



# Numerical Solution of Variable-Order Fractional Bagley-Torvik Equation Using Multi-Domain Legendre Collocation Approach

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## Abstract

This paper introduces a novel multi-domain Legendre-collocation method for the numerical solution of the variable-order fractional Bagley-Torvik equation. To address the computational challenges posed by variable-order fractional derivatives and their weakly singular kernels, an operational matrix formulation is developed, providing a highly efficient alternative to classical three-term recurrence schemes based on Legendre polynomials. The domain decomposition strategy enables local evaluation of the memory integral within each subdomain, thereby preserving spectral accuracy while mitigating the stability degradation and round-off error accumulation often encountered in global spectral methods over extended intervals. Numerical experiments confirm the proposed method's spectral convergence, stability, and computational efficiency. The results highlight the method's potential as a reliable and accurate framework for solving a broad class of variable-order fractional differential equations.

**Keywords.** Variable order fractional Bagley-Torvik equation, Multi-domain Legendre-collocation method, Domain decomposition.

**2010 Mathematics Subject Classification.** 34A08, 65M70, 26A33, 65L05.

## 1. INTRODUCTION

Variable-order fractional differential equations have emerged as a powerful mathematical framework for representing complex dynamical systems in which the underlying processes exhibit intrinsic non-locality, hereditary memory, and spatial or temporally heterogeneous dynamics [6, 10, 29]. In contrast to constant-order formulations, the order of differentiation in variable order fractional differential equations is treated as a continuous function of space and/or time, thereby enabling a more faithful characterization of systems whose dynamic response evolves its changing physical or geometrical conditions. This intrinsic flexibility allows these equations to accurately describe a wide range of real-world phenomena, including oscillatory transport with coupled advection-diffusion mechanisms [27], linear and nonlinear viscoelastic oscillators [10], geospatial data processing [11], constitutive laws in viscoelastic continuum mechanics [28], adaptive fractional derivative-based bio-metric identification [36], anomalous diffusion and disperse transport processes [16, 41], and chloride ion migration in porous concrete media [37]. Consequently, variable order fractional differential equations provide an advanced mathematical paradigm for modeling non-Markovian and memory-dependent systems that can not be adequately captured by conventional integer-order or constant-order fractional models.

The Bagley-Torvik equation represents one of the classical models in fractional calculus, originally formulated to describe the dynamics of a rigid plate immersed in a Newton fluid. This equation incorporates both integer- and fractional-order derivatives to characterize viscoelastic damping behavior, bridging the gap between purely elastic and purely viscous responses [35]. In recent years, the extension of this model to variable-order fractional derivatives has gained considerable attention due to its enhanced capacity to capture non-stationary and time-varying memory effects inherent in many complex systems. This generalization provides a more flexible framework for modeling dynamic processes whose constitutive behavior evolves over time, such as in adaptive viscoelastic materials, diffusion in heterogeneous media, and control systems with time-dependent memory. The variable order Bagley-Torvik equation thus represents a significant advancement over its constant-order counterpart, offering a powerful mathematical framework

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for analyzing systems with continuously changing fractional dynamics. Investigating its analytical and numerical properties is therefore of both theoretical and practical importance in advancing the understanding and simulation of real-world processes governed by non-local and time-varying effects.

A wide range of analytical studies have investigated the mathematical structure of the Bagley-Torvik under constant fractional order. Closed-form or semi-analytical solutions have been derived for specific cases using Laplace and Fourier transforms, Green's function techniques, and Mittag-Leffler function representations. Existence, uniqueness, and stability analyses have been established through equivalent Volterra-type integral formulations and fixed-point approaches, providing a solid theoretical foundation for subsequent numerical developments [7, 15, 38]. In contrast, a comprehensive theoretical framework for the variable-order counterpart remains underdeveloped. Nevertheless, general existence and uniqueness results for variable-order fractional differential equations have been obtained in the literature under standard regularity assumptions on the variable order and the source term (cf. [2, 29, 30, 33, 39]). Although these results provide a plausible theoretical foundation for the well-posedness of problem (1.1), the present study is exclusively devoted to its numerical treatment.

Given the complexity of fractional operator, considerable attention has also been devoted to the numerical approximation of the Bagley-Torvik equation. Prominent strategies include spectral and collocation schemes based on Jacobi polynomials [24], finite element and spline based formulations [17], operational matrix techniques [25, 26, 40]. These approaches have been refined to attain high-order accuracy and computational efficiency while effectively handling the non-local memory term and the weakly singular kernel inherent to fractional operators.

Despite extensive research on the constant-order Bagley-Torvik equation, the variable-order Bagley-Torvik equation has received limited attention. Only a few recent studies have addressed this generalization explicitly. Amin et al. [3, 4, 25, 34] proposed a shifted Chebyshev Gauss collocation scheme for the nonlinear variable-order Bagley-Torvik equation, demonstrating the feasibility of high-order spectral approximations for time-dependent derivative orders. Mogaddam et al. [23] employed an integro-quadratic spline interpolation framework for variable-order fractional models involving Bagley-Torvik and Basset-type operators. Zaky [42] further developed recursive spectral formulations to approximate variable-order fractional operators and applied them to prototype variable-order Bagley-Torvik equation systems. Although these efforts mark significant progress, the available literature remains sparse, particularly regarding rigorous computational efficiency for variable-order Bagley-Torvik equation models. The present study aims to contribute to this growing field by formulating an efficient multi-domain Legendre collocation approach for the following model

$$\begin{cases} u''(t) + {}_0\mathcal{D}_t^{\alpha(t)}u(t) + u(t) = f(t), & t \in (0, T], \\ u(0) = u_0, \quad u'(0) = u'_0, \end{cases} \quad (1.1)$$

where  ${}_0\mathcal{D}_t^{\alpha(t)}$  denotes the variable-order Caputo fractional derivative with order  $1 < \alpha(t) < 2$ , and is defined as

$${}_0\mathcal{D}_t^{\alpha(t)}u(t) = \frac{1}{\Gamma(2 - \alpha(t))} \int_0^t (t - \tau)^{1 - \alpha(t)} u''(\tau) d\tau, \quad t > 0,$$

where  $\Gamma(\cdot)$  is the Gamma function. Here  $\alpha(t)$ ,  $f(t)$  represent the smooth variable-order and source term functions, and  $u_0, u'_0$  are given initial data. Throughout this work, the variable-order function  $\alpha(t)$  is assumed to satisfy

$$1 < \alpha_{\min} \leq \alpha(t) \leq \alpha_{\max} < 2, \quad \forall t \in [0, T],$$

The exact solution  $u(t)$  is assumed to be continuous on the interval  $[0, T]$ . This minimal regularity condition is crucial for two reasons. First, it guarantees that the Caputo variable-order fractional derivative is rigorously well-defined. Second, it ensures the preservation of spectral convergence, which could otherwise be compromised by abrupt variations in  $\alpha(t)$ . Importantly, this level of smoothness is fully compatible with the requirements of physical models in viscoelasticity and anomalous transport phenomena.

The multi-domain Legendre collocation method provides a highly accurate and flexible framework for the numerical approximation of functional differential equations such as the variable-order Bagley-Torvik Equation (1.1). By decomposing the computational interval into multiple subdomains, the method efficiently captures local variations in the solution and in the fractional order, which are intrinsic to variable-order systems. The use of Legendre-Gauss-Lobatto collocation points ensures high accuracy within each subdomain while maintaining numerical stability and avoiding the Runge phenomenon typically associated with global spectral schemes. Compared to single-domain discretization



methods, this approach offers superior convergence behavior, reduced condition numbers and dimensions of the resulting algebraic systems, and enhanced computational efficiency for large-scale stiff fractional models. These advantages stem from the proposed operational matrix formulation, which replaces recursive evaluations of the weakly singular kernel with an explicit, non-recursive matrix-vector product—thereby reducing computational cost and suppressing round-off error accumulation. By analytically expressing the variable-order Caputo derivative via incomplete beta and Gauss hypergeometric functions and coupling this representation with a domain decomposition strategy, the method preserves spectral accuracy locally while ensuring global stability over extended time intervals. These features make the multi-domain Legendre collocation scheme an effective and robust tool for solving the variable-order Bagley-Torvik Equation (1.1) with high precision.

The paper is structured as follows: Section 2 introduces the multi-domain Legendre collocation method and the operational matrix approach for the variable-order fractional Bagley-Torvik Equation (1.1). Section 3 presents numerical results, highlighting the convergence behavior of the proposed method. Finally, Section 4 summarizes the findings and outlines directions for future research.

## 2. MULTI-DOMAIN LEGENDRE COLLOCATION APPROACH

This section develops a computational framework for the numerical solution of the variable-order fractional Bagley-Torvik Equation (1.1). The proposed scheme is based on a multi-domain Legendre collocation strategy that concurrently aims to ensure high accuracy while preserving computational tractability.

Partitioning  $\Lambda = [0, T]$  into  $M$  non-overlapping sub-intervals

$$0 = t_0 < t_1 < \dots < t_{M-1} < t_M = T, \quad \Lambda_e = [t_{e-1}, t_e], \quad e = 1, \dots, M,$$

and defining the following finite-dimensional approximation space  $V_N$  by

$$V_N = \left\{ v \in C(\Lambda) : v|_{\Lambda_e} \in \mathbb{P}_N^{(\Lambda_e)}, \quad e = 1, \dots, M \right\},$$

where  $\mathbb{P}_N^{(\Lambda_e)}$  denotes the space of polynomials from degree at most  $N$  on sub-interval  $\Lambda_e$ , the approximate solution  $u_N^e(t) \in V_N$  on each sub-domain  $\Lambda_e$  is considered as a linear combination of shifted Legendre polynomials as follows

$$u_N^e(t) = \sum_{n=0}^N C_n^e L_n^e(t) = \mathbf{L}^e(t) \mathbf{C}^e, \quad t \in \Lambda_e, \tag{2.1}$$

such that

$$\begin{aligned} &L_n^e(t) \text{ is the } n\text{-th shifted Legendre polynomial on } \Lambda_e, \\ &\mathbf{L}^e(t) = [L_0^e(t), L_1^e(t), \dots, L_N^e(t)]^T \text{ is the vector of basis functions,} \\ &\mathbf{C}^e = [C_0^e, C_1^e, \dots, C_N^e] \text{ denotes the unknown coefficients.} \end{aligned}$$

Clearly, for  $t \in \Lambda_e$ , the main Equation (1.1) can be restated in the following form

$$\begin{cases} u''(t) + \sum_{k=1}^{e-1} {}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t)} u(t) + {}_{t_{e-1}}\mathcal{D}_t^{\alpha(t)} u(t) + u(t) = f(t), & t \in \Lambda_e, \\ u(0) = u_0, \quad u'(0) = u'_0, \end{cases} \tag{2.2}$$

where

$${}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t)} u(t) = \frac{1}{\Gamma(2 - \alpha(t))} \left[ \int_{t_{k-1}}^{t_k} (\zeta - \tau)^{1-\alpha(t)} u''(\tau) d\tau \right]_{\zeta=t}, \quad t > t_k.$$

The auxiliary operator  ${}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t)}$  ( $t > T$ ) captures the accumulated non-local history contribution of the left-sided variable-order fractional derivative over the past interval  $(0, T]$ , evaluated at  $t$  beyond the integration domain. Being constant with respect to the current local element, it allows an efficient separation of global non-local interactions from local computations in multi-domain spectral collocation schemes.



At this stage, we first select  $N + 1$  Legendre Gauss-Lobatto collocation points  $\{t_j^e\}_{j=0}^N \subset \Lambda_e$  on each subdomain  $\Lambda_e$ , yielding the global set

$$\mathbb{N}_c = \{t_j^e : e = 1, 2, \dots, M; j = 0, 1, \dots, N\},$$

and then substitute the representation (2.1) into (2.2). Thus, evaluating the resulting equation at  $\mathbb{N}_c$ , gives

$$(u_N^e)''(t_j^e) + \sum_{k=1}^{e-1} {}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t_j^e)} u_N^k(t_j^e) + {}_{t_{e-1}}\mathcal{D}_t^{\alpha(t_j^e)} u_N^e(t_j^e) + u_N^e(t_j^e) = f(t_j^e), \quad (2.3)$$

in view of  $e = 1, 2, \dots, M; j = 0, 1, \dots, N$ . Using (2.1), we can write

$$\sum_{k=1}^{e-1} {}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t_j^e)} u_N^k(t_j^e) + {}_{t_{e-1}}\mathcal{D}_t^{\alpha(t_j^e)} u_N^e(t_j^e) = \left[ \underline{\mathbf{C}}^e \mathbf{M}^e(t) + \sum_{k=1}^{e-1} \underline{\mathbf{C}}^k \tilde{\mathbf{M}}^k(t) \right]_{t=t_j^e}, \quad (2.4)$$

where

$$\mathbf{M}^e(t) = {}_{t_{e-1}}\mathcal{D}_t^{\alpha(t)} \underline{\mathbf{L}}^e(t), \quad \tilde{\mathbf{M}}^k(t) = {}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t)} \underline{\mathbf{L}}^k(t). \quad (2.5)$$

In order to characterize (2.5), we express the shifted Legendre basis in the following form

$$\underline{\mathbf{L}}^e(t) = \mathbf{\Upsilon}^e \underline{\mathbf{T}}(t),$$

where  $\underline{\mathbf{T}}(t) = [1, t, \dots, t^N]^T$ , and  $\mathbf{\Upsilon}^e$  is a  $(N + 1) \times (N + 1)$  lower-triangular coefficient matrix. Consequently, we have

$$\mathbf{M}^e(t) = \mathbf{\Upsilon}^e ({}_{t_{e-1}}\mathcal{D}_t^{\alpha(t)} \underline{\mathbf{T}}(t)), \quad \tilde{\mathbf{M}}^k(t) = \mathbf{\Upsilon}^k ({}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t)} \underline{\mathbf{T}}(t)). \quad (2.6)$$

The action of the variable-order Caputo derivative on  $\underline{\mathbf{T}}(t)$  can be expressed explicitly in terms of the incomplete beta and the Gauss hyper-geometric functions. Precisely, we obtain

$$\tilde{D}_e(t) = {}_{t_{e-1}}\mathcal{D}_t^{\alpha(t)} \underline{\mathbf{T}}(t), \quad \hat{D}_k(t) = {}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t)} \underline{\mathbf{T}}(t), \quad (2.7)$$

where

$$\tilde{D}_e(t) = \left[ \frac{p(p-1)}{\Gamma(2-\alpha(t))} t^{p-\alpha(t)} \left( B\left[\frac{t_{e-1}}{t}, p-1, 2-\alpha(t)\right] + \frac{\Gamma(-\alpha(t))\Gamma(2-\alpha(t))}{\Gamma(1+p-\alpha(t))} \right) \right]_{p=0}^N,$$

$$\hat{D}_k(t) = \frac{1}{\Gamma(2-\alpha(t))} \left[ \left( \psi^{p-1} \left(1 - \frac{\psi}{t}\right)^{\alpha(t)} (t-\psi)^{-\alpha(t)} {}_2F_1\left[p-1, \alpha(t)-1, p, \frac{\psi}{t}\right] \right) \right]_{t_{k-1}}^{t_k} \Big|_{p=0}^N,$$

where  $S(\psi) \Big|_{t_{k-1}}^{t_k} = S(t_{k-1}) - S(t_k)$ . Furthermore,  $B(\cdot, \cdot, \cdot)$  and  ${}_2F_1(\cdot, \cdot, \cdot, \cdot)$  are the incomplete beta and the Gaussian hyper-geometric functions, respectively [1]. The above formulas have been derived directly from the definitions of the variable-order Caputo operator and the auxiliary operator acting on monomial functions, using Mathematica. By leveraging the software's symbolic computation capabilities, the resulting expressions have been reformulated in terms of well-known special functions—namely, the incomplete beta function and the Gaussian hyper-geometric function. All computations were carried out in machine precision, ensuring that the derived analytical expressions are both mathematically exact (up to floating-point representation) and numerically stable. This approach guarantees a rigorous foundation for the spectral convergence of the proposed method. Inserting (2.7) into (2.6) concludes

$$\mathbf{M}^e(t) = \mathbf{\Upsilon}^e \tilde{D}_e(t), \quad \tilde{\mathbf{M}}^k(t) = \mathbf{\Upsilon}^k \hat{D}_k(t),$$

and thereby from (2.4) we obtain

$$\sum_{k=1}^{e-1} {}_{t_{k-1}}\mathbf{D}_{t_k}^{\alpha(t_j^e)} u_N^k(t_j^e) + {}_{t_{e-1}}\mathcal{D}_t^{\alpha(t_j^e)} u_N^e(t_j^e) = \left[ \underline{\mathbf{C}}^e \mathbf{\Upsilon}^e \tilde{D}_e(t) + \sum_{k=1}^{e-1} \underline{\mathbf{C}}^k \mathbf{\Upsilon}^k \hat{D}_k(t) \right]_{t=t_j^e}. \quad (2.8)$$



Inserting (2.8) into (2.3), and incorporating continuity conditions at subdomain interfaces, for any  $t \in \Lambda_e$  we achieve the following linear algebraic system

$$\begin{cases} \underline{\mathbf{C}}^e \left[ (\underline{\mathbf{L}}^e)''(t) + \mathbf{\Upsilon}^e \tilde{D}_e(t) + \underline{\mathbf{L}}^e(t) \right]_{t=t_j^e} = \left[ f(t) - \sum_{k=1}^{e-1} \underline{\mathbf{C}}^k \mathbf{\Upsilon}^k \hat{D}_k(t) \right]_{t=t_j^e}, \\ u_N^e(t_j^e) = v_e, \quad (u_N^e)'(t_j^e) = w_e, \end{cases} \quad (2.9)$$

such that  $e = 1, 2, \dots, M$ ,  $j = 0, 1, \dots, N$ , and the interface values are considered by

$$v_e = \begin{cases} u_0, & e = 1, \\ u_N^{e-1}(t_{e-1}), & e > 1, \end{cases} \quad w_e = \begin{cases} u'_0, & e = 1, \\ (u_N^{e-1})'(t_{e-1}), & e > 1. \end{cases}$$

The solution procedure proceeds sequentially across the subdomains. After obtaining the coefficient vectors  $[\underline{\mathbf{C}}^1, \dots, \underline{\mathbf{C}}^{e-1}]$  from the preceding subdomains, the  $(N + 1) \times (N + 1)$  linear system (2.9) corresponding to the  $e$ -th subdomain is constructed and solved to determine  $\underline{\mathbf{C}}^e$ . This domain-by-domain progression ensures continuity of the approximate solution and its derivatives at the interfaces, thereby maintaining global consistency. The composite piece-wise polynomial  $u_N(t)$  obtained in this manner constitutes a high accurate approximation to the exact solution, effectively capturing both local and global behaviors of the underlying fractional system.

### 3. NUMERICAL RESULTS

In this section, the proposed multi domain Legendre collocation method is implemented on several test problems to evaluate its efficiency. To this end, the convergence behavior of the method for different values of  $N$  and  $M$ , the computational CPU-time in relation to the achieved accuracy, and the comparative results with existing methods are monitored and analyzed.

**Example 3.1.** Consider the following variable order fractional Bagley-Torvik equation

$$\begin{cases} u''(t) + {}_0\mathcal{D}_t^{\alpha(t)} u(t) + u(t) = \frac{6}{\Gamma(4-\alpha(t))} t^{3-\alpha(t)} + t^3 + 7t + 1, \quad t \in (0, 1.57], \\ u(0) = u'(0) = 1, \end{cases}$$

where  $\alpha(t) = 1 + 0.5 \sin(t)$ , and the exact solution is  $u(t) = t^3 + t + 1$ .

Figure 1, illustrates the convergence behavior of the proposed multi domain Legendre collocation method when the polynomial degree  $N$  is held fixed and the number of subdomains  $M$  is increased. The horizontal and vertical axes represent the number of subdomains ( $M = 2, 4, \dots, 256$ ) and the maximum-norm error  $\|u - u_N\|_{L_\infty}$  on a logarithmic scale, respectively. The nearly linear trend of the data in loglog plot indicates an exponentially rate of convergence for the proposed approach. A black dashed reference line connecting the first and last points highlights this behavior. The monotonic error decay  $\mathcal{O}(10^{-11})$ , without any sign of instability confirms the robustness of the domain decomposition framework and its ability to maintain high accuracy over extended temporal domains.

Figure 2 illustrates the variation of the maximum-norm numerical error  $\|u - u_N\|_{L_\infty}$  with respect to the polynomial degree  $N$  for a fixed number of subdomains ( $M = 32$ ). As evident, the error exhibits exponential decay for low-degree approximations, consistent with the expected spectral convergence resulting from the smoothness of the exact solution. However, for  $N \geq 6$ , the error begins to increase and convergence deteriorates. This behavior can be attributed to the accumulation of round-off errors and the ill-conditioning inherent in spectral discretization at higher degrees. In spectral methods, increasing the polynomial degree  $N$  leads to more severely ill-conditioned discretization matrices, which amplify round-off errors and ultimately degrade numerical stability. The proposed multi-domain approach mitigates this issue by restricting the polynomial degree within small subdomains, thereby reducing both matrix ill-conditioning and sensitivity to numerical errors. Nevertheless, increasing  $N$  beyond an optimal threshold even within a multi-domain framework triggers the Runge phenomenon, resulting in severe oscillations and error growth near subdomain boundaries. Consequently, the existence of such an optimal  $N$  is not only unsurprising but fully consistent with spectral approximation theory.

Figure 3 illustrates the balance between numerical accuracy and computational cost for the proposed multi-domain Legendre collocation method. The maximum-norm error  $\|u - u_N\|_{L_\infty}$  is plotted against CPU time on a log-log scale,



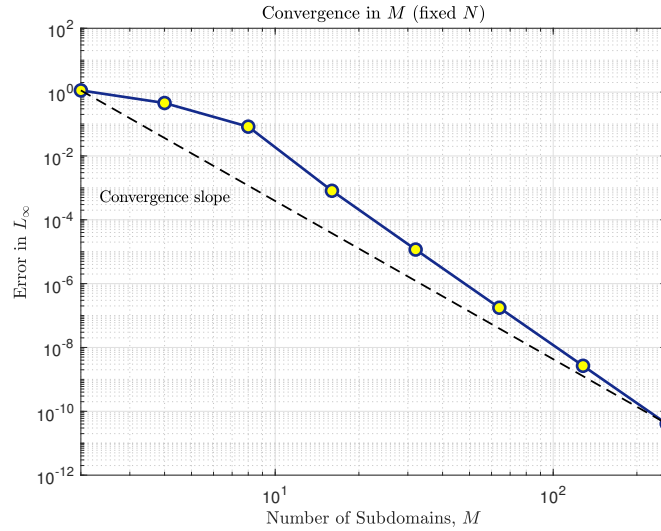


FIGURE 1. Maximum norm error  $\|u - u_N\|_{L^\infty([0, 1.57])}$  versus the number of temporal subdomains  $M$  for a fixed polynomial degree  $N = 3$ . The near linear trend in the log-log plot demonstrates spectral convergence of the proposed method.

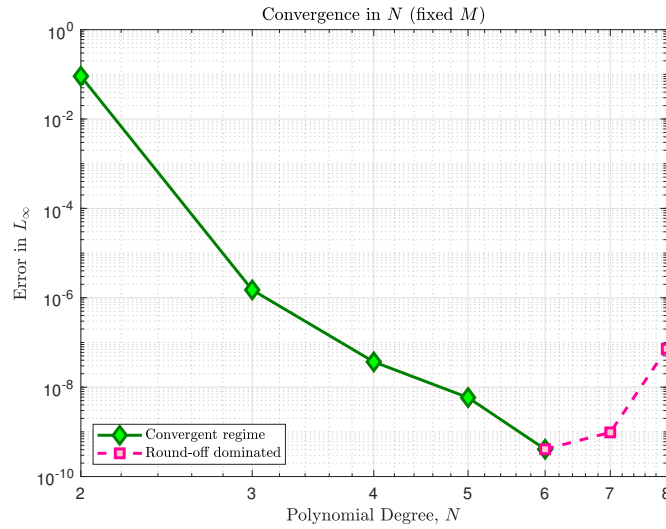


FIGURE 2. Maximum norm error  $\|u - u_N\|_{L^\infty([0, 1.57])}$  as a function of the polynomial degree  $N$  for a fixed number of temporal subdomains  $M = 32$ . The plot illustrates the spectral convergence of the method for moderate values of  $N$ , followed by error stagnation due to the onset of round-off effects and ill-conditioning at higher polynomial degrees.

where accuracy improves with temporal refinement (increasing  $M$ ) while the polynomial degree remains fixed ( $N = 3$ ). The error decays rapidly and monotonically, reaching  $10^{-11}$ , which demonstrates that very high accuracy is achieved with moderate computational effort. The near-exponential reduction of error relative to the roughly linear increase in CPU time underscores the method's high efficiency. This performance stems from two key aspects:



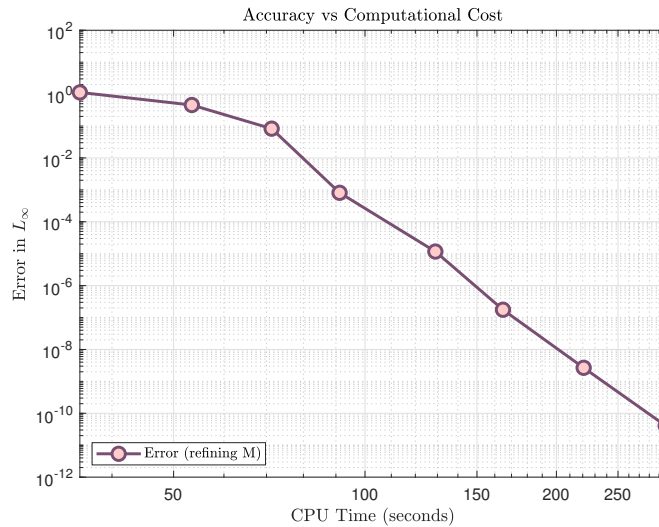


FIGURE 3. Accuracy versus computational cost: maximum norm error  $\|u - u_N\|_{L^\infty}$  plotted against CPU time (in seconds) for a fixed local polynomial degree  $N = 3$  and an increasing number of temporal subdomains  $M$ . The curve illustrates how temporal refinement enhances solution accuracy while incurring only a moderate increase in computational effort, reflecting the efficiency of the multi domain spectral framework.

- high-order spectral representations within each subdomain, minimizing the number of elements needed to resolve smooth solutions;
- an operational matrix formulation that eliminates recursive computations and limits error propagation.

All CPU times were measured on a desktop computer equipped with an Intel Core i5 processor (quad-core, base frequency 2.9 GHz), 8 GB RAM, running MATLAB R2022b under Windows 10. All computations were performed in serial mode using double-precision arithmetic.

Figure 4 displays the computational cost of the proposed method as a function of the number of subdomains  $M$ , with the polynomial degree fixed at  $N = 3$ . The CPU time increases nearly linearly with  $M$ , demonstrating excellent scalability a key attribute for large scale simulations. This linear scaling stems from the domain decomposition framework, in which each subdomain is processed independently, and the per-element computational cost dominated by matrix assembly and the solution of a small dense linear system remains essentially constant. To further substantiate this behavior and illustrate the impact of refined temporal discretization on computational expense, we have extended the results in Figure 4 to include  $M = 512$ . This additional data point confirms that the near linear growth of CPU time persists even at significantly larger values of  $M$ . Combined with the high accuracy observed in Figures 1 and 2, these findings confirm that the proposed framework achieves an optimal balance between numerical precision and computational efficiency, rendering it particularly well suited for long-time dynamical analysis of variable-order fractional differential equations.

TABLE 1. Comparison of maximum error  $\|u - u_N\|_{L^\infty}$  for the proposed method against the M-algorithm [23] and IM-algorithm [22] with different values of  $M$ .

$M$	Our method	$\frac{T}{M}$	M-algorithm [23]	IM-algorithm [22]
64	$1.7569e - 07$	0.02	$7.22e - 03$	$1.53e - 04$
128	$2.6648e - 09$	0.01	$3.70e - 03$	$5.57e - 05$



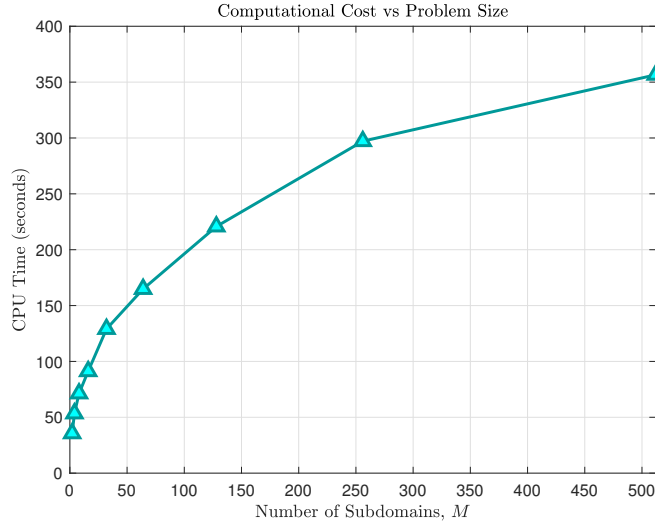


FIGURE 4. Computational cost versus problem size: CPU time (seconds) as a function of the number of temporal subdomains  $M$  for fixed polynomial degree  $N = 3$ . The near-linear growth reflects the linear scalability of the multi domain framework, confirming its efficiency for large-scale or long-time simulations of variable-order fractional systems.

Table 1 presents a comparative evaluation of the maximum-norm error  $\|u - u_N\|_{L^\infty}$  obtained by the proposed multi-domain Legendre collocation method against the errors reported by the M-algorithm [23] and the IM-algorithm [22], for two levels of temporal refinement:  $M = 64$  and  $M = 128$ . The chosen values of  $M$  are selected such that the corresponding time-step size  $T/M$  exactly matches the temporal discretization scales employed in [22, 23], thereby ensuring a fair and directly comparable assessment across all methods under identical temporal resolution criteria. The results unequivocally demonstrate the superior accuracy and scalability of the proposed scheme: even with a low spectral order ( $N = 3$ ), increasing the number of subdomains ( $M = 64, 128$ ) yields a computational error that is significantly lower than those reported in [22, 23]. This remarkable improvement in accuracy stems from the synergistic combination of (i) high-order spectral approximation within each subdomain and (ii) a stable operational matrix formulation specifically tailored for variable-order fractional derivatives.

**Example 3.2.** Consider the following variable order fractional Bagley-Torvik equation with the variable coefficients

$$\begin{cases} u''(t) + (t+1)_0\mathcal{D}_t^{\alpha(t)}u(t) + u(t) = f(t), & t \in (0, 5], \\ u(0) = 1, & u'(0) = 1, \end{cases} \quad (3.1)$$

where  $\alpha(t) = 0.95 + \frac{t+1}{10.5}$ , and the exact solution is  $u(t) = e^t$ .

Figure 5, illustrates the convergence behavior of the proposed method over the large interval  $[0, 5]$ . Despite the substantial increase in temporal extent along with the corresponding amplification of memory effects and round-off error accumulation, the maximum-norm error decreases steadily as  $M$  increases. The nearly linear trend in the loglog plot confirms stable exponential rate of convergence of the proposed approach, demonstrating the robustness of the domain decomposition framework. This result highlights that the proposed scheme preserves high accuracy even over significantly enlarged domains, where conventional spectral approaches typically lose stability or experience notable accuracy degradation.

This robust long-time behavior arises from the local evaluation of the memory integral in each subdomain. In contrast to global spectral methods which accumulate weakly singular kernel contributions over  $[0, t]$  at every evaluation



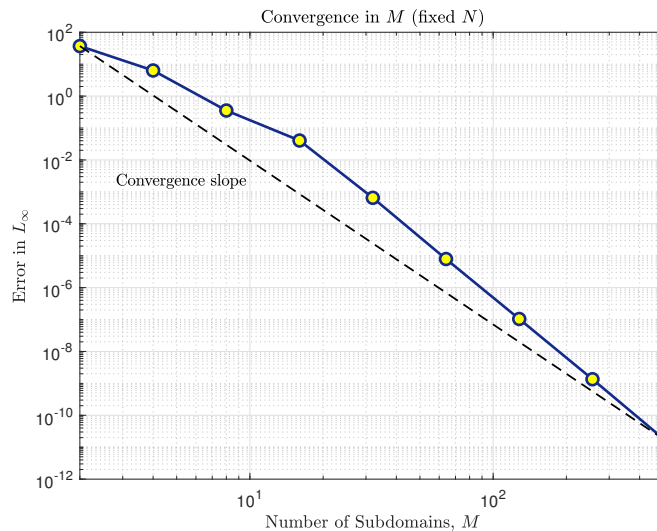


FIGURE 5. Maximum norm error  $\|u - u_N\|_{L^\infty([0,5])}$  versus the number of temporal subdomains  $M$  for a fixed polynomial degree  $N = 3$ . The near linear trend in the log-log plot demonstrates spectral convergence of the proposed method.

point the proposed framework restricts history dependence to a few preceding subdomains, thereby suppressing round-off error growth and ill-conditioning. Furthermore, the analytical form of the variable-order operator, expressed via incomplete beta and Gauss hyper-geometric functions and combined with a non-recursive operational matrix, incorporates memory effects without numerical quadrature over past intervals, preserving both accuracy and stability over long time domains.

Figure 6, monitors the numerical error variations for different values of  $N$  and for a fixed number of subdomains  $M = 32$ . The illustrated results indicates a convergence deterioration especially for  $N \geq 6$  that substantiates the necessity of prioritizing multi-domain methods over classical approaches when selecting appropriate numerical strategies for solving problems over long time intervals.

Figure 7 demonstrates that, despite increased problem complexity, the proposed scheme reduces the error by with modest CPU growth. This efficiency is maintained over a long domain, highlighting the methods ability to deliver high-accuracy solutions over extended time intervals without compromising computational performance, in contrast to standard spectral or finite-difference approaches.

Figure 8 demonstrates the computational scalability of the proposed method over the extended interval  $[0, 5]$ . To further verify the near-linear growth of CPU time with respect to the number of subdomains  $M$ , we include the case  $M = 1024$  in addition to the previously tested values. The results confirm that the linear relationship between problem size and computational cost persists even for large  $M$  and over significantly longer time horizons. This robust scalability unaffected by the accumulation of memory effects or round-off errors highlights a key strength of the multi domain framework: it enables efficient and stable long-time simulations of variable order fractional dynamical systems, where global or recursive methods often suffer from severe computational overhead or numerical instability.

#### 4. CONCLUSION AND FUTURE WORK

In this paper, a novel multi-domain Legendre collocation method is proposed for solving the variable-order fractional Bagley-Torvik equation, effectively combining spectral accuracy with the flexibility of domain decomposition. The operational matrix formulation eliminates recursive evaluations and mitigates round-off error, thereby ensuring high accuracy even over long time intervals. Numerical results confirm the efficiency and robustness of the method. Moreover, the modular structure of the approach naturally permits its extension to nonlinear and multi-dimensional



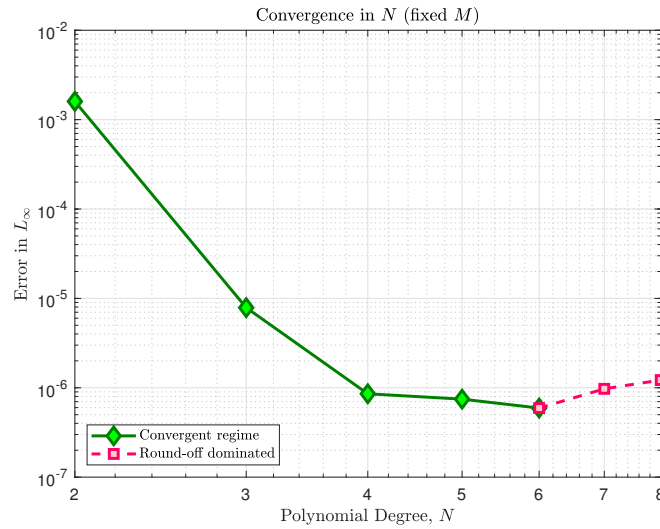


FIGURE 6. Maximum norm error  $\|u - u_N\|_{L^\infty([0,5])}$  as a function of the polynomial degree  $N$  for a fixed number of temporal subdomains  $M = 32$ . The plot illustrates the spectral convergence of the method for moderate values of  $N$ , followed by error stagnation due to the onset of round-off effects and ill-conditioning at higher polynomial degrees.

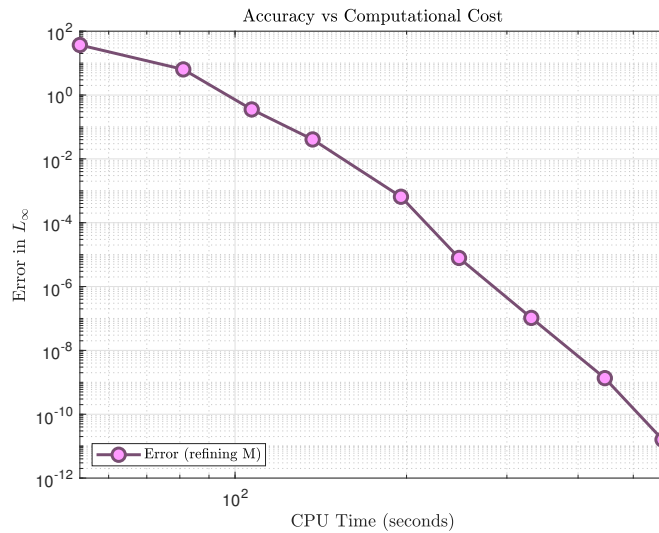


FIGURE 7. Accuracy versus computational cost: maximum norm error  $\|u - u_N\|_{L^\infty}$  plotted against CPU time (in seconds) for a fixed local polynomial degree  $N = 3$  and an increasing number of temporal subdomains  $M$ . The curve illustrates how temporal refinement enhances solution accuracy while incurring only a moderate increase in computational effort, reflecting the efficiency of the multi domain spectral framework.



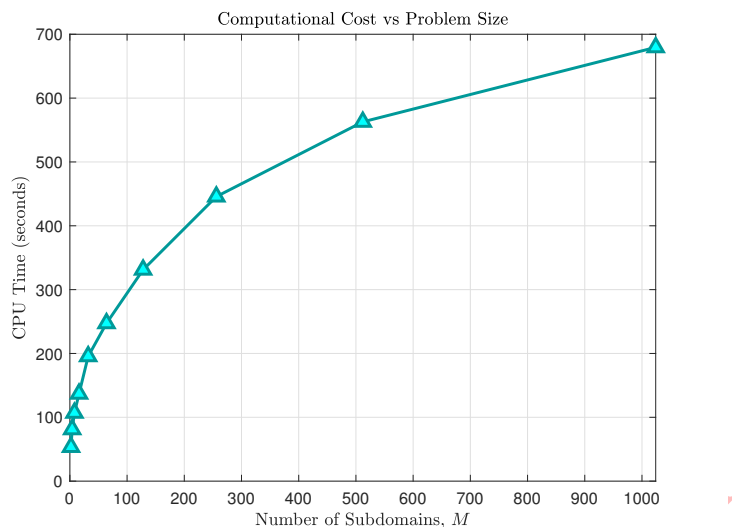


FIGURE 8. Computational cost versus problem size: CPU time (seconds) as a function of the number of temporal subdomains  $M$  for fixed polynomial degree  $N = 3$ . The near-linear growth reflects the linear scalability of the multi domain framework, confirming its efficiency for large-scale or long-time simulations of variable-order fractional systems.

variable-order problems-provided the fractional order function possesses sufficient smoothness-establishing it as a general-purpose framework for modeling complex dynamical systems.

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Uncorrected Proof

