(n-j+1)\right)\right].$$

4.2. **Approximation using the CrankNicholson scheme.** For the discretization, first we consider the following Burgers equation:

$${}_0^{\mathrm{CF}}D_t^{\alpha}u(x_i,t_n)+u\frac{\partial u(x,t)}{\partial x}-\nu\frac{\partial^2 u(x,t)}{\partial x^2}=g(x,t). \tag{4.8}$$

By substituting the derivative from Eq. (4.7) into the above equation and applying the CrankNicholson scheme, we obtain:

$$\begin{aligned} &\frac{M(\alpha)}{\alpha\Delta t}\left[d_{n,n}u_i^n-\sum_{j=1}^{n-1}(d_{j+1,n}-d_{j,n})u_i^j-d_{1,n}u_i^0\right]+\frac{1}{4\Delta x}\left[,u_i^{n-1}(u_{i+1}^n-u_{i-1}^n)+u_i^n(u_{i+1}^{n-1}-u_{i-1}^{n-1})\right] \\ &-\frac{\nu}{2\Delta x^2}\left[(u_{i+1}^n-2u_i^n+u_{i-1}^n)+(u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1})\right]=\frac{1}{2}\left[g_i^n+g_i^{n-1}\right]. \end{aligned} \tag{4.9}$$

After rearranging the terms in Eq. (4.9), we get

$$\begin{aligned} &\left[d_{n,n}+\frac{2\nu\alpha\Delta t}{2M(\alpha)\Delta x^2}+\frac{\alpha\Delta t}{4M(\alpha)\Delta x}\left(u_{i+1}^{n-1}-u_{i-1}^{n-1}\right)\right]u_i^n+\left[\frac{\alpha\Delta t}{4M(\alpha)\Delta x}u_i^{n-1}-\frac{2\nu\alpha\Delta t}{2M(\alpha)\Delta x^2}\right]u_i^{n+1} \\ &-\left[\frac{\nu\alpha\Delta t}{2M(\alpha)\Delta x^2}+\frac{\alpha\Delta t}{4M(\alpha)\Delta x}u_i^{n-1}\right]u_{i-1}^n-\frac{\alpha\Delta t}{2M(\alpha)}g_i^n \\ &=-\frac{2\nu\alpha\Delta t}{2M(\alpha)\Delta x^2}u_i^{n-1}+\frac{\nu\alpha\Delta t}{2M(\alpha)\Delta x^2}u_{i+1}^{n-1}+\frac{\nu\alpha\Delta t}{2M(\alpha)\Delta x^2}u_{i-1}^{n-1}+\sum_{j=1}^{n-1}(d_{j+1,n}-d_{j,n})u_i^j+d_{1,n}u_i^0+\frac{\alpha\Delta t}{2M(\alpha)}g_i^{n-1}. \end{aligned} \tag{4.10}$$

5. NUMERICAL EXAMPLES

In this section, three numerical examples are presented to demonstrate the efficiency and validation of the proposed algorithm for solving the fractional nonlinear Cable and Burgers equations.

In Example 5.1 and Example 5.3, Neumann boundary conditions are applied at both ends of the domain. To implement these conditions in the proposed method, fictitious points outside the domain are utilized. At the left boundary, the value of the fictitious point u_{-1} is calculated using the central difference approximation and the Neumann condition. Similarly, the value of u_{M+1} at the right boundary is determined. This approach enables the use of central difference approximations for the second derivative at the boundary points.

Example 5.1. Consider the following nonlinear Cable equation:

$$ {}_{0}^{\mathrm{CF}}D_{t}^{\alpha}u(x_i,t_n)=\frac{\partial^2 u(x,t)}{\partial x^2}-u(x,t)+g(x,t,u), $$

where

$$ g(x,t,u)=2t^2\sin(x)+2\sin(x)\left(\frac{(\alpha+\alpha t-1)-(\alpha-1)e^{\alpha t/(\alpha-1)}}{\alpha^2}\right). $$

The exact solution of this example is

$$ u(x,t)=t^2\sin(x), $$

with the initial condition

$$ u(x,0)=0, \qquad 0\le x\le 1, $$

and the Neumann boundary conditions

$$ u_x(0,t)=0, \qquad u_x(1,t)=0, \qquad t\ge 0. $$

TABLE 1. Numerical results of solving the Cable equation Example 5.1 for different values of N and α.

α	N	M	Max Absolute Error
0.5	25	10	1.3000E-3
	50	10	3.2418E-4
	100	10	8.2545E-5
0.7	25	10	1.2000E-3
	50	10	3.2015E-4
	100	10	8.2018E-5
0.9	25	10	1.1000E-3
	50	10	3.0103E-4
	100	10	7.9450E-5

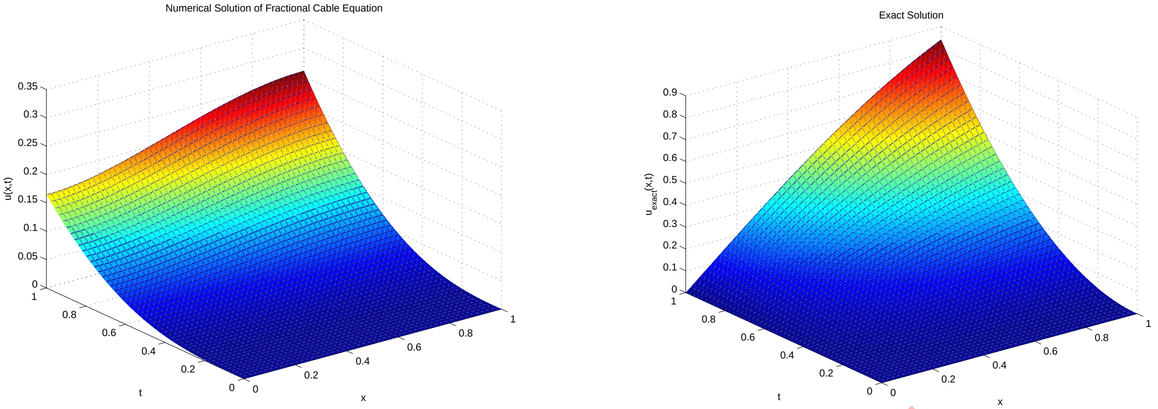
The result of the approximation of solution (left) and exact solution (right) is shown in Figure 1 for $\alpha = 0.5$ ($M = 50$, $N = 50$) when $t = 1$ in Example 5.1($M(\alpha) = 1$). Table 1 gives the approximation absolute errors for the proposed scheme. In Example 5.1, we see that the errors decrease with decreasing time steps. The proposed method provides accurate performance under Neumann boundary conditions.

Example 5.2. We consider the following Burgers equation:

$$ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+u\frac{\partial u(x,t)}{\partial x}-\nu\frac{\partial^2 u(x,t)}{\partial x^2}=g(x,t), $$

with boundary conditions

$$ u(0,t)=t^2, \qquad u(1,t)=-t^2, \qquad t\ge 0, $$



(a) Numerical solution. (b) Exact solution.

FIGURE 1. Solutions of $u(x,t)$ where $\alpha = 0.5$, $\Delta t = 0.02$, and $\Delta x = 0.02$.

and the initial condition

$$u(x,0) = 0, \qquad 0 \leq x \leq 1,$$

where

$$g(x,t) = 2(1-\alpha)\lambda^2 \cos(\pi x)\left(\lambda t - 1 + e^{-\lambda t}\right) - \pi t^4 \cos(\pi x)\sin(\pi x) + \nu \pi^2 t^2 \cos(\pi x),$$

where $\lambda = \frac{\alpha}{1-\alpha}$. The exact solution of this problem is $u(x,t) = t^2 \cos(\pi x)$.

TABLE 2. Numerical results of solving the Burgers equation for different values of N and α with $\nu = 1$.

α	N	M	Max Absolute Error
0.5	10	10	3.7576E-18
	20	10	1.5432E-18
	30	10	5.642E-19
0.7	10	10	9.4694E-19
	20	10	1.4750E-18
	30	10	4.9254E-19
0.9	10	10	2.8040E-18
	20	10	5.7654E-19
	30	10	4.6040E-19

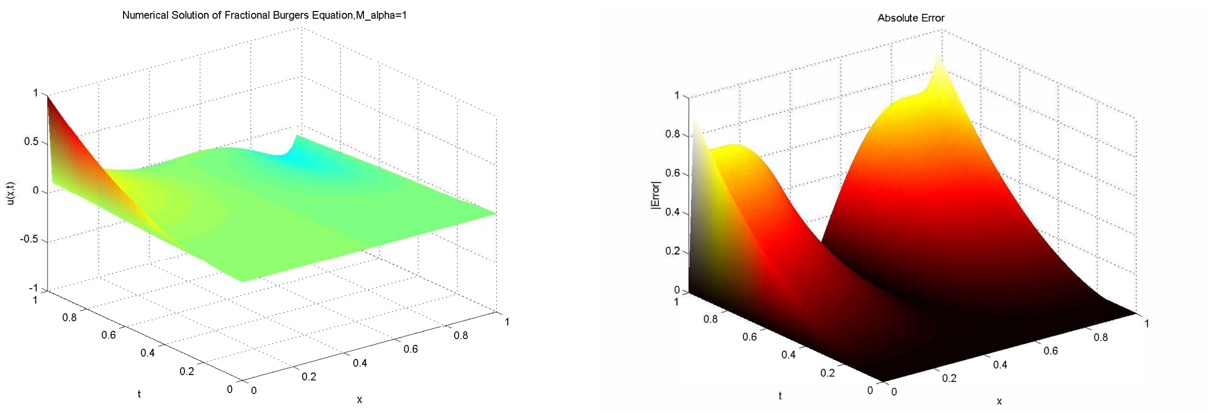
Figure 2 shows the numerical solution (left) and the absolute error (right) for $\alpha = 0.7$ with $M = 50$ and $N = 200$ at $t = 1$. Table 2 reports the maximum absolute errors of the proposed difference scheme for several N and α. In these tests we set $M(\alpha) = 1$ and $\nu = 1$.

Example 5.3. Consider the nonlinear Burgers equation in Example 5.2,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u\frac{\partial u(x,t)}{\partial x} - \nu\frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t),$$

where

$$g(x,t) = \frac{2(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}} - 1 + \frac{\alpha t}{1-\alpha}\right)}{\alpha^2(1-\alpha)} \sin(2\pi x) + 2\pi^4 \sin(2\pi x) + 4\pi^2 \nu t^2 \sin(2\pi x).$$



(a) Numerical solution.

(b) Absolute error.

FIGURE 2. Numerical solution of $u(x,t)$ and the absolute error at $\alpha = 0.9$, $\Delta t = 0.05$, and $\Delta x = 0.1$.

The exact solution of this problem is $u(x,t) = t^2 \sin(2\pi x)$, with the initial condition

$$u(x,0) = 0, \qquad 0 \le x \le 1,$$

and Neumann boundary conditions

$$u_x(0,t) = 0, \qquad u_x(1,t) = 0, \qquad t \ge 0.$$

TABLE 3. Numerical results of solving the Burgers equation Example 5.3 for several values of N, α, and $\nu = 0.01$.

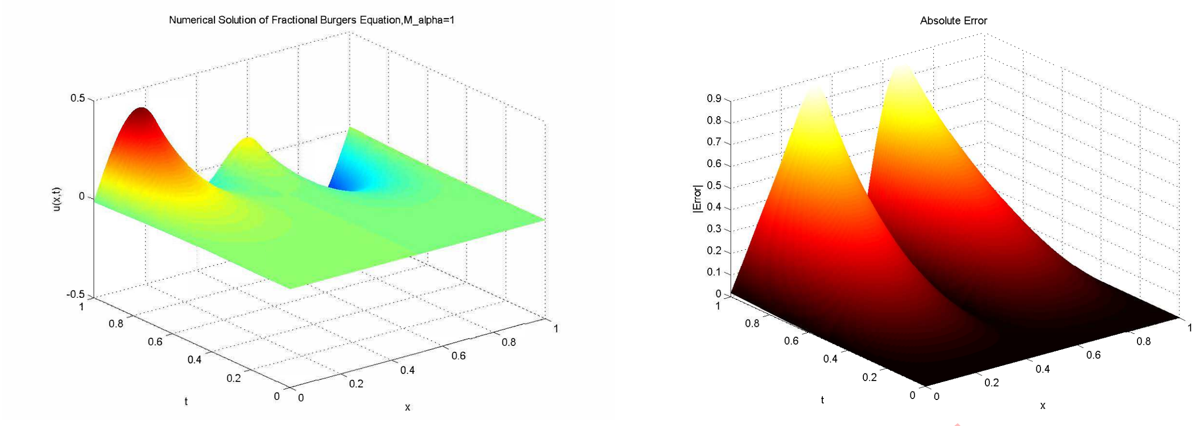
α	N	M	Max Absolute Error
0.1	50	50	3.9323e-4
	100	50	9.9053e-5
	200	50	2.4857e-5
0.5	50	50	3.8974e-4
	100	50	9.8613e-5
	200	50	2.4802e-5
0.9	50	50	3.6016e-4
	100	50	9.4772e-5
	200	50	2.4312e-5

The approximation of solution (left) and the absolute error (right) for $\alpha = 0.1$ ($M = 100$, $N = 1000$, $t = 1$) are shown in Figure 3. Table 3 gives the absolute errors for the Crank–Nicholson scheme.

We observe that the errors decrease with decreasing time steps. The numerical results show that the Neumann boundary condition can model the physical behavior of boundaries more realistically and confirm that the difference method remains effective even with more complex boundary conditions.

6. CONCLUSION

This research presented an implicit finite difference approach based on the Crank–Nicholson scheme combined with the Caputo–Fabrizio fractional time derivative for solving the nonlinear fractional cable equation. A stability analysis based on the Fourier method demonstrated that the proposed difference scheme is unconditionally stable.



(a) Numerical solution.

(b) Absolute error.

Figure 3. The numerical solution of $u(x,t)$ and the absolute error at $\alpha = 0.1$, $\Delta t = 0.001$, and $\Delta x = 0.01$.

The effectiveness of the method was verified through its application to problems with both Dirichlet and Neumann boundary conditions. Numerical experiments showed that the scheme is not only reliable under standard boundary conditions such as Dirichlet, but also maintains high accuracy and stability in more complex cases involving Neumann boundaries.

In addition to the cable equation, the fractional Burgers equation was also investigated within the same numerical framework. The successful results obtained confirm the flexibility and robustness of the proposed method for solving different classes of nonlinear fractional differential equations. The numerical solutions showed strong agreement with the analytical ones, and the absolute error analysis for various values of α and N further confirmed the stability and accuracy of the algorithm for both models.

All numerical expriments were performed in MATLAB, demonstrating the simplicity of implementation and the computational efficiency of the proposed scheme. Overall, the proposed method provides a flexible and efficient numerical tool for solving a wide range of fractional differential problems and establishes a solid foundation for future research on more advanced fractional dynamical systems.

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