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On a discrete collocation method for solving nonlinear system of two-dimensional integral equations

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Abstract

This paper is concentrated on an investigation around a numerical scheme that gives us more precise approximations of the analytic solutions of nonlinear systems of two-dimensional integral equations in comparison with some other numerical strategies cited in the literature during recent past years. In this way, the Lagrange interpolation function together with the Legendre–Gauss quadrature formula are chosen to reach our aim. As will be observed, this method enables us to transform under study nonlinear systems of integral equations into the corresponding nonlinear algebraic systems and consequently, with the help of some standard numerical procedures such as the Newton's method for solving matrix forms of the nonlinear algebraic equations, the solutions of the obtained nonlinear algebraic systems will be obtained. The advantage of the method is that it requires relatively few collocation points to obtain a relatively small error and does not require the calculation of integrals. Thanks to these advantages, we expect to reach small and smaller error bounds for the approximated solution of the two-dimensional integral equations that is presented in frame of the convergence analysis. At the end, some illustrative numerical applications are given to justify the practical efficiency of the proposed collocation technique to numerically solve the system of two-dimensional nonlinear integral equations.

Keywords. System of integral equations, Two-dimensional nonlinear integral equation, Collocation method, Convergence analysis.

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1. INTRODUCTION

In view point of the mathematical sciences connected with the dynamical systems, one can translate the nonlocal differential equation systems into their equivalent systems of the local integral equations. On the other hand, system of integral equations appears in various fields of science and engineering disciplines [1, 17, 20, 43]. In this position, we can concentrate ourselves into a very serious issue that is most of time, obtaining the analytic solutions of these dynamical (differential/integral) systems is a very complicated process and even it maybe impossible. In these situations we attempt to get help from numerical techniques that enable us to extract an approximation instead of the analytic solutions. So, this is logically reasonable that every where we are dealt with the approximation, we have to expect to present more and more accurate numerical schemes that give us possibly the best approximated solutions that are preferred substitution items in our analysis. Here, we are in the approximation theory authority where each appropriate function can be represented by linear combinations of the special families of the simple and fundamental functions. In view point of the function spaces, generally these families are known as the basis of the related function spaces.

We consider the nonlinear systems of two-dimensional Volterra-Fredholm integral equations in general form as

$$\mathbf{V}(s,t)=\mathbf{F}(s,t)+\int_0^t\int_0^1\mathbf{K}\big(s,t,\sigma,\tau\big)\mathbf{h}\big(\mathbf{V}(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau,\quad (s,t)\in\Omega:=[0,1]\times[0,1], \tag{1.1}$$

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and also nonlinear system of two-dimensional Volterra and Fredholm integral equations of the forms

$$\mathbf{V}(s,t) = \mathbf{F}(s,t) + \int_0^t \int_0^s \mathbf{K}\big(s,t,\sigma,\tau\big)\mathbf{h}\big(\mathbf{V}(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \quad (s,t) \in \Omega, \tag{1.2}$$

$$\mathbf{V}(s,t) = \mathbf{F}(s,t) + \int_0^1 \int_0^1 \mathbf{K}\big(s,t,\sigma,\tau\big)\mathbf{h}\big(\mathbf{V}(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \quad (s,t) \in \Omega, \tag{1.3}$$

where the kernel matrix $\mathbf{K}(s,t,\sigma,\tau) = \begin{pmatrix} k_{ij}(s,t,\sigma,\tau) \\ i,j = 1,\dots,d \end{pmatrix}$ and $\mathbf{F}(s,t) = \big(f_1(s,t),\dots,f_d(s,t)\big)^T$, with $d \geq 2$ are sufficiently smooth and $\mathbf{h}(\mathbf{V}(\sigma,\tau)) = \big(h_1(\mathbf{V}(\sigma,\tau)),\dots,h_d(\mathbf{V}(\sigma,\tau))\big)^T$ is nonlinear in $\mathbf{V}(\sigma,\tau)$ where $\mathbf{V}(s,t) = \big(v_1(s,t), \dots, v_d(s,t)\big)^T$ is the unknown vector solution to be determined.

In the past decades, several numerical techniques and basic functions have been devoted to estimate the solution of system of one-dimensional integral equations such as modified homotopy perturbation method [18], semiorthogonal B-spline wavelet collocation method [38], collocation approach [11, 37], analytic method [42], rationalized Haar functions method [27], discrete collocation method [35], Bernstein polynomials and hybrid Bernstein Block-Pulse functions [16], Hybrid Legendre Block-Pulse functions [39] and hat basis functions [7]. Also, in recent years, a variety of numerical methods have been developed for solving multi-dimensional integral equations [4, 5, 10, 19, 29–31, 33, 34]. Several representative approaches are summarized as follows. In [8], a set of two-dimensional orthogonal triangular basis functions was employed to approximate the solutions of nonlinear two-dimensional integral equations. A block-by-block approach for obtaining numerical solutions of nonlinear two-dimensional Volterra integral equations was presented in [32]. Based on the Haar wavelet, a numerical technique for solving two-dimensional nonlinear Fredholm, Volterra, and Volterra–Fredholm integral equations of the first and second kinds was introduced in [3]. A discrete collocation method utilizing radial basis functions for nonlinear two-dimensional Volterra–Fredholm integral equations of the second kind was discussed in [21]. The Bernoulli operational matrix method for solving Hammerstein-type nonlinear two-dimensional Volterra–Fredholm integral equations was investigated in [9]. In [28], a hybrid functions approach was applied to handle two-dimensional nonlinear Volterra–Fredholm integral equations, while the Taylor collocation method was adopted in [22] to address both linear and nonlinear two-dimensional integral equations. In comparison, only a few studies have addressed numerical approaches for solving nonlinear systems of two-dimensional integral equations [6, 14, 26]. In [6], Hes homotopy perturbation method was employed to solve nonlinear systems of two-dimensional Volterra–Fredholm integral equations. Ghasemi et al. [14] proposed an analytical technique for obtaining the solutions of such systems.

On the other hand, the Lagrange interpolation collocation method has attracted considerable attention due to its high accuracy and capability to handle irregular domains [2, 12, 36, 40, 45, 47]. The corresponding interpolation formula exhibits excellent numerical stability and can achieve machine-precision approximation for arbitrary smooth functions. Moreover, extended numerical techniques derived from Lagrange interpolation have been successfully applied to the numerical solution of partial differential equations [24, 46], nonlinear integral equations [23, 25], and time-varying delayed differential systems [44], among others. Among various quadrature techniques, the Legendre–Gauss quadrature formula is one of the most widely employed for solving integral equations. Motivated by these advantages, we aim to develop a numerical scheme that combines the Lagrange interpolation function with the Legendre–Gauss quadrature formula.

The main goal of this paper is to discuss nonlinear systems of two-dimensional integral equations. Under certain conditions, the existence and uniqueness of a solution to these equations will be proved by using the Banach fixed point theorem. After some suitable linear transformations, a discrete collocation method based on Gauss quadrature formulas and Lagrange interpolation function will be constructed to solve the nonlinear system of Volterra–Fredholm, Volterra and Fredholm integral Eqs. (1.1)–(1.3). Then, the error bound and convergence analysis of the proposed method will be given. Finally, the numerical results of some examples will illustrate the effectiveness of the method. It is worth noting that the proposed method utilizes the Gauss-Legendre quadrature formula to approximate the integrals without directly calculating any integral. In comparison with the methods in the literature, our method has simple and easy convergence analysis and achieves desired accuracy by using appropriate values of M and N.

2. EXISTENCE AND UNIQUENESS OF THE SOLUTION

First of all, for a matrix function

$$\mathbf{A}(s,t,\sigma,\tau)=\begin{pmatrix} a_{ij}(s,t,\sigma,\tau)\\ i=1,\dots,I\\ j=1,...,J\end{pmatrix},$$

using matrix norm $\|\cdot\|_\infty$, we define the following norm

$$\begin{aligned}\|\mathbf{A}(s,t,\sigma,\tau)\|_\infty &= \left\|\begin{pmatrix} \|a_{ij}(s,t,\sigma,\tau)\|_\infty\\ i=1,\dots,I\\ j=1,...,J\end{pmatrix}\right\|_\infty\\ &=\max_{1\le i\le I}\Big\{\sum_{j=1}^{J}\sup_{(s,t)\in\Omega}\Big\{\max_{(\sigma,\tau)\in\Omega}|a_{ij}(s,t,\sigma,\tau)|\Big\}\Big\}, \qquad (2.1)\end{aligned}$$

using the norm on the space of sufficiently smooth functions on Ω. Wherever no ambiguity arises, we will use the symbol $\|\cdot\|_\infty$ to denote all three matrix norms, real-valued function norm, and matrix function norm. In case of ambiguity, we specify the type of norm.

In order to describe the key ideas without having to resort to complex notations, we will consider the system (1.1) with $d=2$. Consider the following hypothesis

(i) $v_i:\Omega=[0,1]\times[0,1]\to\mathbb{R},(i=1,2)$, are continuous functions,

(ii) $k_{ij}:\Omega^2\to\mathbb{R},(i,j=1,2)$ are continuous and there exist constants $K_{ij}>0$ in a such way

$$K_{ij}=\sup_{(s,t)\in\Omega}\Big\{\max_{(\sigma,\tau)\in\Omega}|k_{ij}(s,t,\sigma,\tau)|\Big\}<\infty,\quad i,j=1,2.$$

(iii) $h_1,h_2:\Omega\to\mathbb{R}$ are continuous and there exist real nonnegative constants $q_j,(j=1,2)$ such that

$$\begin{aligned}&\big\|h_1(\mathbf{V}_1)-h_1(\mathbf{V}_2)\big\|_\infty\le q_1\|\mathbf{V}_1-\mathbf{V}_2\|_\infty,\\ &\big\|h_2(\mathbf{V}_1)-h_2(\mathbf{V}_2)\big\|_\infty\le q_2\|\mathbf{V}_1-\mathbf{V}_2\|_\infty,\qquad \mathbf{V}_1,\mathbf{V}_2\in\mathbb{R}^2.\end{aligned}$$

Theorem 2.1. *Assuming that the conditions (i)-(iii) hold, and* $\sum_{j=1}^{2}K_{ij}q_j<1,(i=1,2)$. *Then, there is only one solution for Eq.* (1.1).

Proof. Let

$$(\mathcal{G}\mathbf{V})(s,t)=\mathbf{F}(s,t)+\int_0^t\int_0^1\mathbf{K}(s,t,\sigma,\tau)\mathbf{h}\big(\mathbf{V}(s,t)\big)\mathrm{d}\sigma\mathrm{d}\tau,$$

Since $h_1\big(\mathbf{V}(\sigma,\tau)\big),h_2\big(\mathbf{V}(\sigma,\tau)\big)\in C(\Omega\times\mathbb{R})$, therefore $\mathcal{G}$ is a continuous mapping of $C(\Omega)$ into $C(\Omega)$. Now, for $\mathbf{V}_1,\mathbf{V}_2\in C(\Omega)$, we have

$$\begin{aligned}\|\mathcal{G}\mathbf{V}_1-\mathcal{G}\mathbf{V}_2\|_\infty&=\left\|\int_0^t\int_0^1\mathbf{K}(s,t,\sigma,\tau)\big(\mathbf{h}(\mathbf{V}_1(s,t))-\mathbf{h}(\mathbf{V}_2(s,t))\big)\mathrm{d}\sigma\mathrm{d}\tau\right\|_\infty\\ &\le\left\|\int_0^t\int_0^1|\mathbf{K}(s,t,\sigma,\tau)||\mathbf{h}(\mathbf{V}_1(s,t))-\mathbf{h}(\mathbf{V}_2(s,t))|\mathrm{d}\sigma\mathrm{d}\tau\right\|_\infty\\ &\le\left\|\begin{pmatrix}K_{11}&K_{12}\\K_{21}&K_{22}\end{pmatrix}\begin{pmatrix}q_1&0\\0&q_2\end{pmatrix}\|\mathbf{V}_1-\mathbf{V}_2\|_\infty\right\|_\infty\\ &\le\left\|\begin{pmatrix}K_{11}q_1&K_{12}q_2\\K_{21}q_1&K_{22}q_2\end{pmatrix}\right\|_\infty\|\mathbf{V}_1-\mathbf{V}_2\|_\infty.\end{aligned}$$

Consequently, by assumptions $\sum_{j=1}^{2}K_{ij}q_j<1,(i=1,2)$, operator $\mathcal{G}$ is a contraction mapping and the nonlinear system (1.1) has a unique solution. □

3. IMPLEMENTATION OF THE NUMERICAL METHOD

This section provides a thorough theoretical examination of the numerical method that relies on approximating solutions of the different categories of nonlinear system of two-dimensional integral equations using two-dimensional Lagrange interpolation, collocation method and numerical integration. It is important to note that the numerical approach will be applied to systems with dimension $d = 2$, and the unknown vector function $V(s,t) \in L^2([0,1] \times [0,1])$ is a solution that can be expressed as

$$\mathbf{V}(s,t) = \big(v_1(s,t), v_2(s,t)\big)^T.$$

3.1. **System of two-dimensional nonlinear Fredholm integral equations.** We consider again the system of two-dimensional nonlinear Fredholm integral equation

$$\mathbf{V}(s,t) = \mathbf{F}(s,t) + \int_0^1 \int_0^1 \mathbf{K}\big(s,t,\sigma,\tau\big)\mathbf{h}\big(\mathbf{V}(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \quad (s,t) \in \Omega. \tag{3.1}$$

First of all, in order to use the numerical integration formula, we need to transform the integrals over $[0,1]$ to the integrals over $[-1,1]$ using the following linear mappings

$$s = \frac{1+x}{2}, \quad t = \frac{1+y}{2}, \quad x, y \in [-1,1],$$
$$\sigma = \frac{1+\xi}{2}, \quad \tau = \frac{1+\eta}{2}, \quad \xi, \eta \in [-1,1].$$

So, Eq. (3.1) converts to the following equation

$$\mathbf{U}(x,y) = \mathbf{G}(x,y) + \int_{-1}^1 \int_{-1}^1 \mathbf{\Phi}\big(x,y,\xi,\eta\big)\mathbf{H}\big(\mathbf{U}(\xi,\eta)\big)\mathrm{d}\xi\mathrm{d}\eta, \tag{3.2}$$

where

$$\mathbf{U}(x,y) = \mathbf{V}\left(\frac{1+x}{2}, \frac{1+y}{2}\right),$$
$$\mathbf{G}(x,y) = \mathbf{F}\left(\frac{1+x}{2}, \frac{1+y}{2}\right),$$
$$\mathbf{\Phi}\big(x,y,\xi,\eta\big) = \left(\frac{1}{2}\right)^2 \mathbf{K}\left(\frac{1+x}{2}, \frac{1+y}{2}, \frac{1+\xi}{2}, \frac{1+\eta}{2}\right),$$
$$\mathbf{H}(\mathbf{U}(\xi,\eta)) = \mathbf{h}\left(\mathbf{V}\left(\frac{1+\xi}{2}, \frac{1+\eta}{2}\right)\right).$$

In order to discretize Eq. (3.2), the first step is to identify an appropriate collocation points system. Here we use the zeros of first kind Chebyshev polynomials over the interval $[-1,1]$, that is we will use $(M+1)(N+1)$ collocation points

$$(x_i, y_j) = \left(\cos\left(\frac{2i+1}{2M+2}\pi\right), \cos\left(\frac{2j+1}{2N+2}\pi\right)\right), \; i = 0,1,\dots,M, \; j = 0,1,\dots,N.$$

Imposing the above collocation points into the Eq. (3.2), we arrive at the following nonlinear system:

$$\mathbf{U}(x_i,y_j) = \mathbf{G}(x_i,y_j) + \int_{-1}^1 \int_{-1}^1 \Phi\big(x_i,y_j,\xi,\eta\big)\mathbf{H}\big(\mathbf{U}(\xi,\eta)\big)\mathrm{d}\xi\mathrm{d}\eta, \qquad i = 0,1,\dots,M, \; j = 0,1,\dots,N. \tag{3.3}$$

In this position, we approximate the integrals in recent nonlinear system using the Gauss-Legendre quadrature formula, yields the following nonlinear algebraic system:

$$\widetilde{\mathbf{U}}(x_i,y_j) = \mathbf{G}(x_i,y_j) + \sum_{l=0}^{N} \sum_{k=0}^{M} w_l w_k \Phi\big(x_i,y_j,\xi_k,\eta_l\big)\mathbf{H}\big(\widetilde{\mathbf{U}}(\xi_k,\eta_l)\big), \tag{3.4}$$

such that ξ_k and η_l denote the zeros of the Legendre polynomials having orders $M+1$ and $N+1$, respectively. Also, w_k and w_l stand for the corresponding weights. Then, we commence with $I_{MN}[\mathbf{U}](x,y)$ to indicate the approximate value for $\widetilde{\mathbf{U}}(x_i,y_j)$ as

$$I_{MN}[\widetilde{\mathbf{U}}](x,y) = \big(I_{MN}[\tilde{u}_1](x,y), I_{MN}[\tilde{u}_2](x,y)\big)^T, \tag{3.5}$$

in which, each component takes the following form

$$\begin{aligned} I_{MN}[\tilde{u}_1](x,y) &= \sum_{n=0}^{N}\sum_{m=0}^{M} \ell_m(x)\ell_n(y)\tilde{u}_1(x_m,y_n), \\ I_{MN}[\tilde{u}_2](x,y) &= \sum_{n=0}^{N}\sum_{m=0}^{M} \ell_m(x)\ell_n(y)\tilde{u}_2(x_m,y_n), \end{aligned}$$

where $\ell_m(x), \ell_n(y)$ are well-known Lagrange polynomials. Combining Eqs. (3.4) and (3.5), we will have

$$\widetilde{\mathbf{U}}(x_i,y_j) = \mathbf{G}(x_i,y_j) + \sum_{l=0}^{N}\sum_{k=0}^{M} w_l w_k \Phi\big(x_i,y_j,\xi_k,\eta_l\big) H\big(I_{MN}[\widetilde{\mathbf{U}}](\xi_k,\eta_l)\big). \tag{3.6}$$

The aforementioned nonlinear algebraic system gives $(M+1)(N+1)$ nonlinear algebraic equations in hand, with $(M+1)(N+1)$ unknowns to be identified precisely. To this aim, we use the Newton's iterative method to solve nonlinear system (3.6). Afterward, the approximated solution $I_{MN}[\mathbf{U}]$ of Eq. (3.2) will be characterized exactly from Eq. (3.6), and the approximate solution $\mathbf{V}_{MN}(s,t)$ of Eq. (3.1) can be obtained by $\mathbf{V}_{MN}(s,t) = I_{MN}[\widetilde{\mathbf{U}}](2s-1, 2t-1)$, where

$$\mathbf{V}_{MN}(s,t) = \big(v^1_{MN}(s,t), v^2_{MN}(s,t)\big)^T.$$

3.2. **System of two-dimensional nonlinear Volterra integral equations.** In this section, we are dealt with the nonlinear system of two-dimensional Volterra integral equations of the second kind. We reduce this system into a nonlinear system of algebraic equations by the same method presented in Section 3.1. For this purpose, we consider desired nonlinear system of two-dimensional Volterra integral equations that will be evaluated later in this section, as

$$\mathbf{V}(s,t) = \mathbf{F}(s,t) + \int_0^t\int_0^s \mathbf{K}\big(s,t,\sigma,\tau\big)\mathbf{h}\big(\mathbf{V}(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \quad (s,t)\in\Omega. \tag{3.7}$$

Similar to the previous section, in order to use the numerical integration formula, we need to transform the integrals over $[0,s]$ and $[0,t]$ to the integrals over $[-1,1]$ using the following linear transformations

$$\begin{aligned} &s = \frac{1+x}{2}, \quad t = \frac{1+y}{2}, \quad x,y\in[-1,1], \\ &\sigma = \frac{1+\xi}{2}, \quad \tau = \frac{1+\eta}{2}, \quad \xi\in[-1,x], \ \ \eta\in[-1,y]. \end{aligned}$$

then (3.7) is converted as the following equivalent equation

$$\mathbf{U}(x,y) = \mathbf{G}(x,y) + \int_{-1}^{y}\int_{-1}^{x} \Phi\big(x,y,\xi,\eta\big)\mathbf{H}\big(\mathbf{U}(\xi,\eta)\big)\mathrm{d}\xi\mathrm{d}\eta, \tag{3.8}$$

where the vector functions $\mathbf{U}, \mathbf{G}$ and Φ are given like previous section.
Next, we shall impose the collocation system of the points (x_i,y_j), $i=0,1,2,\dots,M,\ j=0,1,2,\dots,N$, that are the zeros of the first kind Chebyshev polynomials over the interval $[-1,1]$, as previous section. In this case, we reach the following nonlinear two-dimensional semi-discretized system:

$$\mathbf{U}(x_i,y_j) = \mathbf{G}(x_i,y_j) + \int_{-1}^{y_j}\int_{-1}^{x_i} \Phi\big(x_i,y_j,\xi,\eta\big)\mathbf{H}\big(\mathbf{U}(\xi,\eta)\big)\mathrm{d}\xi\mathrm{d}\eta, \tag{3.9}$$

Now, in order to obtain a nonlinear algebraic system, we need another step to fully discretized the semi-discretized system (3.9), that is we have to transform the integration intervals $[-1, x_i]$ and $[-1, y_j]$ into the new symmetric integration intervals $[-1, 1]$. To this aim, we make use the following linear mappings:

$$\xi = \xi(x_i, \theta) = \frac{1+x_i}{2}\theta + \frac{x_i - 1}{2}, \quad \theta \in [-1, 1]\,,$$
$$\eta = \eta(y_j, \rho) = \frac{1+y_j}{2}\rho + \frac{y_j - 1}{2}, \quad \rho \in [-1, 1]\,.$$

After implementing all of the above mentioned settings, the Eq. (3.9) is transformed into the following secondary semi-discretized nonlinear system:

$$\mathbf{U}(x_i, y_j) = \mathbf{G}(x_i, y_j) + \int_{-1}^{1}\int_{-1}^{1} \widetilde{\Phi}\big(x_i, y_j, \xi(x_i, \theta), \eta(y_j, \rho)\big)\mathbf{H}\big(\mathbf{U}(\xi(x_i, \theta), \eta(y_j, \rho))\big)\mathrm{d}\theta\mathrm{d}\rho, \tag{3.10}$$

where

$$\widetilde{\Phi}\big(x, y, \xi, \eta\big) = \frac{1+x}{2}\frac{1+y}{2}\Phi\big(x, y, \xi, \eta\big).$$

In this position, it is the Gauss-Legendre quadrature rule that is needed to apply in the nonlinear system (3.10) to reach the following nonlinear algebraic system:

$$\widetilde{\mathbf{U}}(x_i, y_j) = \mathbf{G}(x_i, y_j) + \sum_{l=0}^{N}\sum_{k=0}^{M} w_l w_k \widetilde{\Phi}\big(x_i, y_j, \xi(x_i, \theta_k), \eta(y_j, \rho_l)\big)\mathbf{H}\big(\widetilde{\mathbf{U}}\big(\xi(x_i, \theta_k), \eta(y_j, \rho_l)\big)\big), \tag{3.11}$$

where the quadrature nodes $\{\theta_k\}_{k=0}^{M}$ and $\{\rho_l\}_{l=0}^{N}$ are the zeros of Legendre polynomials of orders $M+1$ and $N+1$, respectively and w_k and w_l are the corresponding weights. Similar to the previous section, thanks to the approximation (3.5), we get

$$\widetilde{\mathbf{U}}(x_i, y_j) = \mathbf{G}(x_i, y_j) + \sum_{l=0}^{N}\sum_{k=0}^{M} w_l w_k \widetilde{\Phi}\big(x_i, y_j, \xi(x_i, \theta_k), \eta(y_j, \rho_l)\big)\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}]\big(\xi(x_i, \theta_k), \eta(y_j, \rho_l)\big)\big), \tag{3.12}$$

which is a nonlinear system of algebraic equations. To solve the above system, we apply the Newtons iterative technique to calculate all of the unknown parameters precisely, that is the approximated solution $I_{MN}[\mathbf{U}]$ of Eq. (3.7) will be obtained from Eq. (3.12), and the approximate solution $\mathbf{V}_{MN}(s, t)$ of Eq. (3.7) can be resulted by $\mathbf{V}_{MN}(s, t) = I_{MN}[\widetilde{\mathbf{U}}](2s-1, 2t-1)$.

3.3. **System of two-dimensional nonlinear Volterra-Fredholm integral equations.** As will be seen, since in this section we are dealt with the nonlinear system of two-dimensional mixed Volterra-Fredholm integral equations

$$\mathbf{V}(s, t) = \mathbf{F}(s, t) + \int_{0}^{t}\int_{0}^{1} \mathbf{K}\big(s, t, \sigma, \tau\big)\mathbf{h}\big(\mathbf{V}(\sigma, \tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \quad (s, t) \in \Omega. \tag{3.13}$$

so this section can be considered as an extension of the previous sections. By using the relevant linear transformations, we will get the following system over $[-1, 1]$,

$$\mathbf{U}(x, y) = \mathbf{G}(x, y) + \int_{-1}^{y}\int_{-1}^{1} \Phi\big(x, y, \xi, \eta\big)\mathbf{H}\big(\mathbf{U}(\xi, \eta)\big)\mathrm{d}\xi\mathrm{d}\eta, \tag{3.14}$$

where the vector functions $\mathbf{U}, \mathbf{G}, \Phi$ and $\mathbf{H}$ are given as in section 3.1. Then, Eq. (3.14) at collocation points (x_i, y_j), $i = 0, 1, 2, \dots, M$, $j = 0, 1, 2, \dots, N$, can be written as

$$\mathbf{U}(x_i, y_j) = \mathbf{G}(x_i, y_j) + \int_{-1}^{y_j}\int_{-1}^{1} \Phi\big(x_i, y_j, \xi, \eta\big)\mathbf{H}\big(\mathbf{U}(\xi, \eta)\big)\mathrm{d}\xi\mathrm{d}\eta. \tag{3.15}$$

In this position, by using the linear mapping

$$\eta = \eta(y_j, \rho) = \frac{1+y_j}{2}\rho + \frac{y_j - 1}{2}, \quad \rho \in [-1, 1]\,.$$

Eq. (3.15) is transformed into the following semi-discretized nonlinear system:

$$\mathbf{U}(x_i, y_j) = \mathbf{G}(x_i, y_j) + \int_{-1}^{1}\int_{-1}^{1} \widetilde{\Phi}(x_i, y_j, \xi, \eta(y_j, \rho))\mathbf{H}\left(\mathbf{U}(\xi, \eta(y_j, \rho))\right) \mathrm{d}\xi \mathrm{d}\rho, \tag{3.16}$$

where

$$\widetilde{\Phi}(x, y, \xi, \eta) = \frac{1+y}{2}\Phi(x, y, \xi, \eta).$$

Replacing the integrals on the right by the Gauss-Legendre quadrature formula and disregarding the quadrature errors, obtains

$$\widetilde{\mathbf{U}}(x_i, y_j) = \mathbf{G}(x_i, y_j) + \sum_{l=0}^{N}\sum_{k=0}^{M} w_l w_k \widetilde{\Phi}(x_i, y_j, \xi_k, \eta(y_j, \rho_l))\mathbf{H}\big(\widetilde{\mathbf{U}}(\xi_k, \eta(y_j, \rho_l))\big), \tag{3.17}$$

where $\widetilde{\mathbf{U}}(x_i, y_j)$ is the approximation for $\mathbf{U}(x_i, y_j)$, $\{\xi_k\}_{k=0}^{M}, \{\rho_l\}_{l=0}^{N}$ are quadrature nodes and w_k, w_l are the corresponding weights.
Now, thanks to the Eq. (3.5), we get

$$\widetilde{\mathbf{U}}(x_i, y_j) = \mathbf{G}(x_i, y_j) + \sum_{l=0}^{N}\sum_{k=0}^{M} w_l w_k \widetilde{\Phi}(x_i, y_j, \xi_k, \eta(y_j, \rho_l))\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi_k, \eta(y_j, \rho_l))\big), \tag{3.18}$$

which is a nonlinear system of algebraic equations. By solving the above system by a suitable method such as Newton's iterative method, the approximate solution of (3.13) can be obtained by taking $\mathbf{V}_{MN}(s, t) = I_{MN}[\widetilde{\mathbf{U}}](2s - 1, 2t - 1)$.

4. CONVERGENCE ANALYSIS AND ERROR ESTIMATE

In this section we finalize the main theoretical results of this paper. To this end, we present an error bound for the approximated solutions of the two-dimensional nonlinear Volterra-Fredholm integral equations system (3.13). We first review some lemmas of the interpolation approximation which make up the main errors of the proposed method.

Lemma 4.1 ([13]). *Suppose that* $u(x, y) = v\left(\frac{1+x}{2}, \frac{1+y}{2}\right)$ *stands for a sufficiently smooth function defined on* $[-1, 1]^2$ *and* $I_{MN}[u](x, y)$ *is the interpolating polynomial to* u *at points* (x_i, y_j). *Then, for* $\sigma, \tau, \bar{\sigma}, \bar{\tau} \in [0, 1]$, *the following interpolation error bound is satisfied:*

$$\|u - I_{MN}[u]\|_\infty \le \frac{C_1\pi}{2^{M+3}(M+1)^{M+1}} + \frac{C_2\pi}{2^{N+3}(N+1)^{N+1}} + \frac{C_3\pi^2}{2^{M+N+6}(M+1)^{M+1}(N+1)^{N+1}},$$

where

$$C_1 = \sup_{t\in[0,1]}\left\{\max_{\sigma\in[0,1]}\left|\frac{\partial^{M+1}v(\sigma, t)}{\partial s^{M+1}}\right|\right\},$$

$$C_2 = \sup_{s\in[0,1]}\left\{\max_{\tau\in[0,1]}\left|\frac{\partial^{N+1}v(s, \tau)}{\partial t^{N+1}}\right|\right\},$$

$$C_3 = \sup_{(s,t)\in\Omega}\left\{\max_{(\bar{\sigma},\bar{\tau})\in\Omega}\left|\frac{\partial^{M+N+2}v(\bar{\sigma}, \bar{\tau})}{\partial s^{M+1}\partial t^{N+1}}\right|\right\}.$$

Lemma 4.2 ([41]). *Let* $Q(f) = \int_{-1}^{1} f(x)dx$. *Then the error by the Gauss-Legendre quadrature formula,* $Q_N(f) = \sum_{i=0}^{N} w_i f(x_i)$ *approximating* $Q(f)$ *is given by*

$$R_N(f) = Q(f) - Q_N(f) = \frac{2^{2N+1}(N!)^2(\Gamma(N+1))^2 f^{(2N)}(\xi)}{(2N+1)(2N!)^2(\Gamma(2N+1))^2}, \quad \xi \in (-1, 1). \tag{4.1}$$

Theorem 4.3. *Assume that the conditions (i)-(iii) hold for the system* (3.13) *with* $d = 2$, *let* $\mathbf{U}(x,y)$ *be the exact solution of Eq.* (3.14) *and* $I_{MN}[\widetilde{\mathbf{U}}](x,y)$ *be the approximate solution obtained by presented method* (3.18). *If* $\Big(\max\limits_{i=1,2}\sum\limits_{j=1}^{2} K_{ij}q_j\Big) < 1$, *then we have the following error bound*

$$\Big\|\mathbf{U}(x,y) - I_{MN}[\widetilde{\mathbf{U}}](x,y)\Big\|_\infty \le \frac{\alpha_1\pi}{2^{M+3}(M+1)^{M+1}} + \frac{\alpha_2\pi}{2^{N+3}(N+1)^{N+1}} + \frac{\alpha_3\pi^2}{2^{M+N+6}(M+1)^{M+1}(N+1)^{N+1}}$$
$$+ \frac{\frac{1}{2}\Big(\max\limits_{i=1,2}\sum\limits_{j=1}^{2}K_{ij}q_j\Big)\|\mathbf{R}_M(\mathbf{U}(\xi,y))\|_\infty + \frac{1}{4}\Big(\max\limits_{i=1,2}\sum\limits_{j=1}^{2}K_{ij}q_j\Big)\|\mathbf{R}_N(\mathbf{U}(\xi,\eta))\|_\infty}{1-\Big(\max\limits_{i=1,2}\sum\limits_{j=1}^{2}K_{ij}q_j\Big)},$$

where

$$\mathbf{R}_M(\mathbf{U}(\xi,y)) = \frac{2^{2M+1}(M!)^2(\Gamma(M+1))^2}{(2M+1)(2M!)^2(\Gamma(2M+1))^2}\frac{\partial^{2M}\mathbf{U}(\xi,y)}{\partial x^{2M}}, \quad \xi \in (-1,1),$$
$$\mathbf{R}_N(\mathbf{U}(\xi,\eta)) = \frac{2^{2N+1}(N!)^2(\Gamma(N+1))^2}{(2N+1)(2N!)^2(\Gamma(2N+1))^2}\frac{\partial^{2N}\mathbf{U}(\xi,\eta)}{\partial y^{2N}}, \quad \xi,\eta \in (-1,1).$$

Proof. Multiplying both sides of Eq. (3.18) by $\ell_i(x)\ell_j(y)$ and taking summation over $i = 0,1,\ldots,M$, $j = 0,1,\ldots,N$, we derive the following equation:

$$I_{MN}[\widetilde{\mathbf{U}}](x,y) = I_{MN}[\mathbf{G}](x,y) + \sum_{l=0}^{N}\sum_{k=0}^{M} w_l w_k I_{MN}[\widetilde{\Phi}]\big(x,y,\xi_k,\eta(y_j,\rho_l)\big)\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}]\big(\xi_k,\eta(y_j,\rho_l)\big)\big). \tag{4.2}$$

Letting $\mathbf{E}(x,y) = \mathbf{U}(x,y) - I_{MN}[\widetilde{\mathbf{U}}](x,y)$ and subtracting both sides of Eq. (4.2) from equation

$$\mathbf{U}(x,y) = \mathbf{G}(x,y) + \int_{-1}^{1}\int_{-1}^{1}\widetilde{\Phi}\big(x,y,\xi,\eta(y,\rho)\big)\mathbf{H}\left(\mathbf{U}(\xi,\eta(y,\rho))\right)\mathrm{d}\xi\mathrm{d}\rho, \tag{4.3}$$

we acquire

$$\begin{aligned}
\mathbf{E}(x,y) &= \mathbf{G}(x,y) - I_{MN}[\mathbf{G}](x,y) + \int_{-1}^{1}\int_{-1}^{1}\widetilde{\Phi}\big(x,y,\xi,\eta(y,\rho)\big)\mathbf{H}\left(\mathbf{U}(\xi,\eta(y,\rho))\right)\mathrm{d}\xi\mathrm{d}\rho \\
&\quad - \sum_{l=0}^{N}\sum_{k=0}^{M} w_l w_k I_{MN}[\widetilde{\Phi}]\big(x,y,\xi_k,\eta(y_j,\rho_l)\big)\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}]\big(\xi_k,\eta(y_j,\rho_l)\big)\big) \\
&= \mathbf{G}(x,y) - I_{MN}[\mathbf{G}](x,y) \\
&\quad + \int_{-1}^{1}\int_{-1}^{1}\widetilde{\Phi}\big(x,y,\xi,\eta(y,\rho)\big)\Big(\mathbf{H}\left(\mathbf{U}(\xi,\eta(y,\rho))\right) - \mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}]\big(\xi,\eta(y,\rho)\big)\big)\Big)\mathrm{d}\xi\mathrm{d}\rho \\
&\quad + \int_{-1}^{1}\int_{-1}^{1}\Big(\widetilde{\Phi}\big(x,y,\xi,\eta(y,\rho)\big) - I_{MN}[\widetilde{\Phi}]\big(x,y,\xi_k,\eta(y_j,\rho_l)\big)\Big)\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}]\big(\xi,\eta(y,\rho)\big)\big)\mathrm{d}\xi\mathrm{d}\rho \\
&\quad + \int_{-1}^{1}\int_{-1}^{1} I_{MN}[\widetilde{\Phi}]\big(x,y,\xi_k,\eta(y_j,\rho_l)\big)\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}]\big(\xi,\eta(y,\rho)\big)\big)\mathrm{d}\xi\mathrm{d}\rho \\
&\quad - \sum_{l=0}^{N}\sum_{k=0}^{M} w_l w_k I_{MN}[\widetilde{\Phi}]\big(x,y,\xi_k,\eta(y_j,\rho_l)\big)\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}]\big(\xi_k,\eta(y_j,\rho_l)\big)\big).
\end{aligned} \tag{4.4}$$

Taking norm on both sides, we have

$$\|\mathbf{E}(x,y)\|_\infty \le \|\mathbf{G}(x,y) - I_{MN}[\mathbf{G}](x,y)\|_\infty$$

$$
\begin{aligned}
&+\left\|\int_{-1}^{1}\int_{-1}^{1}|\widetilde{\Phi}(x,y,\xi,\eta(y,\rho))||\mathbf{H}\left(\mathbf{U}(\xi,\eta(y,\rho))\right)-\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi,\eta(y,\rho))\big)|\mathrm{d}\xi\mathrm{d}\rho\right\|_\infty\\
&+\int_{-1}^{1}\int_{-1}^{1}\|\widetilde{\Phi}(x,y,\xi,\eta(y,\rho))-I_{MN}[\widetilde{\Phi}](x,y,\xi_k,\eta(y_j,\rho_l))\|_\infty\|\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi,\eta(y,\rho))\big)\|_\infty\mathrm{d}\xi\mathrm{d}\rho\\
&+\Big\|\int_{-1}^{1}\int_{-1}^{1}I_{MN}[\widetilde{\Phi}](x,y,\xi_k,\eta(y_j,\rho_l))\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi,\eta(y,\rho))\big)\mathrm{d}\xi\mathrm{d}\rho\\
&-\sum_{l=0}^{N}\sum_{k=0}^{M}w_lw_kI_{MN}[\widetilde{\Phi}](x,y,\xi_k,\eta(y_j,\rho_l))\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi_k,\eta(y_j,\rho_l))\big)\Big\|_\infty.
\end{aligned}
\tag{4.5}
$$

For the first term in the right hand side of (4.5), since $I_{MN}[\mathbf{G}](x,y)$ is the interpolation polynomial of $\mathbf{G}(x,y)$, then according to the Lemma 4.1, we have the bound

$$
\|\mathbf{G}(x,y)-I_{MN}[\mathbf{G}](x,y)\|_\infty\leq\frac{C_1\pi}{2^{M+3}(M+1)^{M+1}}+\frac{C_2\pi}{2^{N+3}(N+1)^{N+1}}+\frac{C_3\pi^2}{2^{M+N+6}(M+1)^{M+1}(N+1)^{N+1}}. \tag{4.6}
$$

The second term satisfies inequality

$$
\begin{aligned}
&\left\|\int_{-1}^{1}\int_{-1}^{1}|\widetilde{\Phi}(x,y,\xi,\eta(y,\rho))||\mathbf{H}\left(\mathbf{U}(\xi,\eta(y,\rho))\right)-\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi,\eta(y,\rho))\big)|\mathrm{d}\xi\mathrm{d}\rho\right\|_\infty\\
&\leq4\left\|(\frac{1}{2})^2\begin{pmatrix}K_{11}&K_{12}\\K_{21}&K_{22}\end{pmatrix}\begin{pmatrix}q_1&0\\0&q_2\end{pmatrix}\|\mathbf{E}(x,y)\|_\infty\right\|_\infty\\
&\leq\Big(\max_{i=1,2}\sum_{j=1}^{2}K_{ij}q_j\Big)\|\mathbf{E}(x,y)\|_\infty.
\end{aligned}
\tag{4.7}
$$

In the third term, using again the interpolation error in Lemma 4.1, we have

$$
\begin{aligned}
\|\widetilde{\Phi}(x,y,\xi,\eta(y,\rho))-I_{MN}[\widetilde{\Phi}](x,y,\xi_k,\eta(y_j,\rho_l))\|_\infty&\leq\frac{C_1'\pi}{2^{M+3}(M+1)^{M+1}}+\frac{C_2'\pi}{2^{N+3}(N+1)^{N+1}}\\
&+\frac{C_3'\pi^2}{2^{M+N+6}(M+1)^{M+1}(N+1)^{N+1}},
\end{aligned}
\tag{4.8}
$$

where C_1',C_2' and C_3' are some constants independent of M and N. Then we derive

$$
\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1}\|\widetilde{\Phi}(x,y,\xi,\eta(y,\rho))-I_{MN}[\widetilde{\Phi}](x,y,\xi_k,\eta(y_j,\rho_l))\|_\infty\|\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi,\eta(y,\rho))\big)\|_\infty\mathrm{d}\xi\mathrm{d}\rho\\
&\leq\frac{C_1''\pi}{2^{M+3}(M+1)^{M+1}}+\frac{C_2''\pi}{2^{N+3}(N+1)^{N+1}}+\frac{C_3''\pi^2}{2^{M+N+6}(M+1)^{M+1}(N+1)^{N+1}},
\end{aligned}
\tag{4.9}
$$

where C_1'',C_2'' and C_3'' are some constants independent of N and M. The fourth term is the error of the Gauss-Legendre quadrature rule, and since $\sum_{k=0}^{M}|w_k|=2,\sum_{l=0}^{N}|w_l|=2$, from Lemma 4.2, we have the following bound:

$$
\begin{aligned}
&\Big\|\int_{-1}^{1}\int_{-1}^{1}I_{MN}[\widetilde{\Phi}](x,y,\xi_k,\eta(y_j,\rho_l))\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi,\eta(y,\rho))\big)\mathrm{d}\xi\mathrm{d}\rho\\
&-\sum_{l=0}^{N}\sum_{k=0}^{M}w_lw_kI_{MN}[\widetilde{\Phi}](x,y,\xi_k,\eta(y_j,\rho_l))\mathbf{H}\big(I_{MN}[\widetilde{\mathbf{U}}](\xi_k,\eta(y_j,\rho_l))\big)\Big\|_\infty\\
&\leq\frac{1}{2}\Big(\max_{i=1,2}\sum_{j=1}^{2}K_{ij}q_j\Big)\|\mathbf{R}_M(\mathbf{U}(\xi,y))\|_\infty+\frac{1}{4}\Big(\max_{i=1,2}\sum_{j=1}^{2}K_{ij}q_j\Big)\|\mathbf{R}_N(\mathbf{U}(\xi,\eta))\|_\infty
\end{aligned}
\tag{4.10}
$$

Applying inequalities (4.6), (4.7), (4.9), and (4.10) in (4.5), we obtain the following error bound:

$$\|\mathbf{E}(x,y)\|_\infty \leq \frac{\alpha_1 \pi}{2^{M+3}(M+1)^{M+1}} + \frac{\alpha_2 \pi}{2^{N+3}(N+1)^{N+1}} + \frac{\alpha_3 \pi^2}{2^{M+N+6}(M+1)^{M+1}(N+1)^{N+1}}$$
$$+ \frac{\frac{1}{2}\Big(\max\limits_{i=1,2}\sum\limits_{j=1}^{2} K_{ij}q_j\Big)\|\mathbf{R}_M(\mathbf{U}(\xi,y))\|_\infty + \frac{1}{4}\Big(\max\limits_{i=1,2}\sum\limits_{j=1}^{2} K_{ij}q_j\Big)\|\mathbf{R}_N(\mathbf{U}(\xi,\eta))\|_\infty}{1-\Big(\max\limits_{i=1,2}\sum\limits_{j=1}^{2} K_{ij}q_j\Big)},$$

where α_1, α_2 and α_3 are some constants independent of N and M. $\square$

Corollary 4.4. *As $M, N \to \infty$, $\Big\|\mathbf{U}(x,y) - I_{MN}[\widetilde{\mathbf{U}}](x,y)\Big\|_\infty \to 0$, it means that the approximate solution $I_{MN}[\widetilde{\mathbf{U}}](x,y)$ is convergent to the exact solution $\mathbf{U}(x,y)$.*

Remark 4.5. By using the above technique, we can prove the convergence of the presented method for Eqs. (3.1) and (3.7).

5. NUMERICAL ILLUSTRATIONS

The theoretical findings presented in the preceding sections must be thoroughly analyzed in this section to determine the validity and precision of the proposed numerical technique. To assess the dependability of the numerical method, a comparison will be made between the precise solutions of the specified two-dimensional integral equations system and the corresponding approximations. To this aim, it is needed to define an error function of the following form:

$$e^r_{MN}(s,t) = \Big|v_r(s,t) - v^r_{MN}(s,t)\Big|, \ r = 1,2, \quad (s,t) \in [0,1]\times[0,1], \tag{5.1}$$

in which $v_r(s,t)$ stands for the exact solution, while $v^r_{MN}(s,t) = I_{MN}[u_r](2s-1, 2t-1)$ denotes the approximated solution of an under study two-dimensional nonlinear system of the Fredholm, Volterra or Volterra-Fredholm integral equations. In order to more better understanding of the numerical assessments, both of the absolute values of the errors in selected points and the L^∞-norm will be demonstrated in frame of tables. Prior to present the numerical examples, let us clarify that by L^∞-norm of an error function, we mean the following:

$$\|e^r_{MN}(s,t)\|_\infty = \max_{s_i,t_j\in[0,1]} \{e^r_{MN}(s_i,t_j)\}, \ r = 1,2. \tag{5.2}$$

All experimental results are performed by running some code written in Mathematica software.

Example 5.1. As the first example, consider the following system of the two-dimensional nonlinear Volterra type integral equations

$$\begin{cases} v_1(s,t) = f_1(s,t) + \displaystyle\int_0^t \int_0^s \cos(\tau)\big(v_1(\sigma,\tau)v_2(\sigma,\tau) - v_2(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \\ v_2(s,t) = f_2(s,t) + \displaystyle\int_0^t \int_0^s \big(v_1(\sigma,\tau) + v_2^2(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \end{cases} \tag{5.3}$$

in which

$$f_1(s,t) = s\cos(t) - \frac{1}{9}s^2\sin(\frac{t}{2})^2(-9+3s+2s\cos(t)+s\cos(2t)),$$
$$f_2(s,t) = s\sin(t) - \frac{1}{6}s^2(st+3\sin(t) - scos(t)sin(t)).$$

The exact solution of this nonlinear system is $v_1(s,t) = s\cos(t)$ and $v_2(s,t) = s\sin(t)$. As stated above, the numerical aims including the absolute value of errors and L^∞-norms of the error functions are illustrated in Tables 1 and 2. Furthermore, Figures 1 and 2 illustrate the absolute value error function $e^r_{MN}(s,t)$, $r = 1,2$, with $M = N = 2$ and

$M = N = 6$. Better approximation is expected by choosing $M = N = 10$, which we get $\|e^1_{MN}(s,t)\|_\infty = 4.11 \times 10^{-15}$ and $\|e^2_{MN}(s,t)\|_\infty = 7.94 \times 10^{-15}$.

Table 1. Error date related to $v^1_{MN}(s,t)$ for Example 5.1.

(s,t)	$M=N=2$	$M=N=4$	$M=N=6$
$(0.1, 0.1)$	$7.05E-05$	$6.82E-07$	$2.22E-10$
$(0.2, 0.2)$	$4.09E-04$	$1.34E-07$	$2.11E-09$
$(0.3, 0.3)$	$6.71E-04$	$1.91E-06$	$8.15E-10$
$(0.4, 0.4)$	$6.04E-04$	$2.58E-06$	$4.49E-09$
$(0.5, 0.5)$	$1.00E-04$	$1.07E-07$	$1.53E-10$
$(0.6, 0.6)$	$7.60E-04$	$3.95E-06$	$6.89E-09$
$(0.7, 0.7)$	$1.66E-03$	$5.12E-06$	$2.33E-09$
$(0.8, 0.8)$	$2.02E-03$	$2.36E-07$	$9.45E-09$
$(0.9, 0.9)$	$9.47E-04$	$7.44E-06$	$2.04E-09$
$\|e^1_{MN}(s,t)\|_\infty$	$2.02E-03$	$7.44E-06$	$9.45E-09$

Table 2. Error data related to $v^2_{MN}(s,t)$ for Example 5.1.

(s,t)	$M=N=2$	$M=N=4$	$M=N=6$
$(0.1, 0.1)$	$1.66E-04$	$1.46E-06$	$4.44E-10$
$(0.2, 0.2)$	$8.70E-04$	$2.06E-07$	$4.21E-09$
$(0.3, 0.3)$	$1.29E-03$	$3.84E-06$	$1.67E-09$
$(0.4, 0.4)$	$1.01E-05$	$4.83E-06$	$8.38E-09$
$(0.5, 0.5)$	$4.69E-03$	$1.86E-08$	$5.90E-11$
$(0.6, 0.6)$	$1.58E-03$	$7.14E-06$	$1.25E-08$
$(0.7, 0.7)$	$2.99E-03$	$8.61E-06$	$3.82E-09$
$(0.8, 0.8)$	$3.36E-03$	$8.30E-07$	$1.61E-08$
$(0.9, 0.9)$	$1.55E-03$	$1.22E-05$	$3.74E-09$
$\|e^2_{MN}(s,t)\|_\infty$	$3.36E-03$	$1.22E-05$	$1.61E-08$

Example 5.2. Second example, concentrates on the nonlinear system of two-dimensional Fredholm integral equations of the form:

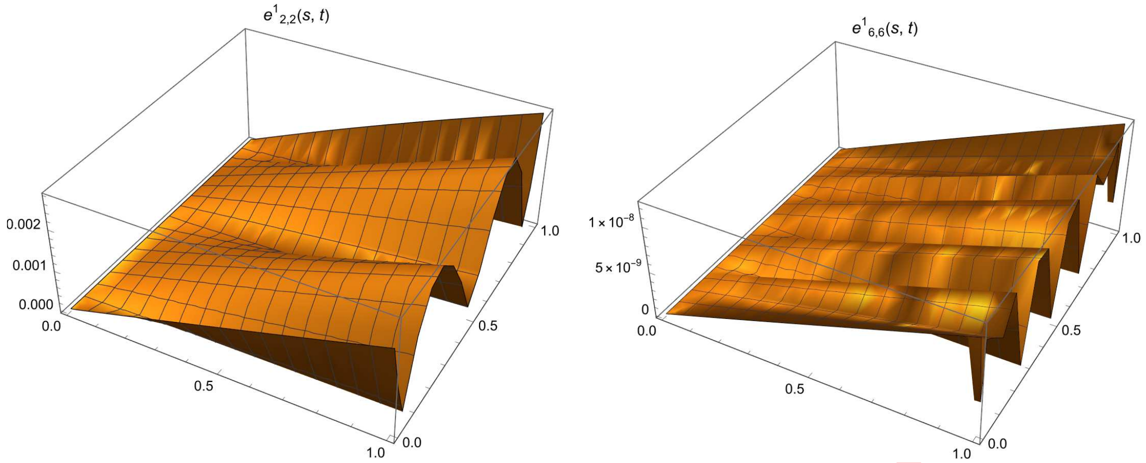
$$\begin{cases} v_1(s,t) = f_1(s,t) + \displaystyle\int_0^1\int_0^1 \Big(\frac{s(1+\sigma+\tau)}{1+t} v_1^2(\sigma,\tau) + v_2(\sigma,\tau)\Big)\mathrm{d}\sigma\mathrm{d}\tau, \\ v_2(s,t) = f_2(s,t) + \displaystyle\int_0^1\int_0^1 \big(v_1(\sigma,\tau) - (\sigma\sin\tau + 1)v_2^3(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \end{cases} \tag{5.4}$$

such that

$$f_1(s,t) = \frac{1}{(1+s+t)^2} - \frac{1}{6}\left(\frac{s}{1+t} + 3\sin 1\right),$$

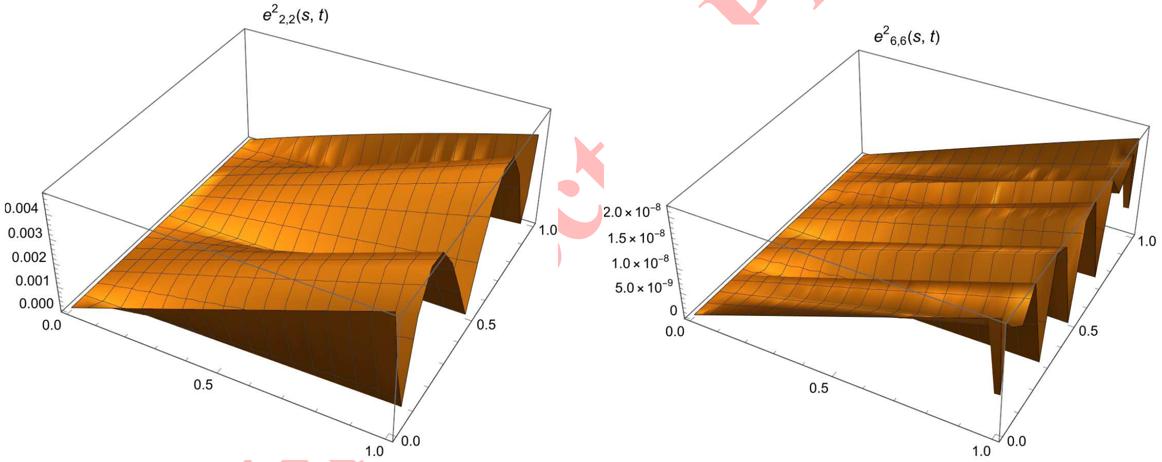
$$f_2(s,t) = s\cos t - \log\left(\frac{4}{3}\right) + \frac{1}{40}(3+\cos 2)\sin^2 1 + \frac{1}{48}(9\sin 1 + \sin 3).$$

This nonlinear system admits the exact solution $V(s,t) = (v_1(s,t), v_2(s,t))$ given by $v_1(s,t) = \dfrac{1}{(1+s+t)^2}$ and $v_2(s,t) = s\cos t$. Similar to the first example, we applied the presented method to this system and the numerical data for the error functions is given in Tables 3 and 4, and Figures 3 and 4. Better approximation is expected by choosing $M = N = 10$, which we get $\|e^1_{MN}(s,t)\|_\infty = 3.35 \times 10^{-08}$ and $\|e^2_{MN}(s,t)\|_\infty = 2.02 \times 10^{-10}$.



(a) $M = N = 2$ (b) $M = N = 6$

FIGURE 1. Graph of the function $e^1_{MN}(s,t)$ related to $v^1_{MN}(s,t)$ in Example 5.1.



(a) $M = N = 2$ (b) $M = N = 6$

FIGURE 2. Graph of the function $e^2_{MN}(s,t)$ related to $v^2_{MN}(s,t)$ in Example 5.1.

Example 5.3. In this opportunity we present a numerical application to show effectiveness of the proposed collocation method to approximate solutions of the nonlinear two-dimensional integral equations essentially of Volterra-Fredholm type. So, let us take focus on the following system of the two-dimensional nonlinear mixed Volterra-Fredholm integral equations [6]

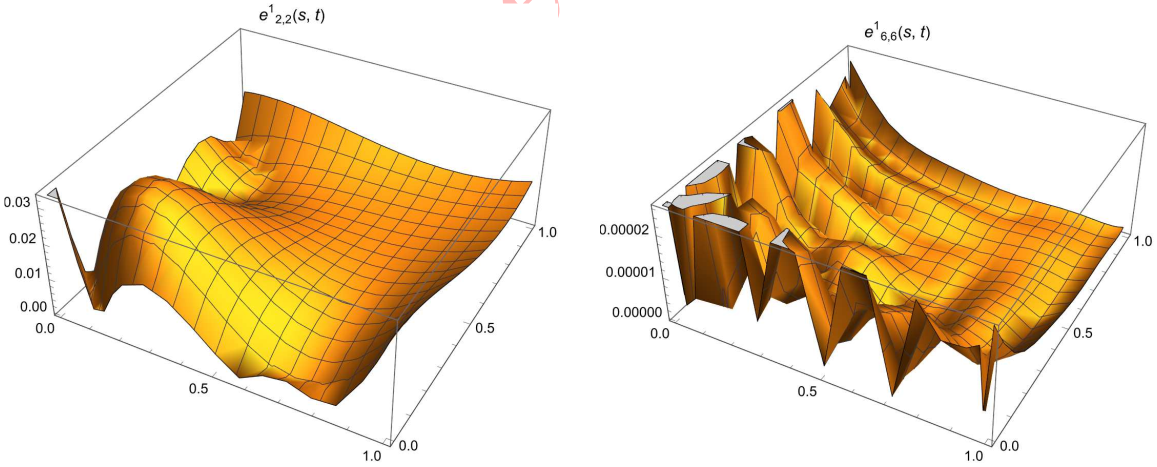
$$\begin{cases} v_1(s,t) = f_1(s,t) + \displaystyle\int_0^t \int_0^1 \sigma\tau(t^2+s^2)v_1(\sigma,\tau)\mathrm{d}\sigma\mathrm{d}\tau, \\ v_2(s,t) = f_2(s,t) + \displaystyle\int_0^t \int_0^1 \sigma(s-t)v_2^2(\sigma,\tau)\mathrm{d}\sigma\mathrm{d}\tau. \end{cases} \tag{5.5}$$

Table 3. Error data related to $v^1_{MN}(s,t)$ in Example 5.2.

(s,t)	$M=N=2$	$M=N=4$	$M=N=6$
$(0.1,0.1)$	$2.00E-02$	$1.51E-03$	$1.09E-05$
$(0.2,0.2)$	$3.17E-02$	$1.24E-04$	$2.35E-05$
$(0.3,0.3)$	$2.47E-02$	$3.70E-04$	$4.37E-06$
$(0.4,0.4)$	$1.53E-02$	$1.89E-04$	$8.44E-06$
$(0.5,0.5)$	$8.92E-03$	$7.72E-05$	$1.05E-06$
$(0.6,0.6)$	$6.07E-03$	$1.99E-04$	$1.49E-06$
$(0.7,0.7)$	$5.87E-03$	$1.70E-04$	$7.31E-07$
$(0.8,0.8)$	$7.23E-03$	$8.33E-05$	$2.24E-06$
$(0.9,0.9)$	$9.58E-03$	$4.76E-05$	$1.10E-06$
$\Vert e^1_{MN}(s,t)\Vert_\infty$	$3.17E-02$	$1.51E-03$	$2.35E-05$

Table 4. Error data related to $v^2_{MN}(s,t)$ in Example 5.2.

(s,t)	$M=N=2$	$M=N=4$	$M=N=6$
$(0.1,0.1)$	$5.02E-03$	$4.50E-05$	$6.01E-07$
$(0.2,0.2)$	$4.69E-03$	$4.44E-05$	$6.04E-07$
$(0.3,0.3)$	$4.46E-03$	$4.24E-05$	$6.01E-07$
$(0.4,0.4)$	$4.56E-03$	$4.18E-05$	$5.97E-07$
$(0.5,0.5)$	$5.09E-03$	$4.43E-05$	$6.02E-07$
$(0.6,0.6)$	$5.97E-03$	$4.83E-05$	$6.09E-07$
$(0.7,0.7)$	$6.87E-03$	$4.94E-05$	$6.04E-07$
$(0.8,0.8)$	$7.19E-03$	$4.38E-05$	$5.92E-07$
$(0.9,0.9)$	$6.01E-03$	$3.65E-05$	$6.04E-07$
$\Vert e^2_{MN}(s,t)\Vert_\infty$	$7.19E-03$	$4.94E-05$	$6.09E-07$

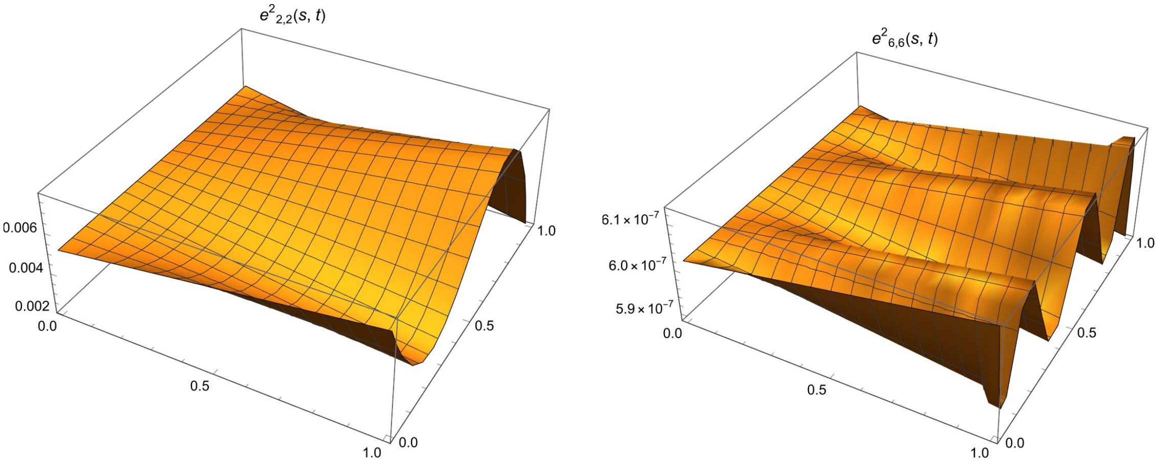


(a) $M=N=2$ (b) $M=N=6$

Figure 3. Graph of the function $e^1_{MN}(s,t)$ related to $v^1_{MN}(s,t)$ in Example 5.2.

In this system, we have

$$f_1(s,t) = -\frac{1}{6}(t^2+s^2)(t\cos(t)-\sin(t)) - \frac{1}{2}s\sin(t),$$



(a) $M = N = 2$ (b) $M = N = 6$

FIGURE 4. Graph of the function $e^2_{MN}(s,t)$ related to $v^2_{MN}(s,t)$ in Example 5.2.

$$f_2(s,t) = 0.14726\ t^3(t-s) + t\tan(s).$$

The exact solution of this system is $v_1(s,t) = -\frac{1}{2}s\sin(t), v_2(s,t) = t\tan(s)$. Like the previous examples, Tables 5 and 6, and Figures 5 and 6 indicate the computational data related to the corresponding error functions. As can be seen, taking a look at these tables it finds out that the proposed technique has more reliable performance in comparison with the homotopy perturbation method given in [6] and the Adomian decomposition method given in [15] (using five terms), where the errors are zero for $v_2(s,t)$ in both homotopy perturbation and Adomian decomposition methods. Better approximation is expected by choosing $M = N = 10$, which we get $\|e^1_{MN}(s,t)\|_\infty = 4.02\times10^{-15}$ and $\|e^2_{MN}(s,t)\|_\infty = 3.42\times10^{-07}$.

TABLE 5. Error data related to $v^1_{MN}(s,t)$ in Example 5.3.

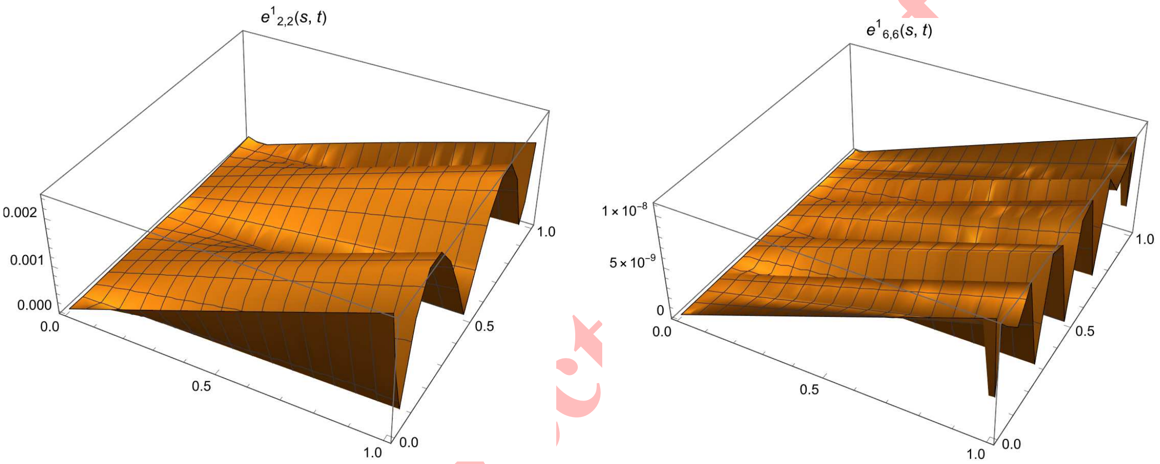
(s,t)	$M = N = 2$	$M = N = 4$	$M = N = 6$	Method in [6]	Method in [15]
$(0.1, 0.1)$	$8.86E-05$	$7.54E-07$	$2.29E-10$	$4.80E-07$	$8.60E-07$
$(0.2, 0.2)$	$4.57E-04$	$1.15E-07$	$2.14E-09$	$8.21E-06$	$1.95E-05$
$(0.3, 0.3)$	$6.85E-04$	$1.95E-06$	$8.28E-10$	$4.51E-05$	$8.84E-05$
$(0.4, 0.4)$	$5.53E-04$	$2.49E-06$	$4.26E-09$	$1.54E-04$	$1.24E-04$
$(0.5, 0.5)$	$3.04E-05$	$6.58E-08$	$6.48E-11$	$4.05E-04$	$3.71E-04$
$(0.6, 0.6)$	$7.42E-04$	$3.55E-06$	$6.26E-09$	$8.75E-04$	$2.82E-03$
$(0.7, 0.7)$	$1.46E-03$	$4.40E-06$	$2.05E-09$	$1.60E-03$	$1.05E-02$
$(0.8, 0.8)$	$1.67E-03$	$2.13E-07$	$8.01E-09$	$2.45E-03$	$3.04E-02$
$\|e^1_{MN}(s,t)\|_\infty$	$1.67E-03$	$4.40E-06$	$8.01E-09$	–	–

Example 5.4. Let us consider another system of nonlinear two-dimensional Volterra-Fredholm integral equations of the form [6, 15]

$$\begin{cases} v_1(s,t) = f_1(s,t) + \displaystyle\int_0^t\int_0^1 \big(t^2 + v_1^2(\sigma,\tau)\big)\mathrm{d}\sigma\mathrm{d}\tau, \\ v_2(s,t) = f_2(s,t) + \displaystyle\int_0^t\int_0^1 (s-t^2)v_2^2(\sigma,\tau)\mathrm{d}\sigma\mathrm{d}\tau, \end{cases} \tag{5.6}$$

TABLE 6. Error data related to $v^2_{MN}(s,t)$ in Example 5.3.

(s,t)	$M=N=2$	$M=N=4$	$M=N=6$
$(0.1,0.1)$	$7.60E-04$	$1.30E-04$	$1.62E-06$
$(0.2,0.2)$	$4.44E-04$	$2.12E-05$	$1.65E-05$
$(0.3,0.3)$	$7.25E-03$	$3.98E-04$	$7.02E-06$
$(0.4,0.4)$	$6.28E-03$	$5.49E-04$	$3.89E-05$
$(0.5,0.5)$	$5.55E-03$	$1.11E-16$	$5.55E-17$
$(0.6,0.6)$	$1.12E-17$	$9.89E-04$	$7.02E-05$
$(0.7,0.7)$	$2.41E-02$	$1.34E-03$	$2.39E-05$
$(0.8,0.8)$	$3.07E-02$	$1.48E-04$	$1.17E-04$
$\|e^2_{MN}(s,t)\|_\infty$	$3.07E-02$	$1.34E-03$	$1.17E-04$



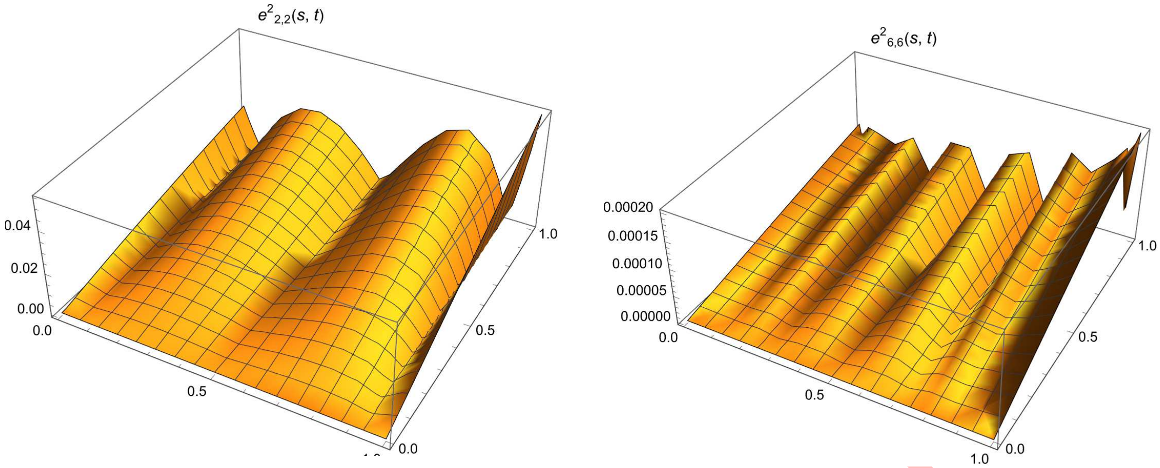
(a) $M=N=2$ (b) $M=N=6$

FIGURE 5. Graph of the function $e^1_{MN}(s,t)$ related to $v^1_{MN}(s,t)$ in Example 5.3.

where

$$f_1(s,t) = 2st - t^2 - \frac{1}{90}t^3(130 - 45t + 18t^2),$$
$$f_2(s,t) = 1 + t^2\sin s - \frac{1}{60}t(t^2 - s)\Big(3t^4(\sin(2) - 2) - 80t^2\sin^2(\frac{1}{2}) - 60\Big).$$

The exact solution of this system is $v_1(s,t) = 2st - t^2, v_2(s,t) = 1 + t^2\sin s$. In [6] and [15], this example is solved numerically by homotopy perturbation and Adomian decomposition methods (using five terms), respectively. Like the previous examples, we solved this problem using the method presented in Section 3 and the computational data related to the corresponding error functions together with the results given in [6, 15] are reported in Tables 7 and 8, and Figures 7 and 8. As can be seen, taking a look at these tables it finds out that the presented method has more reliable performance in comparison with the homotopy perturbation and Adomian decomposition methods.



(a) $M = N = 2$ (b) $M = N = 6$

FIGURE 6. Graph of the function $e^2_{MN}(s,t)$ related to $v^2_{MN}(s,t)$ in Example 5.3.

TABLE 7. Error data related to $v^1_{MN}(s,t)$ in Example 5.4.

(s,t)	$M=N=2$	$M=N=4$	Method in [6]	Method in [15]
$(0.5, 0.5)$	$2.77E-17$	0	$3.95E-06$	$4.33E-03$
$(0.25, 0.25)$	$7.14E-18$	$3.25E-19$	$1.13E-09$	$4.34E-03$
$(0.125, 0.125)$	$3.67E-18$	$8.90E-18$	$1.89E-13$	$7.24E-04$
$(0.0625, 0.0625)$	$1.34E-18$	$8.76E-19$	$2.66E-17$	$1.00E-04$
$\Vert e^1_{MN}(s,t)\Vert_\infty$	$2.77E-17$	$8.90E-18$	$-$	$-$

TABLE 8. Error data related to $v^2_{MN}(s,t)$ in Example 5.4.

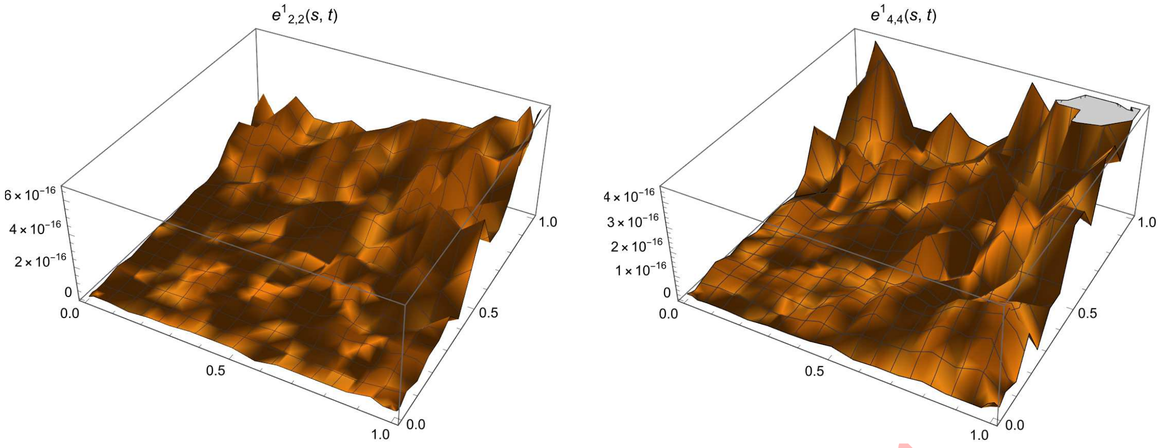
(s,t)	$M=N=2$	$M=N=4$	Method in [6]	Method in [15]
$(0.5, 0.5)$	$2.44E-07$	$1.34E-10$	$3.83E-05$	$2.53E-02$
$(0.25, 0.25)$	$2.87E-04$	$4.52E-07$	$4.33E-06$	$5.45E-04$
$(0.125, 0.125)$	$3.90E-05$	$2.03E-07$	$1.89E-06$	$1.31E-05$
$(0.0625, 0.0625)$	$7.54E-07$	$4.63E-08$	$1.67E-07$	$2.62E-06$
$\Vert e^2_{MN}(s,t)\Vert_\infty$	$2.87E-04$	$4.52E-07$	$-$	$-$

Example 5.5. As our final example, we consider the following nonlinear system of two-dimensional Volterra-Fredholm integral equations

$$\begin{cases} v_1(s,t) = f_1(s,t) + \displaystyle\int_0^t \int_0^1 \Big(\frac{s(1-\sigma^2)}{(1+t)(1+\tau^2)} \big(1 - exp(-v_1(\sigma,\tau))\big) + v_2(\sigma,\tau) \Big) \mathrm{d}\sigma \mathrm{d}\tau, \\ v_2(s,t) = f_2(s,t) + \displaystyle\int_0^t \int_0^1 \big(exp(-v_1(\sigma,\tau)) + (1+\sigma+\tau) v_2^2(\sigma,\tau) \big) \mathrm{d}\sigma \mathrm{d}\tau, \end{cases} \tag{5.7}$$

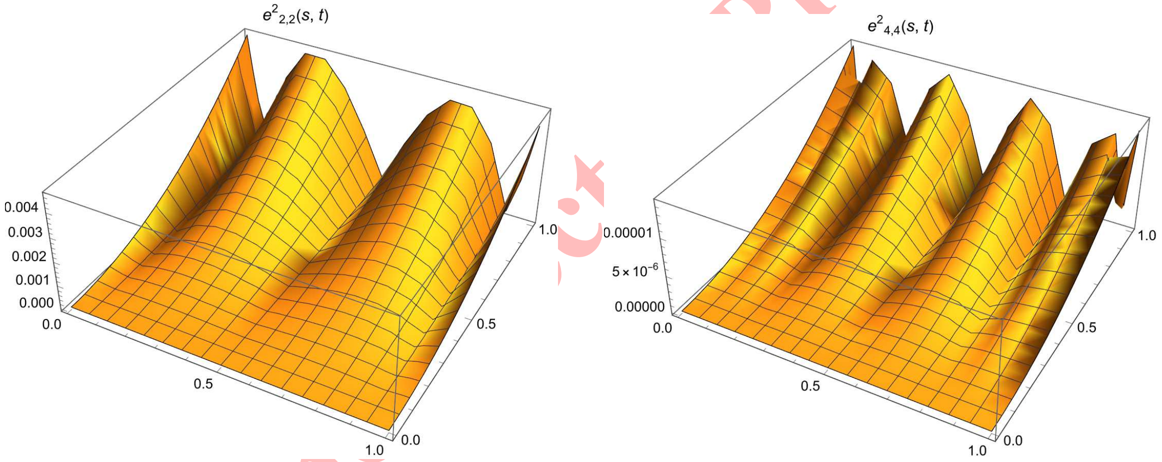
in which

$$f_1(s,t) = \frac{st^2}{8(1+t)(1+t^2)} - \log\left(1 + \frac{st}{1+t^2}\right) + \log(2+t) - \log(2+2t),$$



(a) $M = N = 2$ (b) $M = N = 4$

FIGURE 7. Graph of the function $e^1_{MN}(s,t)$ related to $v^1_{MN}(s,t)$ in Example 5.4.



(a) $M = N = 2$ (b) $M = N = 4$

FIGURE 8. Graph of the function $e^2_{MN}(s,t)$ related to $v^2_{MN}(s,t)$ in Example 5.4.

$$f_2(s,t) = \frac{1}{(1+s+t)^2} + \frac{1}{4}\Big(-1-4t+\frac{2}{2+3t+t^2} - \log(1+t^2)\Big).$$

This nonlinear system has the exact solution $v_1(s,t) = -\log\left(1+\frac{st}{1+t^2}\right)$ and $v_2(s,t) = \dfrac{1}{(1+s+t)^2}$. In the same way as the previous examples, the error data is given for different chosen setting M and N in frame of Tables 9, 10 and Figures 9, 10. Better approximation is expected by choosing $M = N = 12$, which we get $\|e^1_{MN}(s,t)\|_\infty = 7.08 \times 10^{-10}$ and $\|e^2_{MN}(s,t)\|_\infty = 7.06 \times 10^{-10}$.

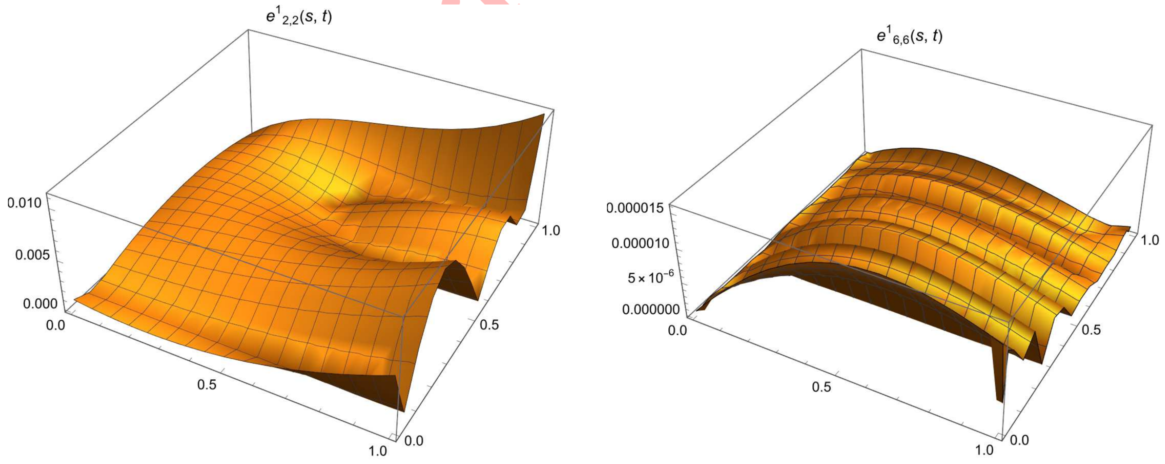
At the end, regarding the proposed collocation method in this paper, CPU times for all of the five numerical Examples 5.1–5.5 are collocated in Table 11.

TABLE 9. Error data related to $v^1_{MN}(s,t)$ in Example 5.5.

(s,t)	$M=N=2$	$M=N=4$	$M=N=6$
$(0.1,0.1)$	$3.23E-05$	$3.24E-05$	$1.45E-06$
$(0.2,0.2)$	$1.23E-03$	$2.93E-05$	$7.43E-06$
$(0.3,0.3)$	$2.84E-03$	$2.22E-04$	$2.09E-06$
$(0.4,0.4)$	$3.69E-03$	$2.59E-04$	$9.77E-06$
$(0.5,0.5)$	$2.82E-03$	$1.94E-05$	$7.69E-07$
$(0.6,0.6)$	$2.85E-04$	$2.79E-04$	$6.56E-06$
$(0.7,0.7)$	$2.65E-03$	$2.95E-04$	$1.36E-06$
$(0.8,0.8)$	$3.77E-03$	$7.83E-05$	$2.88E-06$
$(0.9,0.9)$	$2.16E-04$	$4.22E-04$	$4.33E-07$
$\|e^1_{MN}(s,t)\|_\infty$	$3.77E-03$	$4.22E-04$	$9.77E-06$

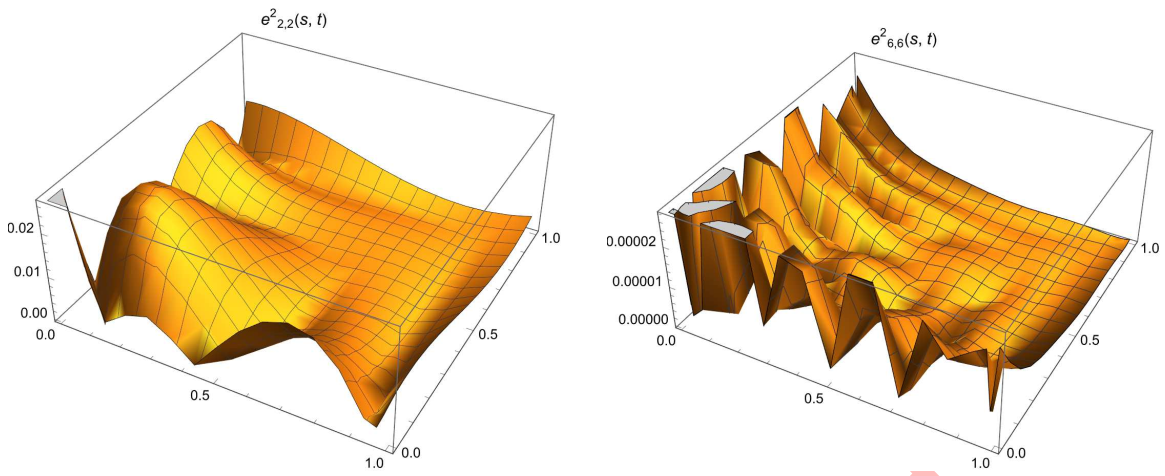
TABLE 10. Error data related to $v^2_{MN}(s,t)$ in Example 5.5.

(s,t)	$M=N=2$	$M=N=4$	$M=N=6$
$(0.1,0.1)$	$1.39E-02$	$1.54E-03$	$1.03E-05$
$(0.2,0.2)$	$2.55E-02$	$1.61E-04$	$2.42E-05$
$(0.3,0.3)$	$1.83E-02$	$3.83E-04$	$3.66E-06$
$(0.4,0.4)$	$8.77E-03$	$2.59E-04$	$7.60E-06$
$(0.5,0.5)$	$2.09E-03$	$2.37E-05$	$7.38E-08$
$(0.6,0.6)$	$1.09E-03$	$1.05E-04$	$2.55E-06$
$(0.7,0.7)$	$1.72E-03$	$1.13E-04$	$3.77E-07$
$(0.8,0.8)$	$9.22E-04$	$6.90E-05$	$1.12E-06$
$(0.9,0.9)$	$7.26E-04$	$3.52E-05$	$7.53E-08$
$\|e^2_{MN}(s,t)\|_\infty$	$2.55E-02$	$1.54E-03$	$2.42E-05$



(a) $M=N=2$ (b) $M=N=6$

FIGURE 9. Graph of the function $e^1_{MN}(s,t)$ related to $v^1_{MN}(s,t)$ in Example 5.5.



(a) $M = N = 2$ (b) $M = N = 6$

FIGURE 10. Graph of the function $e^2_{MN}(s,t)$ related to $v^2_{MN}(s,t)$ in Example 5.5.

TABLE 11. CPU times for Examples 5.1–5.5.

	$M = N = 2$	$M = N = 4$	$M = N = 6$
Example 5.1	3.219	9.859	77.642
Example 5.2	3.453	6.501	43.875
Example 5.3	3.297	7.516	47.329
Example 5.4	3.579	7.328	–
Example 5.5	3.531	8.860	56.563

6. CONCLUSIONS

In summary, we studied special systems of two-dimensional Volterra–Fredholm integral equations. Under certain conditions, we proved the existence and uniqueness of a solution to these systems by using the Banach fixed point theorem. After some suitable linear transformations, a discrete collocation method based on Gauss quadrature formulas and Lagrange interpolation function has been constructed to solve the nonlinear system of Volterra–Fredholm, Fredholm and Volterra integral Eqs. (1.1)–(1.3). Then, the existence and uniqueness of the solutions of discrete equations have been proved by using the compact operator theory. We also carried out the error bound and convergence analysis of the method. Finally, the numerical results of some examples illustrate the effectiveness of the proposed method. It is observed that in comparison with the other methods in the literature, our method is easy to implement and has a high precision.

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